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ABSTRACT

We describe the physical situation in which the consideration of the fermionic vacuum bundle over gauge group manifold is useful. The bundle curvature turns out to be connected with the commutator gauge anomaly. This allows to consider the problem where this curvature is manifested.

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I. Introduction.

It is well known that two types of anomalies arise for the quantum chiral fermions in an external nonabelian gauge field. The first type is connected with the violation of the classical conservation law $\nabla^{\mu} J^{a}_{\mu} = 0$ for the gauge current [I]. Mathematically, the appearance of this anomaly results from the existence of the nontrivial one-cocycle in the gauge group cohomology [2,3]. On the other hand, the existence of a nontrivial two-cocycle leads under quantization to violation of the gauge charges algebra for chiral fermions and to the appearance of central terms [3]. Gauge charges generate the gauge transformations on the time-like hypersurface, thus, their quantum algebra is not closed.

It was shown by Bowick and Rajeev [4] for the boson string case that the commutator anomaly means noninvariance of the Fock space under the gauge transformations. Thus one may say about nontrivial bundle of the Fock spaces over the gauge group (for boson string the gauge group is $\mathrm{Diff}\, S_1 / S_1$), curvature of which coincides with the central term of the charge algebra. Pilch and Warner have shown that it is sufficient to consider the vacuum bundle [5]. This point of view is widely discussed in the literature [6-9].

So one comes to the following geometrical picture. One can consider the classical algebra of gauge charges as an algebra of the tangent vectors over the gauge group manifold. The quantization then consists in substitution of the vector field by the covariant derivative along this vector field. Its curvature determines the charge algebra central extension [7,8].

The first type anomaly manifests itself in the particle creation in the external nontrivial gauge field, for example in the baryon decay catalysis by monopole [IO].

We show here that the commutator anomaly also leads to the particle creation. The external gauge field in this case is a "pure gauge" at arbitrary moment of time:

$$i A_\mu = g^{-1} \partial_\mu g , \qquad (I.I)$$

where μ is a space index and we use the gauge $A_0^a = 0$.

The gauge potential (I.I) at any subsequent time moment is the gauge transformation of gauge potential at previous moment:

$$\begin{aligned} i A_\mu (t+\Delta t) &= \delta g^{-1} i A_\mu (t) \delta g + \delta g^{-1} \partial_\mu \delta g , \\ \delta g &= g^{-1}(t) g(t+\Delta t) = \alpha^a(t) \Delta t \lambda^a , \end{aligned} \qquad (I.2)$$

where λ^a are generators of group G (we limit ourselves to the case of unitary groups).

Functions $\alpha^a(t)$ determine the curvature on the gauge group manifold. We define this manifold as (for two-dimensional cylindrical space-time) $MG = \{ g(x) | x \in S_1 , g \in G \}$. In any point of MG we have vacuum and corresponding Fock space. Transition from one vacuum to another means particle production. The method of ref.[4,5] allows to connect the number of produced particles with curvature of vacuum bundle over MG, i.e. with anomaly of the charge algebra. Especially, we are interested in the acse of closed curve on MG (i.e. $g(t)$ is the same for $t=-\infty$ and $t=+\infty$). However out-vacuum may differ from in-vacuum, this difference is determined by integral of the vacuum bundle curvature 2-form.

We shall demonstrate here the abovementioned effect for 2D space-time and non-abelian gauge group case. This paper is organized as follows. For the sake of completeness, in Sect.2 we give the calculations of commutator anomaly as curvature using the method of ref.[5] (this section is based on results of [9]). In Sect.3 we consider realization of the time-dependent gauge transformation as a curve on the manifold MG. For closed curve the $\langle out | in \rangle$ -amplitude is expressed in terms of integral of the vacuum bundle curvature two-form. In section 4 we discuss monopole-like external gauge field. In this case it is necessary to take into account the nontrivial topology of the manifold MG, which can be decomposed into separate sets corresponding to the elements of the homotopy group $\pi_1(G)$. In Sect.5 we discuss results obtained.

2. Commutator anomaly as curvature.

In this section we consider the geometrical treatment of the commutator anomaly in chiral non-abelian charge algebra in the spirit of papers [4,5]. We calculate the anomaly as curvature of fermion vacuum bundle over the gauge group MG.

Let us consider two-dimensional Weyl fermions, interacting with an external non-abelian gauge field A_μ^a . The Lagrangian has the form:

$$L = i \bar{\psi} \gamma^\mu \nabla_\mu \psi , \qquad (2.I)$$

where $\nabla_\mu \psi = \partial_\mu \psi + i A_\mu^a \lambda^a \psi$, λ^a are hermitian matrices.

We suppose that the chiral fermions satisfy the condition $\gamma_5 \psi = -\psi$ and we use the following representation for two-dimensional γ-matrices:

$$\gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \gamma_0 \gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

From (2.I) we find the expression for canonical momentum:

$$P_\psi = i \psi^+$$

and obtain the Hamiltonian in the gauge $A_0^a = 0$

$$H = \int_0^{2\pi} d\sigma \left[i \bar{\psi} \gamma_1 \partial_\sigma \psi - \bar{\psi} \gamma_1 A_1^a \lambda^a \psi \right].$$

Because of chirality we have

$$\Psi = \begin{pmatrix} 0 \\ \psi \end{pmatrix},$$

so the Hamiltonian reads

$$H = -i \int_0^{2\pi} \psi^+ (\partial_\sigma + i \lambda^a A^a_0) \psi \, d\sigma . \qquad (2.2)$$

Note that hereafter we imply that two-dimensional space-time is cylindrical. Therefore all field functions are periodical in σ with the period 2π.

It can easily be seen from (2.2) that the fermion Hamiltonian is not diagonal. One should diagonalize it in order to determine the creation and annihilation operators which enable us to define correctly zero-energy vacuum.

Let us substitute the following ansatz into (2.2)

$$\psi(\sigma) = F^{-1}(\sigma)\, \phi(\sigma),$$
$$F(\sigma) = P \exp\left(-i \int_0^\sigma A(\sigma')\, d\sigma'\right). \qquad (2.3)$$

Here, $A = A^a \lambda^a$, and P stands for path-ordered exponent.

Then we obtain

$$H = -i \int_0^{2\pi} d\sigma\, \phi^+(\sigma)\, \partial_\sigma \phi(\sigma).$$

For function $\phi(\sigma)$ one has the Fourier expansion:

$$\phi(\sigma) = \frac{1}{\sqrt{2\pi}} \sum_{n>0} \left(a_n e^{in\sigma} + b_n^+ e^{-in\sigma} \right)$$

and finally for the Hamiltonian (2.2) we find

$$H = \sum_{n>0}' n \left(a_n^+ a_n - b_n b_n^+ \right)$$

Thus the ansatz (2.3) diagonalizes the Hamiltonian. It should be

noted that the above calculations were made on the constant time (t=0) hypersurface.

The approach of ref. [4,5] is based on the method of geometrical quantization, which results in the quantum Hilbert space of functions $\phi(a_n^+, b_n^+)$ of anticommutating variables with the following measure

$$\langle \phi_1 | \phi_2 \rangle = \int \exp\left[-\sum \left(a_n^+ a_n + b_n b_n^+ \right) \right] \phi_1^* \phi_2 \cdot \prod_n da_n\, da_n^+\, db_n\, db_n^+ . \qquad (2.4)$$

The operators

$$\hat{a}_n = \frac{\delta}{\delta a_n^+}, \quad \hat{b}_n = \frac{\delta}{\delta b_n^+}, \quad \hat{a}_n^+ = a_n^+, \quad \hat{b}_n^+ = b_n^+$$

act on this Hilbert space.

The vacuum state ϕ_0 is determined by the conditions

$$\hat{a}_n \phi_0 = 0, \quad \hat{b}_n \phi_0 = 0. \qquad (2.5)$$

From the field ψ one can construct by standard procedure charges which generate gauge transformation for ψ :

$$\psi' = \psi + \{\psi, G[u]\}$$
$$G[u] = i \int_0^{2\pi} \psi^+ \lambda^a u^a \psi \, d\sigma , \qquad (2.6)$$

where u^a is an infinitesimal transformation parameter.

Under canonical transformation generated by $G[u]$ (2.6) we have

$$a_k' = a_k + \{a_k, G[u]\}, \quad b_k' = b_k + \{b_k, G[u]\} \qquad (2.7)$$

etc.

One can also consider the geometrical quantization for new variables (a'_k, $a^{+\prime}_k$, b'_k , $b^{+\prime}_k$). Then the new vacuum state ϕ'_0 will be determined by the conditions

$$\hat{a}'_n \, \phi'_0 = 0 , \quad \hat{b}'_n \, \phi'_0 = 0 . \tag{2.8}$$

Substituting (2.6), (2.7) into (2.8) and taking into account (2.3) one gets for the new vacuum

$$\phi'_0 = \exp\left(a^+_m \, \gamma_{mn} \, b^+_n \right) \phi_0 . \tag{2.9}$$

The vacuum bundle over the gauge group MG is an homogenous space. Therefore, all calculations can be limited by the vicinity of unit. Thus in the expression (2.9) for γ_{mn} one is interested only in the first order terms in variables u^a .

By direct calculations one obtains

$$\gamma_{mn} \approx - \frac{i}{4\pi} \int_0^{2\pi} d\sigma \, e^{-i(n+m)\sigma} \, F \cdot u F^{-1} . \tag{2.10}$$

Let us note that old vacuum state ϕ_0 is constant which is determined from condition $\langle \phi_0 | \phi_0 \rangle = 1$. Therefore the new vacuum state has the norm

$$\langle \phi'_0 | \phi'_0 \rangle = \int \exp \left(a^+_m \gamma_{mn} b^+_n + b_n \gamma^+_{nm} a_m - a^+_m a_m - b_m b^+_m \right) \prod_n d a_n \, d a^+_n \, d b_n \, d b^+_n .$$

After calculation of the functional integral one gets

$$\langle \phi'_0 | \phi'_0 \rangle = \det (1 - \bar{\gamma} \gamma) .$$

Thus we have the Hermitian metric in the vacuum bundle over MG:

$$g = \det (1 - \bar{\gamma} \gamma) . \tag{2.11}$$

The metrical connection one-form Γ is determined as usual

$$\nabla g \equiv dg + \Gamma g = 0 ,$$
$$\Gamma = - g^{-1} dg .$$

The curvature $R = d\Gamma$ formally should be equal to zero. However as it is shown in Appendix A, calculation of R by point-splitting regularization method gives the following result :

$$R = - \frac{i}{4\pi} \int_0^{2\pi} \mathrm{Tr} \left[\frac{d}{d\sigma} \left(F du F^{-1} \right) \wedge \left(F du F^{-1} \right) \right] d\sigma . \tag{2.12}$$

Since $\frac{dF}{d\sigma} = i F A$, so one has finally

$$R = - \frac{i}{4\pi} \int_0^{2\pi} \mathrm{Tr} \left[\frac{d}{d\sigma} (du) \wedge du \right] d\sigma + \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Tr} \left[A \, du \wedge du \right] . \tag{2.13}$$

It should be noted that the algebra of classical charges $G_a(x)$ (2.6) can be considered as an algebra of tangent vectors on MG in vicinity of unit. The quantization then corresponds to covariant derivatives along these vector fields [7,8] with curvature 2-form (2.13).

Hence we receive the algebra of quantum anomalous charges (2.6):

$$\left[G^a(x), G^b(y) \right] = f^c_{ab} \, G_c(x) + R_{ab}(x,y) \tag{2.14}$$

with central term

$$R_{ab}(x,y) = \frac{i}{4\pi} \mathrm{Tr} \left[\lambda_a \lambda_b \frac{d}{dx} \delta(x-y) \right] + \frac{1}{4\pi} f^d_{ab} \, \mathrm{Tr} \left[\lambda_c \lambda_d A_c \right] \delta(x-y) \tag{2.15}$$

Thus working with Weyl fermions in the external nonabelian gauge field one should consider the fermion vacuum bundle over gauge group manifold MG. The nontriviality of this bundle means appearance anomalous term in the quantum algebra of gauge charges.

3. Time-dependent gauge transformation.

The idea considered in this section consists in realization of time-dependent gauge transformations as curves C on the manifold MG; Then at every time moment the reconstruction of the quantum fermion vacuum state takes place and the time-dependent vacuum $\phi_c(t)$ will be crossection of the vacuum bundle over MG. Since this bundle is not trivial, vacuum state $\phi_c(t)$ depends on corresponding path C on MG. Especially, we are interested in closed curve case. Then the system is in the same point of MG at time moments $T=-\infty$ and $t=+\infty$. However the corresponding in- and out-vacuums will be differed and the difference will be determined by the curvature 2-form R integral (i.e. by holonomy group):

$$\int_S R$$

on the surface S with boundary C.

Let us consider the Weyl 2D-fermions interecting with external nonabelian gauge field A^a_μ. The Lagrangian (2.I) for the chiral fermions in the gauge A^a_0 is as follows

$$L = i\psi^+\partial_0\psi + i\psi^+(\partial_\sigma + iA)\psi, \qquad (3.I)$$

where $A = A^a_1\lambda^a$.

From (3.I) we have equation of motion for ψ:

$$(\partial_0 + \partial_\sigma + iA)\psi(\sigma,t) = 0. \qquad (3.2)$$

Let us suppose that the external gauge field has the following form:

$$A(\sigma,t) = \bar{A}(\sigma) + \nabla_\sigma\alpha(\sigma,t), \qquad (3.3)$$

where $\nabla_\sigma\alpha = \frac{d\alpha}{d\sigma} + i(\bar{A}\alpha - \alpha\bar{A})$ is covariant derivative with respect to a background time-independent field $\bar{A}(\sigma)$.

The potential (3.3) can be rewritten also as

$$iA = g^{-1}\partial_\sigma g + ig^{-1}\bar{A}g, \qquad (3.4)$$

where $g = \exp(i\lambda^a\alpha^a)$ is the element of gauge group MG.

Hence we come to the equation

$$(\partial_0 + \partial_\sigma + i\bar{A}(\sigma) + i\nabla_\sigma\alpha(\sigma,t))\psi = 0 \qquad (3.5)$$

with periodical boundary conditions on the functions $\bar{A}$ and ψ.

Substituting

$$\psi(\sigma,t) = F^{-1}(\sigma)\,\phi(\sigma,t) \qquad (3.6)$$

with

$$F^{-1}(\sigma) = P\exp\left(-i\int_0^\sigma \bar{A}(\sigma')d\sigma'\right) \qquad (3.7)$$

into (3.5) we obtain the equation for $\phi(\sigma,t)$:

$$(\partial_0 + \partial_\sigma + iF\nabla_\sigma\alpha F^{-1})\phi = 0. \qquad (3.8)$$

Since

$$F\nabla_\sigma\alpha F^{-1} = \frac{d}{d\sigma}V(\sigma,t) \qquad (3.9)$$

where

$$V(\sigma,t) = F\alpha F^{-1} \qquad (3.IO)$$

the equation (3.8) has the form:

$$\left(\partial_0 + \partial_\sigma + i\frac{dV}{d\sigma}\right)\phi(\sigma,t) = 0. \qquad (3.II)$$

For the Fourier expansion we have

$$\phi(\sigma,t) = \sum_{n=-\infty}^{+\infty}\phi_n(t)e^{in\sigma} \qquad (3.I2)$$

$$V(\sigma,t) = \beta(t)\sigma + \sum_{n=-\infty}^{+\infty}V_n(t)e^{in\sigma} \qquad (3.I3)$$

where $\beta = \beta^a\lambda^a$.

One should impose the periodicity condition on the derivative $\frac{d}{d\sigma}V(\sigma,t)$. As a consequence, function $V(\sigma,t)$ in general has the form (3.I3) with nonperiodical (monopole) term.

The monopole term will be considered in the next section. Here we suppose that function $V(\sigma,t)$ is periodical and search the solution of (3.II) in the following form

$$\phi(\sigma,t)= e^{-iV(\sigma,t)}\mathcal{F}(\sigma,t) \qquad (3.I4)$$

Substituting (3.I4) into (3.II) one gets

$$\left[\partial_t+\partial_\sigma-i\frac{dV}{dt}\right]\mathcal{F}=0. \qquad (3.I5)$$

We shall suppose that at first the system is in the unit on MG, i.e. $V(\sigma,0)=0$.

The solution of equation (3.I5) is found in Appendix B so $\phi(\sigma,t)$ reads:

$$\phi(\sigma,t)=e^{-iV}\frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{+\infty}d_n(t)\,e^{in(\sigma-t)}=$$

$$=e^{-iV}\frac{1}{\sqrt{2\pi}}\sum_{n>0}\left(a_n(t)\,e^{in(\sigma-t)}+b_n^+(t)\,e^{-in(\sigma-t)}\right), \qquad (3.I6)$$

where $d_n(t)=A_{nm}(t)d_m^c$ are creation and annihilation operators at the moment t (d_m^c are operators at t=0). Matrix $A_{nm}(t)$ has the form (B.23).

After substituting (3.6) the Hamiltonian corresponding to the Lagrangian (3.I)

$$H=-i\int_0^{2\pi}d\sigma\,\psi^+(\partial_\sigma+i\bar{A}+i\nabla_\sigma\alpha)\psi \qquad (3.I7)$$

transforms into

$$H=-i\int_0^{2\pi}d\sigma\,\phi^+(\partial_\sigma+iV)\phi \qquad (3.I8)$$

and for solution (3.I6) one gets

$$H=\sum_{n>0}n\left(a_n^+(t)\,a_n(t)-b_n(t)\,b_n^+(t)\right). \qquad (3.I9)$$

Hence the solution (3.I6) diagonalizes the Hamiltonian at every time moment.

As it is seen from (B.20), (B.23) $A_{nm}(t)$ is unitary matrix:

$$A^+(t)\,A(t)=\mathbb{1} \qquad (3.20)$$

Hence at every time moment we have the unitary equivalent set of creation and annihilation operators. It should be noted that one can consider the above problem from the Berry phase point of view [II], i.e. one can suppose V_n to be the adiabaticaly time-dependent variables. However the above results differ from the standard results by Berry, because $A_{nm}(t)$ is unitary operator but not simply a phase factor.

As it is seen from (3.I4), (3.I6) the solution (3.5) can be rewritten as follows

$$\psi(\sigma,t)=e^{-i\alpha(\sigma,t)}\psi_c(\sigma,t), \qquad (3.2I)$$

where $\psi_c(\sigma,t)$ is solution of equation (3.5) for external potential $\left[\bar{A}(\sigma)-\frac{d\alpha}{dt}\right]$:

$$\psi_0(\sigma,t)=F^{-1}(\sigma)\,\mathcal{F}(\sigma,t). \qquad (3.22)$$

Thus if $\alpha(\sigma,t)$ is varying slowly (adiabatically) with time one obtains solution (3.2I) at every time moment as time-dependent gauge transformation, realizing the curve $C\equiv\{g=e^{i\alpha(\sigma,t)}, t=-\infty,+\infty\}$ on the manifold MG.

Let us now consider the time evolution of variables a_n, b_n, a_n^+, b_n^+. From (B.I5) one has

$$d\,d_n(t)=i\,dB_{n-m}(t)\,d_m(t), \qquad (3.23)$$

where $dB_k(t)=dV_k(t)e^{ikt}$.

For variables a_n, b_n, a_n^+, b_n^+ we obtain

$$da_n(t)=i\,dB_{n-m}a_m+i\,dB_{n+m}b_m^+ \qquad (3.24)$$

$$db_n(t)=-i\,dB_{n+m}a_m^+-i\,d\bar{B}_{m-n}b_m$$

etc.

From (3.24) it is seen that the time evolution of variables a_m, b_m represents the time-dependent gauge transformation (2.7).

Under quantization one should consider the momentum variables $a_n(t), b_n(t), \ldots$, diagonalizing the Hamiltonian (3.17)-(3.19). Thus we come to the Hilbert space of functions $\Phi(a_n^+(t), b_k^+(t))$ with measure (2.4). The following operators act on this space:

$$\hat{a}_k(t) = \frac{\delta}{\delta a_k^+(t)}, \quad \hat{b}_k(t) = \frac{\delta}{\delta b_k^+(t)} \tag{3.25}$$

$$\hat{a}_k^+(t) = a_k^+(t), \quad \hat{b}_k^+(t) = b_k^+(t).$$

The fermionic vacuum state $\Phi_c(t)$ corresponding to the zero energy at the moment t is determined by conditions:

$$\hat{a}_k(t)\Phi_c(t) = 0, \quad \hat{b}_k(t)\Phi_c(t) = 0. \tag{3.26}$$

It follows from (3.20) that if the operators $\hat{a}_k$ satisfy anti-commutative relations $[\hat{a}_k, \hat{a}_n^+]_+ = \delta_{k,n}$ at the moment t=0, then they would satisfy these relations at any other moment.

Consequently, each point in curve C defined by functions $\alpha^a(t)$ on manifold MG is associated with vacuum, Fock space and the creation and annihilation operators.

Let us consider the condition (3.26)

$$\hat{a}_n(t)\Phi_c(t) = 0$$

and its time variation

$$d\hat{a}_n(t)\Phi_c + \hat{a}_n(t)\, d\Phi_c = 0.$$

Inserting (3.24) into the last expression we get

$$i\, dB_{n+m}\, b_m^+ \Phi_c + \frac{\delta}{\delta a_n^+} d\Phi_c = 0.$$

Then we obtain

$$d\Phi_c = \sum_{n,m>c} \left[-i\, a_n^+(t)\, dB_{n+m}(t)\, b_m^+(t)\right]\Phi_c \tag{3.27}$$

and for vacuum at the moment t we have

$$\Phi_c(t) = \exp\left(-i\int_0^t a_n^+(\tau)\, dB_{n+m}(\tau)\, b_m^+(\tau)\right)\Phi_0. \tag{3.28}$$

It should be noted here that this formula is analogous to the exponential factor for the Berry phase [11]. However the integrand in (3.28) is not number function but the operator one. Notice also that $\Phi_c(t)$ generally is path-dependent (on curve C) function on MG:

$$\Phi_c(C) = \left[\exp\int_C \check{\Gamma}\right]\Phi_0 \tag{3.29}$$

where

$$\check{\Gamma} = -i\,\hat{a}_n^+\, dB_{n+m}\,\hat{b}_m^+. \tag{3.30}$$

Supposing that operators $\hat{a}_n^+, \hat{b}_n^+$ are functions on MG, one can consider $\check{\Gamma}$ as a connection one-form on MG. One should note that $\check{\Gamma}$ differs from connection Γ considered in Section 2, though (3.29) is analogous to (2.9).

For the closed curve C one gets

$$\oint_C \check{\Gamma} = \int_S d\check{\Gamma},$$

where $\partial S = C$.

Let us calculate $\check{F} = d\check{\Gamma}$, under condition that differentials da_n^+ and db_n^+ satisfy relations (3.24) in each point on S:

$$\check{F} = b_k\, d\bar{B}_{n+k} \wedge dB_{n+m}\, b_m^+ + a_n^+\, dB_{n+m} \wedge d\bar{B}_{k+m}\, a_k + a_k^+\, dB_{k-n} \wedge dB_{n+m}\, b_m^+ + a_n^+\, dB_{n+m} \wedge dB_{k-m}\, b_m^+. \tag{3.31}$$

Consequently for closed curve C one has

$$\Phi_c(C) = \left[\exp\int_S \check{F}\right]\Phi_0$$

Since $a_k = \frac{\delta}{\delta a_k^+}$, $b_k = \frac{\delta}{\delta b_k^+}$ and Φ_0 doesn't depend on a_k^+, b_k^+ we obtain:

$$\Phi_c(C) = \left[\exp - \int_{\Sigma} \left(d\bar{B}_{m+k} \wedge dB_{m+k} - \right.\right.$$
$$\left.\left. - a_k^+ dB_{k-n} \wedge dB_{n+m} b_m^+ - a_n^+ dB_{n+m} \wedge dB_{k-m} b_m \right)\right] \Phi_c \quad (3.32)$$

Hence for vacuum-vacuum amplitude one has

$$< \Phi_c | \Phi_c(C) > = \exp - \int_{\Sigma} R \,, \qquad (3.33).$$

where

$$R = \sum_{k>0} k \, d\bar{V}_k \wedge dV_k \qquad (3.34)$$

is curvature 2-form (2.2I). (In Appendix C we give another proof for formula (3.33), which illustrates, in particular the connection of result (3.33) with first-type (one-cocycle) gauge anomaly.)

As it is shown in Appendix B, one can rewrite the curvature (3.34) as a function on MG (from element $g(\sigma) \in MG$) (B.26):

$$R = -\frac{i}{4\pi} \int_0^{2\pi} \mathrm{Tr}\left[\frac{d}{d\sigma}(g\,dg^{-1}) \wedge g^{-1} dg \right] d\sigma \qquad (3.35)$$

Let us now discuss the question of particle production. As it was noted, we suppose that operators $\hat{a}_n^+$, $\hat{a}_n$ are functions (from $\alpha(\sigma)$ in vicinity of unit) on MG. Consequently, one can also consider particle number operator as a function on MG:

$$\hat{N}(\alpha) = \sum_{n>0} \hat{a}_n^+(\alpha) \, \hat{a}_n(\alpha) \qquad (3.36)$$

Under variation (3.24) we have for $\hat{N}$:

$$d\hat{N}(\alpha) = i\left[\hat{a}_n^+ dB_{n-m} \hat{a}_m + \hat{a}_n^+ dB_{n+m} \hat{b}_m^+ \right] - \qquad (3.37)$$
$$- i\left[\hat{a}_m^+ d\bar{B}_{n-m} \hat{a}_n + \hat{b}_m \, d\bar{B}_{n+m} \hat{a}_n \right].$$

Let $\Phi(\alpha)$ be vacuum in point α on MG, then for

$$dN \equiv < \Phi(\alpha) | d\hat{N}(\alpha) | \Phi(\alpha) > \qquad (3.38)$$

we receive from (3.37) that

$$dN = 0.$$

On the other hand calculating the second differential for $\hat{N}$ one obtains

$$d^2 N \equiv < \Phi(\alpha) | d^2\hat{N}(\alpha) | \Phi(\alpha) > = 2 \sum_{n>0} n \, d\bar{V}_n \, dV_n . \qquad (3.39)$$

Let us consider the case of the adiabatic vacuum arrangement. Then the created particles are adiabatically removed, so at every moment of time the state of quantum fermionic field is returned to vacuum state corresponding to the external gauge potential at this moment of time.

Hence for the total number of created particles N_A from (3.38) one has

$$\frac{d^2 N_A}{dt^2} = \iint_0^{2\pi} 2 R_{ab}(x,y) \frac{d\alpha^a(x)}{dt} \cdot \frac{d\alpha^b}{dt}(y,t) \, dx\, dy \,, \qquad (3.40)$$

where $R_{ab}(x,y)$ are components of curvature 2-form (2.I5), (3.34).

On the other hand one can consider N_A , which is function on MG as the Kähler potential for 2-form R. Using complex structure on MG, which is determined by the Fourier expansion $\alpha^a(\sigma) = \sum_n \alpha_n^a e^{in\sigma}$, one can obtain from (3.38), (3.39) for curvature R components:

$$R^{ab}_{km} = \frac{\partial^2 N(\alpha)}{\partial \alpha_k^a \partial \alpha_m^b} \,. \qquad (3.4I)$$

Consequently the commutator anomaly manifests in the particle production (3.40) under adiabatic arrangement of vacuum.

For the time-independent quantum field state one can calculate the number of created out-modes in the state of in-vacuum. It will be determined by matrix (B.23):

$$A_{n+m} = \sum_{k=-\infty}^{+\infty} g_{n+m+k}(+\infty) \, g_k^{-1}(-\infty) \,, \qquad (3.42)$$

where $n, m > 0$.

It is seen from (B.24) that for closed curve on MG ($g(t)$

is the same for $t=-\infty$ and $t=+\infty$) one finds that $A_{n+m}=0$ and particles are not created.

Notice here that adiabatic arrangement of vacuum is analogous to parallel vector transport: after returning to the initial point the parallel transported state differs from initial state.

4. Remarks on the influence of monopole term.

Let us now discuss the monopole term in (3.I3) and the solution of (3.II) with this term. In this case we come to the following equation for $\phi(\sigma,t)$:

$$[\partial_0+\partial_\sigma+i\beta]\,\phi(\sigma,t)=0, \tag{4.I}$$

where $\beta=\beta^a\lambda^a$, $\beta^a=\beta^a(t)$

It should be noted that one cannot use the substitution of (3.14) then, because $e^{i\beta\sigma}$ is not periodical function in general case. Hence substituting the Fourier expansion

$$\phi(\sigma,t)=\sum_n \phi_n(t)\,e^{in\sigma}$$

into equation (4.I) one comes to the equation

$$\dot{\phi}_n+[in+i\beta]\,\phi_n=0. \tag{4.2}$$

At first let us suppose that β doesn't depend on time t. Then inserting $\phi_n(t)=e^{-iE_n t}\phi_n^0$ into (4.2) one gets

$$[E_n-n-\beta]\,\phi_n^0=0. \tag{4.3}$$

This system has a solution if the following condition holds:

$$\det(E_n-n-\beta)=0. \tag{4.4}$$

For $E_n=n+\varepsilon$ the equation (4.4) is equivalent to

$$\det|\varepsilon\mathbb{1}-\beta^a\lambda^a|=0. \tag{4.5}$$

Consequently, ε should be a eigen-value of unitary traceless matrix $\beta=\beta^a\lambda^a$ (we remember that only unitary gauge groups are considered). If rank of matrices λ^a is equal to N, then (4.5) has N real solutions ε_i , $i=1,\dots,N$. Moreover their sum is equal to zero $\sum_{i=1}^{N}\varepsilon_i=0$ (since matrix β is traceless). For every ε_i one has corresponding solution ϕ_n^i of (4.3). Hence the general solution of (4.I) is as follows:

$$\phi(\sigma,t)=\sum_{i=1}^{N}e^{-i\varepsilon_i t}\sum_{n=-\infty}^{+\infty}d_n^i\,\phi_n^i\,e^{in(\sigma-t)}. \tag{4.6}$$

As we see, the monopole potential removes degeneration on group index and gives rise to energy levels splitting. After separating the positive and negative frequency parts in (4.6) one gets:

$$\phi(\sigma,t)=\sum_{i=1}^{N}e^{-i\varepsilon_i t}\Big(\sum_{n>-\varepsilon_i}e^{in(\sigma-t)}a_n^i\,\phi_n^i+\sum_{n<-\varepsilon^i}e^{in(\sigma-t)}b_n^{+i}\,\phi_n^i\Big). \tag{4.7}$$

As it was shown above the static periodical potential $V(\sigma)$ doesn't influence energy spector, since functions $\phi_n=e^{-iV}e^{in\sigma}$ are the Hamiltonian eigen-functions with E=n. But non-periodical monopole potential $V(\sigma)=\beta\sigma$ changes energy spector removing the degeneration on group index.

If β depends on t then the solution of (4.I) reads

$$T\exp\Big(-i\int_0^t\beta(\tau)\,d\tau\Big)\,e^{in(\sigma-t)}\,\phi_n^0.$$

We shall consider only the adiabatic case, then $\int_0^t\beta(\tau)d\tau=\beta(t)\,t$ and the solution of (4.I) will take the form (4.7) where $\varepsilon^i(t)$ are adiabatic solutions of the equation:

$$\det \left| \mathcal{E}(t) - \beta^a \lambda^a \right| = 0. \tag{4.8}$$

As it is seen from (4.7) the redefinition of creation and annihilation operators do not occur for adiabatic $\mathcal{E}^i(t)$. However when the function $\mathcal{E}^i(t)$ becomes an integer $\mathcal{E}^i(t)$ =m for some i, then the momentary redefinition of creation and annihilation operators and corresponding particles production occur.

We shall illustrate the above by two examples.

A. Group U(I).

In this case the equation (4.4) has the unique solution:

$$E_n = n + \beta \tag{4.9}$$

and for solution of (4.I) one obtains

$$\phi(\sigma,t) = e^{-i\beta t}\Big(\sum_{n>0} \big(a_n e^{-in(t-\sigma)} + b_n^+ e^{in(t-\sigma)}\big) + c_0 \Big). \tag{4.I0}$$

Let us assume that at the moment t $p-1 < \beta(t) < p$ for a positive integer (let β be positive) then for Hamiltonian one gets

$$\mathcal{H} = \sum_{n>0} (n+\beta) a_n^+ a_n + \sum_{n=1}^{p-1} (\beta - n) B_n^+ B_n + \beta C_0^+ C_0 - \sum_{n=p}^{+\infty} (n-\beta)\, b_n b_n^+ \tag{4.II}$$

and wave function (4.I0) is as follows

$$\phi(\sigma,t) = e^{-i\beta t}\Big(\sum_{n>0} e^{-in(t-\sigma)} a_n + \sum_{n=1}^{p-1} e^{in(t-\sigma)} B_n^+ + \sum_{n=p}^{+\infty} e^{in(t-\sigma)} b_n^+ + C_0 \Big), \tag{4.I2}$$

where $B_n = b_n^+,\ n = 1,\dots,p-1.$

Consequently when $\beta(t)$ at some moment becomes equal integer p the redefinition of creation and annihilation operators occurs and the new annihilation operator $B_p = b_p^+$ appears. Hence at any such moment a new mode is created from the old vacuum state:

$$N_p = \langle 0 | B_p^+ B_p | 0 \rangle = \langle 0 | b_p b_p^+ | 0 \rangle = 1. \tag{4.I3}$$

Assuming that at the initial moment $\beta(t)=0$, we have for initial Hamiltonian:

$$\mathcal{H}(0) = \sum_{n>0} n\,(a_n^+ a_n - b_n b_n^+)$$

and vacuum state at initial moment:

$$a_n|0\rangle = b_n|0\rangle = C_0|0\rangle = 0. \tag{4.I4}$$

At the moment t_p, $\beta(t_p) = p$, one obtains for the Hamiltonian:

$$\mathcal{H}_p = \sum_{n>0} (n+p) a_n^+ a_n + \sum_{n=1}^{p-1} (p-n) B_n^+ B_n + p\, C_0^+ C_0 - \sum_{n=p}^{+\infty} (n-p)\, b_n b_n^+ \tag{4.I5}$$

and for the wave function

$$\phi(\sigma,t) = \sum_{n>0} e^{-i(p+n)t} e^{in\sigma} a_n + \sum_{n=1}^{p} e^{i(n-p)t} e^{-in\sigma} B_n^+ + \sum_{n=p+1}^{+\infty} e^{i(n-p)t} e^{-in\sigma} b_n^+ + C_0 e^{-ipt}. \tag{4.I6}$$

Consequently, at the moment t_p the vacuum state $|0\rangle_p$ will be determined by the condition:

$$a_n|0\rangle_p = 0,\ n>0 \qquad C_0|0\rangle_p = 0 \tag{4.I7}$$
$$b_n|0\rangle_p = 0,\ n \geq p+1 \qquad B_n|0\rangle_p = 0,\ n = 1,\dots,p$$

Thus at the moment t_p the new mode B_p (4.I3) is created. Notice that when $0<\beta(t)<1$ the zero mode C_0, which does not contribute to initial Hamiltonian, creates as dynamical. Hence if $\beta(t)$ changes from 0 to p then during this time the total number of new particles

$$\mathcal{N}=\sum_{k=0}^{p} \mathcal{N}_k=(p+1) \tag{4.I8}$$

is created in the in-vacuum state.

It is interesting to note that the new mode B_n has a different chirality from that of a_n and b_n. The vacuum reconstruction occurs instantaneously at the moment t_p. Hence the result would be the same (4.I8) if $\beta(t)$ instantaneously changes from 0 to p. We want to note that it is reflected in effectively two-dimensional problem of particle production during the instantaneous solenoid setting on [I2].

B. Group SU(2).

Let us consider (4.I) with $\beta=\beta^i\sigma^i$, where σ^i are the 2×2 Pauli matrices. Then the equation (4.3) takes the form

$$\begin{pmatrix} \beta_3+E_n-n & \beta_2-i\beta_1 \\ \beta_2+i\beta_1 & E_n-n-\beta_3 \end{pmatrix} \begin{pmatrix} \phi^0_{1n} \\ \phi^0_{2n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.I9}$$

The condition (4.4) for this system yields the following eigenvalues $E_n^{\pm}$:

$$E_n^{\pm}=n\pm\hat{\beta}, \tag{4.20}$$

where

$$\hat{\beta}=\sqrt{\beta_1^2+\beta_2^2+\beta_3^2}\,. \tag{4.2I}$$

Thus equation (4.I) has the solution

$$\phi(\sigma,t)=\sum_{-,+} e^{\mp i\hat{\beta}t}\Big(\sum_{n>0}\Big(e^{-in(t-\sigma)}a_n^{(\pm)}+ + e^{in(t-\sigma)}\overset{+}{b}{}_n^{(\pm)}\Big)+C_0^{(\pm)}\Big)v^{\pm}, \tag{4.22}$$

where

$$v^{\pm}=\begin{pmatrix} \dfrac{\hat{\beta}\mp\beta_3}{2\hat{\beta}} \\ \mp\dfrac{\beta_2+i\beta_1}{2\hat{\beta}} \end{pmatrix} \tag{4.23}$$

Hence the solution (4.22) is equivalent to the two solutions (4.I0) with $\beta>0$ and $\beta<0$.

If at the moment t=0 one has β =0 and vacuum satisfies the conditions

$$a_n^{(\pm)}|0\rangle=0,\quad b_n^{(\pm)}|0\rangle=0,\quad C_n^{(\pm)}|0\rangle=0, \tag{4.24}$$

then at the moment $t=t_p$, when $\hat{\beta}=p\in Z$, the solution (4.22) takes the form

$$\begin{aligned}\phi(\sigma,t)=&\sum_{n>0} e^{-i(n+p)t}e^{in\sigma}a_n^{(+)}v^{(+)}+\sum_{n=1}^{p} e^{-i(n-p)t}e^{-in\sigma}B_n^{(+)}v^{(+)}+\\ &+\sum_{n=p+1}^{+\infty} e^{i(n-p)t}e^{-in\sigma}\overset{+}{b}{}_n^{(+)}v^{(+)}+C_0^{(+)}e^{-ipt}v^{(+)}+\\ &+\sum_{n>0} e^{i(n+p)t}e^{-in\sigma}\overset{+}{b}{}_n^{(-)}v^{(-)}+\sum_{n=p+1}^{+\infty} e^{-i(n-p)t}e^{in\sigma}a_n^{(-)}v^{(-)}+\\ &+\sum_{n=1}^{p} e^{i(p-n)t}e^{in\sigma}\overset{+}{A}{}_n^{(-)}v^{(-)}+C_0^{(-)}v^{(-)}e^{ipt},\end{aligned} \tag{4.25}$$

where

$$\begin{aligned} B_n^{(+)}&=\overset{+}{b}{}_n^{(+)},\quad n=1,\dots,p \\ \overset{+}{A}{}_n^{(-)}&=a_n^{(-)},\quad n=1,\dots,p \end{aligned} \tag{4.26}$$

Consequently at the moment $t=t_p$ the vacuum $|0\rangle_p$ is defined by conditions:

$$
\begin{array}{ll}
a_n^{(-)}|0\rangle_p = 0,\ n \geq p+1 & a_n^{(+)}|0\rangle_p = 0,\ n>0 \\
b_n^{(-)}|0\rangle_p = 0,\ n>0 & b_n^{(+)}|0\rangle_p = 0,\ n\geq p+1 \\
C_0^{+(-)}|0\rangle_p = 0 & C_0^{(+)}|0\rangle_p = 0 \\
A_n^{(-)}|0\rangle_p = 0,\ n=1,\dots,p & B_n^{(+)}|0\rangle_p = 0,\ n=1,\dots,p
\end{array}
\qquad (4.27)
$$

Notice that $A_n^{(-)}$ and $B_n^{(+)}$ have an opposite chirality as compared to the initial moment t=0.

Thus the picture is analogous to the case of group U(I). At the moment when $\hat{\beta}(t)$ =p the creation of two new modes in in-vacuum occurs:

$$N_p^{(+)} = \langle 0| B_p^{+(+)} B_p^{(+)} |0\rangle = 1 \qquad (4.28)$$

$$N_p^{(-)} = \langle 0| A_p^{+(-)} A_p^{(-)} |0\rangle = 1 .$$

If during the time t_p function $\hat{\beta}(t)$ has changed from 0 to p then during this time

$$N = \sum_{k=0}^{p} N_k^{(+)} + N_k^{(-)} = 2(p+1) \qquad (4.29)$$

gives number of particles created.

For the group SU(2) one finds

$$g(\sigma) = \exp i(\beta^i \sigma^i)\sigma = \cos(\hat{\beta}\sigma) + \frac{i\beta^i \sigma^i}{\hat{\beta}} \sin(\hat{\beta}\sigma), \qquad (4.30)$$

with $\hat{\beta} = \sqrt{\beta_1^2+\beta_2^2+\beta_3^2}$.

Hence $g(\sigma)$ is periodical function if $\hat{\beta} \in Z$. Moreover for integer $\hat{\beta}$ the expression (4.30) represents the element from the homotopy group π_1 (G).

It should be noted that in this case the potential (3.3) takes the form:

$$A = G^{-1} \frac{d}{d\sigma} G, \qquad (4.3I)$$

where G=F·g , g has the form (4.30).

Thus for integer $\hat{\beta}$, G realizes a nontrivial element of homotopy group π_1(G) and potential (3.3) is a topologically nontrivial configuration. In general, the background field $\bar{A}$ can also have a monopole character, i.e. $\int_0^{2\pi} \bar{A} d\sigma \neq 0$. However such a generalization doesn't give something new to our consideration.

Returning to our picture of the vacua bundle over MG, it should be noted that MG is not connected manifold. MG is decomposed into the non-connected pieces $(MG)_p$ characterized by an integer from the homotopy group π_1(G). Thus the potential (3.I3) with $\hat{\beta}$=p means that we should consider the curve and corresponding vacua bundle over the piece $(MG)_p$. The potential (3.I3) with non-integer $\hat{\beta}$ generally doesn't correspond to any point of MG (since the corresponding matrix $g(\sigma)$ is not periodical). However an adiabatic change of $\hat{\beta}$ from p-I to p one can consider as a transition from one piece $(MG)_{p-I}$ to an other piece $(MG)_p$. This allows to consider the vacua bundle over the total manifold of the gauge group MG and determine the corresponding particle production. Notice that inside the one piece $(MG)_p$ the particles are created continuously, but for a transition from $(MG)_p$ to $(MG)_{p+I}$ particles are created unevenly. The situation is analogous to that of phase transition.

5. Discussion.

We have attempted to describe the physical situation in which the consideration of the fermionic vacua bundle over gauge group manifold MG is useful . The bundle curvature turns out to be connected with the commutator gauge anomaly. This allows to consider the problem where this curvature is manifested. Certainly this is a development of the Berry phase ideas [II] for secondary-quantized systems. Though the role of phase here is played (as it is seen from (3.28), (3.29)) by an operator constructed from the chiral fermionic creation operators. Nevertheless the geometrical background of these effects is the same: this is the nontriviality of a bundle over parametors manifold (in our case over MG).

It should be noted that for (I+I)-space-time our consideration is quite general, since any nonabelian gauge field $A(\sigma, t)$ (in the gauge $A_0 = 0$) is represented in the form (I.I):

$$i A = g^{-1} \frac{d}{d\sigma} g, \tag{5.I}$$

where the group element is determined as

$$g(\sigma, t) = P\exp i \int_0^{\sigma} A(\sigma', t)\, d\sigma'. \tag{5.2}$$

If the gauge field $A(\sigma)$ is nontrivial, i.e. $\int_0^{2\pi} A(\sigma) d\sigma \neq 0$, then $g(\sigma)$ (5.2) realizes a nontrivial element of the homotopy group $\pi_1(G)$.

On the other hand, one should note that the monopole term in (3.I3) takes the form of an angle depending part of the nonabelian string solution [I3]. Therefore the considered effects can take place in the effective two-dimensional problems, for example in the cosmic string field [I3] with the angle dependence changing with time. The quantum effects near such strings may be very important for galaxy formation.

Another application of the results obtained is the ~~c~~ondensed state physics, for example the Hall effect in spirit of known Laughlin's consideration [I4] or anionic mechanism of high-temperature superconductivity [I5]. It seems very attractive, since as shown in ref.[I6] the anomalous Kac-Moody algebras play a role in the quantum Hall effect.

The natural development would be a generalization for the gravitational anomaly case [I7]. The gauge transformations then are coordinate's changing, corresponding to a transition from one reference

system to another.

These and other problems will be considered elsewhere.

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Appendix A.

Let us consider the Hermitian metric (2.II) on the vacua bundle over gauge group MG:

$$g = \det(1 - \gamma^{+}\gamma), \qquad (A.I)$$

where the matrix γ in a vicinity of unit is determined as

$$\gamma_{mn} = -\frac{i}{2\pi}\int_0^{2\pi} d\sigma\, e^{-i(n+m)\sigma} F u F^{-1} \qquad (A.2)$$

The metric (A.I) defines a natural metric connection one-form Γ :

$$\nabla g = dg + \Gamma g = 0,$$
$$\Gamma = -g^{-1} dg$$

with curvature two-form

$$R = d\Gamma.$$

It should be noted that the base space is an homogeneous space and the Hermitian structure is invariant under the action of the gauge group. Then the curvature is a homogeneous 2-form. So it is enough to compute it at one point in MG. The value everywhere else is fixed by this [5]. Hence one can do calculation in a vicinity of unit omitting the high order by u^a terms.

From (A.I) one gets in a vicinity of unit:

$$g = \exp(-\mathrm{Tr}\,\gamma^{+}\gamma).$$

Hence the expression for the curvature R :

$$R = dd\,\mathrm{Tr}\,\gamma^{+}\gamma \qquad (A.3)$$

formally should be equal to zero.

After calculating the trace one gets

$$k = \mathrm{Tr}\,\gamma^{+}\gamma = \sum_{n,m>0}\left(\frac{1}{2\pi}\right)^{2}\iint_0^{2\pi} \mathrm{Tr}\Big[F(\sigma)u(\sigma)F^{-1}(\sigma)\cdot$$
$$\cdot F(\sigma')u(\sigma')F^{-1}(\sigma')\Big] e^{i(n+m)(\sigma'-\sigma)} d\sigma\, d\sigma'.$$

Using the complex variables $z = e^{i\sigma}$ and $y = e^{i\sigma'}$ and choosing

a integrating contour $|z|>|y|$ one can easily rewrite this as follows

$$k = \sum_{n,m>0}\left(\frac{1}{2\pi}\right)^{2}\oint_{|z|>|y|}\left(\frac{y}{z}\right)^{n+m} \mathrm{Tr}\Big[F(z)u(z)F^{-1}(z)F(y)u(y)F^{-1}(y)\Big]\frac{dy}{iy}\frac{dz}{iz}.$$

After calculating the sum one has

$$k = -\left(\frac{1}{2\pi}\right)^{2}\oint_{|z|>|y|}\mathrm{Tr}\Big[F(z)u(z)F^{-1}(z)F(y)u(y)F^{-1}(y)\Big]\frac{y}{z(z-y)^{2}}dz\,dy \qquad (A.4)$$

Let us choose the differential operator on the gauge group manifold MG in the form

$$d = \int dz\, du^{a}(z)\frac{\delta}{\delta u^{a}(z)}$$

and using the point-splitting method:

$$R = ddk = \oint_{|z|>|y|} dz\,dy\,\frac{\delta^{2}k}{\delta u^{a}(z)\,\delta u^{b}(y)}\,du^{a}(z)\wedge du^{b}(y). \qquad (A.5)$$

Inserting (A.4) into (A.5) one finds

$$R = -\frac{i}{2\pi}\int_0^{2\pi}\mathrm{Tr}\Big[\frac{d}{d\sigma}\left(F du F^{-1}\right)\wedge F du F^{-1}\Big]d\sigma. \qquad (A.6)$$

Appendix B.

Let us consider the equation (3.I5):

$$\left(\partial_t + \partial_\sigma - i\frac{dV}{dt}(\sigma,t)\right)\tilde{\Phi}(\sigma,t) = 0. \qquad (B.I)$$

We look for the solution in the form

$$\tilde{\Phi}(\sigma,t) = \hat{A}(\sigma,t)\,\tilde{\Phi}_0, \qquad (B.2)$$

where $\tilde{\Phi}_0$ is a constant column.

Inserting (B.2) into (B.I) one gets for matrix $\tilde{A}$:

$$(\partial_t + \partial_\sigma)\hat{A} = i\frac{dV}{dt}\hat{A}. \qquad (B.3)$$

We suppose that $\hat{A} = \exp iB$, then

$$(\partial_t + \partial_\sigma)B = \frac{dV}{dt}. \qquad (B.4)$$

Taking into account the periodicity of $B(\sigma, t)$ we obtain the general solution of (B.4):

$$B(\sigma,t)=\sum_{n=-\infty}^{+\infty} e^{in(\sigma-t)}B_n+\int_0^t \frac{\partial V}{\partial t}(\tau-t+\sigma,\tau)\,d\tau, \tag{B.5}$$

where the derivative acts on the second argument.

Since

$$V(\sigma,t)=\sum_{n=-\infty}^{+\infty} V_n(t)\,e^{in\sigma},$$

one finds

$$B(\sigma,t)=\sum_{n=-\infty}^{+\infty} e^{in(\sigma-t)}\left(B_n+\int_0^t \dot V_n(\tau)\,e^{in\tau}d\tau\right). \tag{B.6}$$

Thus one obtains for solution of (B.I)

$$\tilde\Phi(\sigma,t)=\exp(iB(\sigma,t))\,\Phi_0 . \tag{B.7}$$

On the other hand the solution of (B.7) is represented as

$$\tilde\Phi(\sigma,t)=\sum_{n=-\infty}^{+\infty} e^{in\sigma}\,\tilde d_n(t), \tag{B.8}$$

where

$$\tilde d_n(t)=\frac{1}{2\pi}\int_0^{2\pi} e^{iB(\sigma,t)}\,e^{-in\sigma}d\sigma\,\tilde\Phi_0 . \tag{B.9}$$

From (B.6) one sees that

$$B(\sigma,t)=\sum_{n=-\infty}^{+\infty} B_k(t)\,e^{ik(\sigma-t)},$$
$$B_k(t)=B_k+\int_0^t \dot V_k(\tau)\,e^{ik\tau}d\tau. \tag{B.10}$$

Substituting (B.10) into (B.9) one gets

$$\tilde d_n(t)=e^{-int}\,d_n(t),$$

where

$$d_n(t)=\frac{1}{2\pi}\int_0^{2\pi}\exp\left(\sum_{k=-\infty}^{+\infty} iB_k(t)\,e^{ik\sigma}\right)e^{-in\sigma}d\sigma\,\tilde\Phi_0 . \tag{B.11}$$

We can rewrite (B.11) in terms of complex variable $y=e^{i\sigma}$:

$$d_n(t)=\frac{1}{2\pi}\oint \exp\left[i\sum_{k=-\infty}^{+\infty} y^k B_k(t)\right]\frac{y^{-n}}{iy}\,dy\,\tilde\Phi_0 . \tag{B.12}$$

One can consider B_k as an element of the gauge group algebra and $g(y)=\exp[i\sum y^k B_k(t)]$ as an element of the gauge group MG, then

$$d_n(t)=\frac{1}{2\pi}\oint g(y,t)\,y^{-n}\frac{dy}{iy}\,\tilde\Phi_0 \tag{B.13}$$

or

$$d_n(t)=g_n(t)\,\tilde\Phi_0 , \tag{B.14}$$

where $g(y)=\sum_{n=-\infty}^{+\infty} g_n(t)\,y^n$.

From (B.12), (B.13) we have for the time differential:

$$d\,d_n(t)=i\,dB_{n-m}(t)\,d_m(t). \tag{B.15}$$

Hence

$$iB_k=\frac{1}{2\pi i}\oint \frac{dy}{iy}\ln g(y)\,y^{-k} \tag{B.16}$$

and

$$i\,dB_k=\frac{1}{2\pi i}\oint \frac{dy}{y}\,g^{-1}dg\,y^{-k}. \tag{B.17}$$

Our aim now is to derive a formula which connects $d_n(t)$ and $d_n(0)$.

One has

$$g(y,t)=A(y,t)\,g(y,0), \tag{B.18}$$

where

$$g(y,t)=\exp i\sum_{k=-\infty}^{+\infty} y^k B_k(t). \tag{B.19}$$

Hence

$$A(y,t)=g(y,t)\,g^{-1}(y,0) \tag{B.20}$$

and

$$g_n(t)=\oint\frac{dy}{2\pi i y}\,y^{-n}A(y,t)\,g(y,0). \tag{B.2I}$$

Assuming $A(y)=\sum A_m y^m$,

$$d_n(t)=A_{n-m}(t)\,d_m(0) \tag{B.22}$$

and

$$A_k=\sum_{m=-\infty}^{+\infty} g_{k+m}(t)\,g_m^{-1}(0). \tag{B.23}$$

From

$$\sum_{m=-\infty}^{+\infty} g_{k+m}(t)\,g_m^{-1}(t)=\delta_{k,0} \tag{B.24}$$

we obtain the natural condition $A_k(0)=\delta_{k,0}$.

Let us consider now the expression for the curvature (2.I2), (3.34)

$$R=\sum_{k>0} k\,\mathrm{Tr}\,d\bar{B}_k\wedge dB_k. \tag{B.25}$$

Substituting (B.I7) into (3.25) one gets

$$R=-\frac{i}{4\pi}\int_0^{2\pi}\mathrm{Tr}\left[\frac{d}{d\sigma}\left(g\,dg^{-1}\right)\wedge g^{-1}dg\right]d\sigma, \tag{B.26}$$

where g(y) is the group MG element (B.I9).

Since $g^{-1}dg=iF\,d\alpha\,F^{-1}$, from (B.26) we get the expression (A.6), (2.I2).

The formula (B.26) gives the curvature two-form R of the bundle over the gauge group MG as a function on MG (from group element g(y)).

Appendix C.

We give here another proof of the formula (3.33). Let us consider the two-dimensional chiral Dirac operator $\hat{D}(A)$ in the external nonabelian gauge field A_μ on the two-dimensional torus with coordinates (x_1 , x_2). We shall assume that

$$A_1=0 \tag{C.I}$$

$$iA_2=g^{-1}i\bar{A}g+g^{-1}\frac{d}{dx_2}g,$$

where $\bar{A}(x_2)$ is a background field independent of x_1 , and g(x_1 , x_2) is a group element:

$$g=\exp i\alpha(x_1,x_2). \tag{C.2}$$

Then for A_2 one has

$$A_2=\bar{A}+i\bar{\nabla}_{x_2}\alpha, \tag{C.3}$$

where $\bar{\nabla}_2\alpha=\frac{d\alpha}{dx_2}+i[\bar{A},\alpha]$ is the covariant derivative with respect to background $\bar{A}$.

The gauge field choice (C.I), (C.3) cooresponds to the potential (3.3 , (3.4) in the pseudoeuclidean case.

Vacuum-vacuum amplitude reads as usual

$$\langle out|in\rangle\equiv Z=\int D\psi\,D\psi^+\exp\int\psi^+\hat{D}(A)\psi\,dx_1dx_2= \tag{C.4}$$

$$=\det\hat{D}(A).$$

Let us find how Z depends on $\alpha(x_1,x_2)$. One can consider the field A_μ (C.I) as an adiabatic pure gauge. The operator $\hat{D}(A)$ determinant is not invariant under a transformation $A_\mu\to A_\mu+\nabla_\mu\alpha$ and is changed as follows (it is the first type anomaly [I,3]):

$$\delta_\alpha\ln\det\hat{D}(A)=\frac{i}{4\pi}\int\mathrm{Tr}\,\alpha\,\epsilon^{\mu\nu}\left(\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]\right)d^2x \tag{C.5}$$

Inserting (C.1) into (C.5) one gets

$$\delta_\alpha \ln Z = \frac{i}{4\pi} \int \mathrm{Tr}\, \alpha\, \partial_1 A_2 \, dx_1 dx_2 . \tag{C.6}$$

Taking into account (C.3),

$$\delta_\alpha \ln Z = \frac{i}{4\pi} \int \mathrm{Tr} \left[\alpha \left(\partial_1 \partial_2 \alpha + i \left[\bar{A}, \partial_1 \alpha \right] \right) \right] dx_1 dx_2 . \tag{C.7}$$

We assume that function $\alpha(x_1)$ gives a closed curve C on the gauge group manifold MG, then (C.7) yields

$$\delta_\alpha \ln Z = \frac{i}{4\pi} \oint_C \int_c^{2\pi} \mathrm{Tr} \left[\alpha \frac{d}{dx_2} d\alpha + i \alpha \left[\bar{A}, d\alpha \right] \right] dx_2 \tag{C.8}$$

where $d\alpha = \frac{\partial \alpha}{\partial x_1} dx_1$ is differential on MG.

For surface S, bounded by curve C on MG, one obtains from (C.8):

$$\ln Z = \frac{i}{4\pi} \int_S \int_c^{2\pi} \left(\mathrm{Tr}\, d\alpha \wedge \frac{d}{dx_2} d\alpha - 2i\, \mathrm{Tr} \left[\bar{A} d\alpha \wedge d\alpha \right] \right) dx_2 \tag{C.9}$$

Hence we find

$$\ln Z = \int_S R ,$$

where the 2-form curvature R has the form (2.I3) and we again arrive at the formula (3.33).

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