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ABSTRACT

In this paper, some contiguous function relations and generating functions have been established for extended Jacobi polynomials. The results obtained here, are of very general nature.

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The object of this note is to furnish some contiguous function relations and generating functions for extended Jacobi polynomials defined as

$$f_n^{(\alpha,\beta)}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_q;y) \equiv \frac{(1+\alpha)_n}{n!}\,{}_{q+2}F_{q+1}\left[\begin{matrix}-n,\ 1+\alpha+\beta+n,\ \alpha_1,\dots,\alpha_q; \\ 1+\alpha,\ \beta_1,\dots,\beta_q;\end{matrix}\ y\right] \tag{1}$$

(where n is a non–negative integer), by using a very simple and easy integral given as Theorem 28 in [4, p.85].

CONTIGUOUS FUNCTION RELATIONS

We have from [4, p.264(9)]

$$(\alpha+\beta+n)\,P_n^{(\alpha,\beta)}(x) = (\beta+n)\,P_n^{(\alpha,\beta-1)}(x) + (\alpha+n)\,P_n^{(\alpha-1,\beta)}(x)\,, \tag{2}$$

from [4, p.265(14)]

$$(\alpha+\beta+2n)\,P_n^{(\alpha,\beta-1)}(x) = (\alpha+\beta+n)\,P_n^{(\alpha,\beta)}(x) + (\alpha+n)\,P_{n-1}^{(\alpha,\beta)}(x) \tag{3}$$

and from [4, p.265(15)]

$$(\alpha+\beta+2n)\,P_n^{(\alpha-1,\beta)}(x) = (\alpha+\beta+n)\,P_n^{(\alpha,\beta)}(x) - (\beta+n)\,P_{n-1}^{(\alpha,\beta)}(x)\,. \tag{4}$$

In these relations, replacing x by $(1-2vx)$, multiplying both sides by $x^{\xi-1}(1-x)^{p-\xi-1}$, and integrating between limits 0 and 1, with respect to x and using Theorem 28 in [4, p.85], we get respectively,

$$(\alpha+\beta+n) H_n^{(\alpha,\beta)}(\xi, p, v)$$
$$= (\beta+n) H_n^{(\alpha,\beta-1)}(\xi, p, v) + (\alpha+n) H_n^{(\alpha-1,\beta)}(\xi, p, v), \tag{5}$$

$$(\alpha+\beta+2n) H_n^{(\alpha,\beta-1)}(\xi, p, v)$$
$$= (\alpha+\beta+n) H_n^{(\alpha,\beta)}(\xi, p, v) + (\alpha+n) H_{n-1}^{(\alpha,\beta)}(\xi, p, v) \tag{6}$$

and

$$(\alpha+\beta+2n) H_n^{(\alpha-1,\beta)}(\xi, p, v)$$
$$= (\alpha+\beta+n) H_n^{(\alpha,\beta)}(\xi, p, v) - (\beta+n) H_{n-1}^{(\alpha,\beta)}(\xi, p, v), \tag{7}$$

where

$$H_n^{(\alpha,\beta)}(\xi, p, v) = \frac{(1+\alpha)_n}{n!}\, {}_3F_2\left[\begin{matrix} -n, 1+\alpha+\beta+n, \xi; \\ 1+\alpha, p; \end{matrix}\; v\right]$$

is a generalized Rice's polynomial [3, p.157]. On combining (6) and (7), we get

$$H_n^{(\alpha,\beta-1)}(\xi, p, v) - H_n^{(\alpha-1,\beta)}(\xi, p, v) = H_n^{(\alpha,\beta)}(\xi, p, v). \tag{8}$$

Repeating $q-1$ times the same procedure as given in the above paragraph, one can easily obtain the following contiguous function relations for extended Jacobi polynomials:

$$(\alpha+\beta+n) f_n^{(\alpha,\beta)}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q; y)$$
$$= (\beta+n) f_n^{(\alpha,\beta-1)}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q; y)$$
$$+ (\alpha+n) f_n^{(\alpha-1,\beta)}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q; y), \tag{9}$$

$$(\alpha+\beta+2n) f_n^{(\alpha,\beta-1)}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q; y)$$
$$= (\alpha+\beta+n) f_n^{(\alpha,\beta)}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q; y)$$
$$+ (\alpha+n) f_{n-1}^{(\alpha,\beta)}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q; y) \tag{10}$$

and

$$(\alpha+\beta+2n) f_n^{(\alpha-1,\beta)}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q; y)$$
$$= (\alpha+\beta+n) f_n^{(\alpha,\beta)}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q; y)$$
$$- (\beta+n) f_{n-1}^{(\alpha,\beta)}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_q; y) \tag{11}$$

On combining (10) and (11), we obtain

$$
\begin{aligned}
&f_n^{(\alpha,\beta-1)}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_q;y) \\
&\quad - f_n^{(\alpha-1,\beta)}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_q;y) \\
&= f_n^{(\alpha,\beta)}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_q;y).
\end{aligned}
\tag{12}
$$

GENERATING FUNCTIONS

We have from [2, p.120]

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta)}(x)\,t^n = (1+t)^{\alpha}\left[1-\tfrac{1}{2}(x-1)t\right]^{-\alpha-\beta-1}, \tag{13}$$

from [2, p.120]

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta-n)}(x)\,t^n = (1-t)^{\beta}\left[1-\tfrac{1}{2}(x-1)t\right]^{-\alpha-\beta-1} \tag{14}$$

and from [1]

$$
\begin{aligned}
&\sum_{n=0}^{\infty} P_n^{(\alpha-\frac{n}{2},\,\beta-\frac{n}{2})}(x)\,t^n \\
&\quad = \left[1+u(t)\right]^{\alpha+1}\left[1+\tfrac{1}{2}u(t)\right]^{-1}\left[1-\tfrac{1}{2}(x-1)u(t)\right]^{-\alpha-\beta-1},
\end{aligned}
\tag{15}
$$

where $u(t)=\frac{1}{2}t[t+\sqrt{t^2+4}]$.

In these relations, replacing x by $(1-2vx)$, multiplying both sides by $x^{\xi-1}(1-x)^{p-\xi-1}$, integrating between the limits 0 and 1, with respect to x and using Theorem 28 in [4, p.28], we get respectively,

$$\sum_{n=0}^{\infty} H_n^{(\alpha-n,\beta)}(\xi,p,v)\,t^n = (1+t)^{\alpha}\,{}_2F_1\left[\begin{matrix}1+\alpha+\beta,\ \xi; \\ p;\end{matrix}\ -vt\right], \tag{16}$$

$$\sum_{n=0}^{\infty} H_n^{(\alpha,\beta-n)}(\xi,p,v)\,t^n = (1-t)^{-\alpha-1}\,{}_2F_1\left[\begin{matrix}1+\alpha+\beta,\ \xi; \\ p;\end{matrix}\ \frac{vt}{t-1}\right] \tag{17}$$

and

$$
\begin{aligned}
&\sum_{n=0}^{\infty} H_n^{(\alpha-\frac{n}{2},\,\beta-\frac{n}{2})}(\xi,p,v)\,t^n \\
&\quad = \left[1+u(t)\right]^{\alpha+1}\left[1+\tfrac{1}{2}u(t)\right]^{-1}{}_2F_1\left[\begin{matrix}1+\alpha+\beta,\ \xi; \\ p;\end{matrix}\ -v\,u(t)\right]
\end{aligned}
\tag{18}
$$

where $u(t)=\frac{1}{2}t[t+\sqrt{t^2+4}]$.

The results (16), (17) and (18) can be obtained from [5, p.591(9)] also, but our approach is entirely different here. Further repeating $q-1$ times the same procedure as given in the above paragraph, we get the following generating functions for extended Jacobi polynomials:

$$
\begin{aligned}
&\sum_{n=0}^{\infty} f_n^{(\alpha-n,\beta)}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_q;y)\,t^n \\
&\quad = (1+t)^{\alpha}\,{}_{q+1}F_q\left[\begin{matrix}1+\alpha+\beta,\ \alpha_1,\dots,\alpha_q; \\ \beta_1,\dots,\beta_q;\end{matrix}\ -yt\right],
\end{aligned}
\tag{19}
$$

$$\sum_{n=0}^{\infty} f_n^{(\alpha,\beta-n)}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_q;y)\,t^n$$

$$=(1-t)^{-\alpha-1}\,{}_{q+1}F_q\left[\begin{matrix}1+\alpha+\beta,\alpha_1,\dots,\alpha_q; \\ \beta_1,\dots,\beta_q;\end{matrix}\ \frac{yt}{t-1}\right] \tag{20}$$

and

$$\sum_{n=0}^{\infty} f_n^{(\alpha-\frac{n}{2},\beta-\frac{n}{2})}(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_q;y)\,t^n$$

$$=[1+u(t)]^{\alpha+1}[1+\tfrac{1}{2}u(t)]^{-1}\,{}_{q+1}F_q\left[\begin{matrix}1+\alpha+\beta,\alpha_1,\dots,\alpha_q; \\ \beta_1,\dots,\beta_q;\end{matrix}\ -y\,u(t)\right] \tag{21}$$

where $u(t) = \frac{1}{2}t[t+\sqrt{t^2+4}]$.

PARTICULAR CASES

In (8), putting $\xi = p$ and $v = \frac{1-x}{2}$, we get a well-known result [4, p.265(17)] for Jacobi polynomials. Similarly, on specializing the paremeters, the results obtained here, can be reduced to so many other well-known polynomials.

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REFERENCES

[1] Brown J.W., "On zero type sets of Lagurre polynomials", Duke Math. J. 35 (1968) 821–827.

[2] Feldheim E., "Relations entre les polynomes de Jacobi, Lagurre et Hermite", Acta Math. 74 (1941) 117–138.

[3] Khandekar P.R., "On a generalization of Rice's polynomial", Proc. Nat. Acad. Sci., India Section A, 34(1964) 157–162.

[4] Rainville E.D., **Special Functions** (MacMillan Co., New York) (1960).

[5] Srivastava H.M., "Generating functions for Jacobi and Lagurre polynomials", Proc. Amer. Math. Soc. 23 (1969) 590–595.