# AN ALTERNATIVE SCHEME OF THE BOGOLYUBOV'S AVERAGE METHOD

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## ABSTRACT

In this paper the average energy and the magnetic moment conservation laws in the Drift Theory of charged particle motion are obtained in a simple way. The approach starts from the energy and magnetic moment conservation laws and afterwards the average is performed. This scheme is more economic from the standpoint of time and algebraic calculations than the usual procedure of Bogolyubov's Method.

## INTRODUCTION

There is nowadays a great interest in developing the Drift Theory to higher approximation order through different methods [1-3], though the development of the Drift Theory to higher order involves laborious algebraic calculations. In view of that, the algebraic simplification methods is important particularly those based on a suitable choice of variables.

The need to find a starting new variables set which permited to simplify the algebraic labour has not been presented before due to that first order Drift Theory involved lenghty calculations -is not a problem- but to higher order.

Our strategy to find suitable variable set consisted in to note that the starting variables of Bogolyubov's Method $(\mathbf{r}, v_{\parallel}, v_{\perp}, \theta)$, where $\mathbf{r}$ is the position vector radius of the particle, $v_{\parallel}$ and $v_{\perp}$ are the parallel and perpendicular velocities respectively and $\theta$ is the rapidly rotation phase about the magnetic field, permited to find the drift equations $(\frac{d\mathbf{r}}{dt}, \frac{dE}{dt}, \frac{dM}{dt})$, where E is the energy and M the magnetic moment. Then it arises the question if it is possible to obtain the drift equations starting from the new variables set $(\mathbf{r}, E, M, \theta)$ that they simplify the tedious algebra and besides it permits to calculate in a simple and staightforward way the drift equations. The comparision between the labour done with these new variables and the older ones shows that the calculations with new variables were as far less laborious than older ones but final results are identical.

## DERIVATION OF DRIFT EQUATIONS IN A NEW APPROACH

The first order drift equations are obtained in a new approach not taking into accout relativistic effects. This approach makes possible the obtaintion of drift equations in straightforward and simple way by using the new variables set $(\mathbf{r}, E, M, \theta)$. Then we start from:

$$
\begin{aligned}
\frac{d\mathbf{r}}{dt} &= \mathbf{v}_1 + \mathbf{v}_2\cos\theta + \mathbf{v}_3\sin\theta \equiv \mathbf{f}_r \\
\frac{dE}{dt} &= E_1 + E_2\cos\theta + E_3\sin\theta \equiv f_E \\
\frac{dM}{dt} &= M_1 + M_2\cos\theta + M_3\sin\theta + M_4\cos 2\theta + M_5\sin 2\theta \equiv f_M \\
\frac{d\theta}{dt} &= \omega + C_1 + C'_3\cos\theta - C'_2\sin\theta + C_5\cos 2\theta - C_4\sin 2\theta \equiv \omega + f_\theta \ ,
\end{aligned}
\tag{1}
$$

where $\omega$ is the cyclotron frequency and $\mathbf{v}$'s. E's, M's and C's are given in Appendix I.

The solution of system (1) is found in a power series through a small parameter $\epsilon$ which is the ratio between the Larmor radius and the characteristic lenght of the inhomegeneity of magnetic field:

$$x_k = \xi_k + \epsilon g_{1k}(\xi_i, t, \alpha) + \cdots \tag{2}$$

$$\theta = \alpha + \epsilon q_1(\xi_i, t, \alpha) + \cdots \tag{3}$$

where $x_k(\mathbf{r}, E, M)$ and the variables $\xi_k$ and $\alpha$ satisfy the equations[4]:

$$\frac{d\xi_k}{dt} = \varphi_{0k}(\xi_i, t) + \epsilon\varphi_{1k}(\xi_i, t) + \cdots \tag{4}$$

$$\frac{d\alpha}{dt} = \frac{1}{\epsilon}\omega(\xi_i, t) + \Omega_0(\xi_i, t) + \epsilon\Omega_1(\xi_i, t) + \cdots \tag{5}$$

The aim of this paper consists in determining the unknown functions $g_{1k}, q_1, \Omega_0, \Omega_1, \varphi_{0k}, \varphi_{1k}$ and so to find the drift equations:

$$
\left.
\begin{aligned}
\frac{d\mathbf{r}}{dt} &= \varphi_{0r} + \epsilon\varphi_{1r} + \cdots \\
\frac{dE}{dt} &= \varphi_{0\mathcal{E}} + \epsilon\varphi_{1\mathcal{E}} + \cdots \\
\frac{dM}{dt} &= \varphi_{0\mu} + \epsilon\varphi_{1\mu} + \cdots
\end{aligned}
\right\}
\tag{6}
$$

The expressions for the unknown functions are obtained substituting equations (2-5) into (1) and equating the expressions of the same order:

$$g_{1\mathbf{r}} = \frac{1}{\omega}\left(\frac{2MB}{m}\right)^{\frac{1}{2}}(\hat{e}_2 \sin\theta - \hat{e}_3\cos\theta)$$

$$g_{1\varepsilon} = \frac{1}{\omega}(E_2\sin\theta - E_3\cos\theta)$$

$$g_{1\mu} = \frac{1}{\omega}\left(M_2\sin\theta - M_3\cos\theta + \frac{M_4}{2}\sin 2\theta - \frac{M_5}{2}\cos 2\theta\right)$$

$$q_1 = \frac{1}{\omega}\left(C_3'\sin\theta + C_2''\cos\theta + \frac{C_5}{2}\sin 2\theta + \frac{C_4}{2}\cos 2\theta - \left(\frac{2MB}{m}\right)^{\frac{1}{2}}\frac{\nabla\omega}{\omega}\cdot\hat{r}\right)$$

$$\varphi_{0\mathbf{r}} = \left[\frac{2(E-MB)}{m}\right]^{\frac{1}{2}}\hat{e}_1$$

$$\varphi_{0\varepsilon} = [2(E-MB)m]^{\frac{1}{2}}(\mathbf{F}\cdot\hat{e}_1) = E_1$$

$$\varphi_{0\mu} = -M\left[\frac{2(E-MB)}{m}\right]^{\frac{1}{2}}(\nabla\cdot\hat{e}_1) - \frac{2M}{B}\left\{\frac{\partial B}{\partial t} + \left[\frac{2(E-MB)}{m}\right]^{\frac{1}{2}}\nabla_1 B\right\} = M_1$$

$$\varphi_{1\mathbf{r}} = \frac{MB}{m\omega}[-\hat{e}_1(\hat{e}_1\cdot\nabla\times\hat{e}_1)] + \frac{1}{\omega}\left\{\hat{e}_1\times\left[\mathbf{F} - \frac{2(E-MB)}{m}\mathbf{T} - \frac{MB}{m}\frac{\nabla\omega}{\omega}\right]\right\}$$

$$\varphi_{1\varepsilon} = \frac{MB}{\omega}\{\hat{e}_1\cdot\nabla\times\mathbf{F} - (\mathbf{F}\cdot\hat{e}_1)(\hat{e}_1\cdot\nabla\times\hat{e}_1)\} - \frac{BM}{\omega^2}[\nabla\omega_1\times\mathbf{F}]\cdot\hat{e}_1 - \frac{2(E-MB)}{\omega}[\mathbf{T}\times\mathbf{F}]\cdot\hat{e}_1$$

$$\varphi_{1\mu} = \frac{M}{\omega}\left\{\hat{e}_1\cdot\nabla\times\mathbf{F} - \frac{2(E-MB)}{m}\hat{e}_1\cdot\nabla\times\mathbf{T} - \left[(\mathbf{F}\cdot\hat{e}_1) - \frac{MB}{m\omega}\nabla_1\omega\right]\hat{e}_1\cdot\nabla\times\hat{e}_1\right\}$$

The system (6) can be simplified with the following change of variables:

$$M' = M + \frac{M}{\omega}\left[\frac{2(E-MB)}{m}\right]^{\frac{1}{2}}\hat{e}_1\cdot(\nabla\times\hat{e}_1)$$

$$E' = E \ ,$$

therefore

$$\frac{d\mathbf{r}}{dt} = \left[\frac{2(E-M)}{m}\right]^{\frac{1}{2}}\hat{e}_1 + \frac{1}{\omega}\left\{\hat{e}_1\times\left(\mathbf{F} - \frac{2(E-MB)}{m}\mathbf{T} - \frac{MB}{m}\frac{\nabla\omega}{\omega}\right)\right\}$$

$$\frac{dE}{dt} = m\mathbf{F}\cdot\frac{d\mathbf{r}}{dt} + \frac{MB}{\omega}\hat{e}_1\cdot(\nabla\times\mathbf{F})$$

$$\frac{dM}{dt} = \frac{M}{\omega}\left[\hat{e}_1\cdot\nabla\times\mathbf{F} - \frac{\partial\omega}{\partial t}\right]$$

in these formulas the apostrophe has been omitted for reasons of clearness.

# CONCLUSIONS

It is shown that it exists an alternative way to obtain the drift equations by choosing a new variables set, this way is simpler than usual one besides it has two advantages: i) it reduces the algebraic calculations ii) it permits to obtain in straightforward and simpler way the energy and magnetic conservation laws.

# APPENDIX I

$$\mathbf{v}_1 = v_{\|}\hat{e}_1$$
$$\mathbf{v}_2 = v_{\perp}\hat{e}_2$$
$$\mathbf{v}_3 = v_{\perp}\hat{e}_3$$
$$\hat{r} = \hat{e}_2\cos\theta + \hat{e}_3\sin\theta$$
$$E_1 = [2m(E-MB)]^{\frac{1}{2}}(\mathbf{F}\cdot\hat{e}_1)$$
$$E_2 = 2mMB(\mathbf{F}\cdot\hat{e}_2)$$
$$E_3 = 2mMB(\mathbf{F}\cdot\hat{e}_3)$$
$$\mathbf{M} = \left(\frac{2Mm}{B}\right)^{\frac{1}{2}}\left\{\mathbf{F} - \left[\frac{2(E-MB)}{m}\right]\mathbf{T} - \left[\frac{2(E-MB)}{m}\right]^{\frac{1}{2}}\frac{\partial\hat{e}_1}{\partial t} - \frac{M}{m}\nabla B\right\}$$
$$M_1 = -M\left[\frac{2(E-MB)}{m}\right]^{\frac{1}{2}}(\nabla\cdot\hat{e}_1) - \frac{2M}{B}\left\{\frac{\partial B}{\partial t} + \left[\frac{2(E-MB)}{m}\right]^{\frac{1}{2}}\nabla_1 B\right\}$$
$$M_2 = \mathbf{M}\cdot\hat{e}_2$$
$$M_3 = \mathbf{M}\cdot\hat{e}_3$$
$$M_4 = -M\left[\frac{2(E-MB)}{m}\right]^{\frac{1}{2}}(T_{221} - T_{331})$$
$$M_5 = -M\left[\frac{2(E-MB)}{m}\right]^{\frac{1}{2}}(T_{231} + T_{321})$$
$$\mathbf{C} = \left(\frac{m}{2MB}\right)^{\frac{1}{2}}\left\{\mathbf{F} - 2\frac{(E-MB)}{m}\mathbf{T} - \left[\frac{2(E-MB)}{m}\right]^{\frac{1}{2}}\frac{\partial\hat{e}_1}{\partial t}\right\}$$
$$C_1 = -\frac{1}{2}\left[\frac{2(E-MB)}{m}\right]^{\frac{1}{2}}(\hat{e}_1\cdot\nabla\times\hat{e}_1 + 2T_{312}) - \frac{1}{2}\left(\frac{\partial\hat{e}_2}{\partial t}\cdot\hat{e}_3 - \frac{\partial\hat{e}_3}{\partial t}\cdot\hat{e}_2\right)$$
$$C_2' = (\mathbf{C}\cdot\hat{e}_2) + \left(\frac{2MB}{m}\right)^{\frac{1}{2}}T_{332}$$
$$C_3' = (\mathbf{C}\cdot\hat{e}_3) + \left(\frac{2MB}{m}\right)^{\frac{1}{2}}T_{223}$$
$$C_4 = -\frac{1}{2}\left[\frac{2(E-MB)}{m}\right]^{\frac{1}{2}}(T_{221} - T_{331})$$
$$C_5 = -\frac{1}{2}\left[\frac{2(E-MB)}{m}\right]^{\frac{1}{2}}(T_{231} + T_{321})$$
$$\mathbf{T} = (\hat{e}_1\cdot\nabla)\,\hat{e}_1$$
$$T_{ijk} = \hat{e}_i\,(\hat{e}_j\cdot\nabla)\,\hat{e}_k$$

# REFERENCES


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