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Spectral decompositions of R matrices for exceptional Lie algebras

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Abstract

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In this paper we present spectral decompositions of R matrices for vector representat ons of exceptional Lie algebras.

Аннотация

С.М.Сергеев. Спектральное разложение R матриц для исключительных алгебр Ли: Препринт ИФВЭ 90-163. — Протвино, 1990. — 7 с., библиогр.: 5.

В настоящей работе мы представляем спектральные разложения R матриц для векторных представлений исключительных алгебр Ли.

1.Introduction.

Quantum group structures are playing an important role in many branches of mathematical physics. They have been a key to understand the intimate relations among the resent developments in conformal field theories, operator algebras, link invariants and exactly solvable models.

Historically quantum group appears in the quantum inverse problem as a specific structure of L operator, obeying

$$R_{12}(x/y)L_1(x)L_2(y) = L_2(y)L_1(x)R_{12}(x/y) \qquad (1.1)$$

for the given intertvining operator R (six-vertex R matrix). Here R itself was found from the Young-Baxter equation that is the associativity condition (Jacoby identity) for (1.1)

$$R_{12}(x/y)R_{13}(x)R_{23}(y) = R_{23}(y)R_{13}(x)R_{12}(x/y) \qquad (1.2)$$

Under some simplifications of L operator (with a loss of generality) quantum deformation appears to have a simple form. So it is convinient to solve the linear problem of type (1.1) instead of cubic relation (1.2) in order to find R matrix, associated with the given Lie algebra.

Now R matrices for several representations for A_n, B_n, C_n, D_n and G_2 algebras have been build [1,2,3]. The aim of our paper is to complete the list of known spectral decompositions by exceptional fundamental ones.

The well known constructions for the quantum group and R matrix are described in section 2. In section 3 we describe briefly the exceptional algebras of type E_n and F_4 and present spectral decomposition for their R matrices.

2.Quantum group and R matrix.

Let $\mathcal{G}^{(1)}$ be an affine Lie algebra and $\mathcal{G}$ be its classical part with a Cartan subalgebra $\mathcal{H}$; $\{\alpha\} = \Pi_0$ be fundamental system of $\mathcal{G}^{(1)}$, and, correspondingly, $\{H_\alpha\}$ be set of co-roots, so that $A_{\alpha\beta} = \alpha(H_\beta)$ be Cartan matrix. It is convinient to choose $\mathcal{H}^* \equiv \mathcal{H}$ so that $\alpha(H) = (\alpha, H)$, where $(.,.)$ being the Killing form. Also let $r(\lambda)$ mean irreducible representation of $\mathcal{G}$ with higest weight λ. Chevalley basis of $\mathcal{G}^{(1)}$ is given by

$$e_0 = tF_\theta, \quad f_0 = t^{-1}E_\theta, \quad e_\alpha = E_\alpha, \quad f_\alpha = F_\alpha$$

We adopt the convention so that $|long\ root|^2 = 2$. Let

$$m(\alpha) = \frac{(\alpha,\alpha)}{(\alpha_s,\alpha_s)}, \quad m = \frac{1}{(\alpha_s,\alpha_s)} \tag{2.1}$$

α_s -short root. Then quantum group $\mathcal{U}_q(\mathcal{G}^{(1)})$ can be described by deformation of Chevalley basis:

$$\begin{aligned} [HE_\alpha] &= \alpha(H)E_\alpha \\ [HF_\alpha] &= -\alpha(H)F_\alpha \\ [E_\alpha F_\beta] &= \delta_{\alpha\beta}\frac{[m(\alpha)H_\alpha]}{[m(\alpha)]} \end{aligned} \tag{2.2}$$

where α, β belong to the fundamental system of roots. Hereafter we adopt the conventional notation:

$$[u] = \frac{q^u - q^{-u}}{q - q^{-1}} \tag{2.3}$$

Formally one can introduce the co-multiplication of the following form:

$$\begin{aligned} \Delta_q(H_\alpha) &= H_\alpha \otimes 1 + 1 \otimes H_\alpha \\ \Delta_q(E_\alpha) &= E_\alpha \otimes q^{-m\alpha} + q^{m\alpha} \otimes E_\alpha \\ \Delta_q(F_\alpha) &= F_\alpha \otimes q^{-m\alpha} + q^{m\alpha} \otimes F_\alpha \\ \Delta_q^{\mu\nu}(E_0) &= t_\mu E_0 \otimes q^{-m\alpha_0} + t_\nu q^{m\alpha_0} \otimes E_0 \end{aligned} \tag{2.4}$$

$$\Delta_q^{\mu\nu}(F_0) = t_\mu^{-1} F_0 \otimes q^{-m\alpha_0} + t_\nu^{-1} q^{m\alpha_0} \otimes F_0$$

Here μ, ν stand for higest weights of corresponding irreducible representations and the ratio $x^2 = t_\mu / t_\nu$ is the spectral parameter.

The first part of (2.4) is a Hopf algebra homomorphism, that allows one to build theory of representations.

The most essential problem is to build the affine generators for irreducible representations of quantum groups, where E_0 and F_θ do not coincide. It is easy to show that for adjoint representations of all algebras (except A_n) E_0 and F_0 do not exist, and irreducible adjoint representation of deformed affine algebra is equivalent to direct sum of adjoint and trivial representations of deformed semisimple algebra. Investigations for every semisimple algebra show that affine generators for fundamental representation μ exist iff $(\mu, \theta) = 1$.

The Yang-Baxter equation is the associativity condition for the linear equations

$$R_{\mu\nu}(x)\, \Delta_q^{\mu\nu}(E_\alpha) = \Delta_{1/q}^{\mu\nu}(E_\alpha) R_{\mu\nu}(x) \tag{2.5}$$

$$x^2 = t_\mu / t_\nu; \quad \alpha \in \{\Pi_0\}$$

An obvious solution for (2.5) reads

$$R_{\mu\nu}(x) = \sum_\lambda (-)^{(\mu+\nu-\lambda, \rho^*)} \rho_\lambda(x) P_\lambda$$

$$where \quad r(\lambda) \in r(\mu) \otimes r(\nu) \tag{2.6}$$

$$P_\lambda = \sum_{m_\lambda} < \nu, m_\nu; \mu, m_\mu | \lambda, m_\lambda >_{1/q} < \lambda, m_\lambda | \nu, n_\nu; \mu, n_\mu >_q$$

Here $|\lambda, m_\lambda>_q$ and $|\lambda, m_\lambda>_{1/q}$ are the vectors of $r(\lambda)$, constructed with a help of $\Delta_q(\mathcal{G})$ and $\Delta_{1/q}(\mathcal{G})$ comultiplications.

Functions $\rho_\lambda(x)$ are to be found by treating equation (2.5) for the affine generator. Namely, let $|\omega_i>_q$ and $_{1/q}<\omega_i|$ are some vectors of $r(\omega_i)_q$ and $r(\omega_i)_{1/q}$. Taking the matrix element of (2.5) between two appropriate vectors of such type one obtains

$$\frac{\rho_{\omega_1}}{\rho_{\omega_2}} = \frac{_q<\omega_2|\,\Delta_q\,(E_0)|\omega_1>_q}{_{1/q}<\omega_2|\,\Delta_{1/q}\,(E_0)|\omega_1>_{1/q}}, \tag{2.7}$$

where numerator and denominator of (2.7) are implied to be nonzero.

In case of exceptional algebras E_0 and F_0 don't depend on q, so relations (2.7) are of the form, providing the unitarity:

$$\frac{\rho_{\omega_i}}{\rho_{\omega_j}} = \frac{xq^{p_{ij}} - x^{-1}q^{-p_{ij}}}{x^{-1}q^{p_{ij}} - xq^{-p_{ij}}} \tag{2.8}$$

and the set of $\{p_{ij}\}$ determines all the functions ρ_λ uniquely.

We have found all these sets by direct calculations of quantum Clebsh-Gordon coefficients for every case involved with the help of analytical calculation system.

Note, that in all cases constant R matrices agree with Reshetikhin's formula [5]:

$$\rho_\lambda^+ \sim (-)^{(\mu+\nu-\lambda,\rho^*)} q^{m(\lambda,\lambda+2\rho)} \tag{2.9}$$

where ρ_λ^+ are the asymptotic values of $\rho_\lambda(x)$ under $x \to +\infty$, and

$$\rho = \frac{1}{2}\sum_{\alpha>0} \alpha, \quad \rho^* = \frac{1}{2}\sum_{\alpha>0} H_\alpha.$$

3.Exceptional Lie algebras.

In this part we follow notations of [4]. $r(\mu)$ means an irreducible representation of weight μ. Representations $r(\mu)$ such that $(\mu,\theta)=1$ we shall call as vector representations. We use notation $\varepsilon_i, i=1\dots n$ for $\mathbf{R}^n$ orthonormal basis. Also we use a spectral parameter u instead of $x=q^u$.

3.1 The F_4 algebra.

We choose the fundamental system of roots of the form

$$\alpha_1=\varepsilon_2-\varepsilon_3,\quad \alpha_2=\varepsilon_3-\varepsilon_4,\quad \alpha_3=\varepsilon_4,$$

$$\alpha_4=\frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)$$

Decomposition of the tensor product the for vector representation is

$$r(\omega_4)\otimes r(\omega_4)=r(0)\oplus r(\omega_4)\oplus r(\omega_1)\oplus r(\omega_3)\oplus r(2\omega_4)$$

and a corresponding spectral decomposition for the fundamental R matrix is

$$\begin{aligned}
\rho_0 &= [1+u][4-u][6+u][9-u]\\
\rho_{\omega_4} &= [1-u][4+u][6-u][9+u]\\
\rho_{\omega_1} &= [1+u][4-u][6+u][9+u]\\
\rho_{\omega_3} &= [1-u][4+u][6+u][9+u]\\
\rho_{2\omega_4} &= [1+u][4+u][6+u][9+u]
\end{aligned}\tag{3.1}$$

3.2 The E_6 algebra.

The fundamental root system is of the form

$$\alpha_1=\frac{1}{2}(\varepsilon_1+\varepsilon_8)-\frac{1}{2}(\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7)$$

$$\alpha_2=\varepsilon_1+\varepsilon_2,\quad \alpha_3=\varepsilon_2-\varepsilon_1,\quad \alpha_4=\varepsilon_3-\varepsilon_2$$

$$\alpha_5=\varepsilon_4-\varepsilon_3,\quad \alpha_6=\varepsilon_5-\varepsilon_4$$

For the vector representation:

$$r(\omega_1) \otimes r(\omega_1) = r(\omega_6) \oplus r(\omega_3) \oplus r(2\omega_1)$$

$$\begin{aligned} \rho_{\omega_6} &= [1-u][4-u] \\ \rho_{\omega_3} &= [1-u][4+u] \\ \rho_{2\omega_1} &= [1+u][4+u] \end{aligned} \tag{3.2}$$

For the product of condjugated representations we have:

$$r(\omega_1) \otimes r(\omega_6) = r(\omega_1 + \omega_6) \oplus r(\omega_2) \oplus r(0)$$

$$\begin{aligned} \rho_{\omega_1+\omega_6} &= [3+u][6+u] \\ \rho_{\omega_2} &= [3-u][6+u] \\ \rho_0 &= [3-u][6-u] \end{aligned} \tag{3.3}$$

3.3 The E_7 algebra.
The fundamental root system is

$$\alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8) - \frac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7)$$

$$\alpha_2 = \varepsilon_1 + \varepsilon_2, \quad \alpha_3 = \varepsilon_2 - \varepsilon_1, \quad \alpha_4 = \varepsilon_3 - \varepsilon_2$$

$$\alpha_5 = \varepsilon_4 - \varepsilon_3, \quad \alpha_6 = \varepsilon_5 - \varepsilon_4, \quad \alpha_7 = \varepsilon_6 - \varepsilon_5$$

For the vector representation we have:

$$r(\omega_7) \otimes r(\omega_7) = r(0) \oplus r(\omega_6) \oplus r(\omega_1) \oplus r(2\omega_7)$$

$$\begin{aligned} \rho_0 &= [1-u][5-u][9-u] \\ \rho_{\omega_1} &= [1-u][5-u][9+u] \\ \rho_{\omega_6} &= [1-u][5+u][9+u] \\ \rho_{2\omega_7} &= [1+u][5+u][9+u] \end{aligned} \tag{3.4}$$

3.4 The E_8 algebra.
There are no vector representation, so the R matrix in standart diagonal form does not exist.

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