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Stochastic Cooling with a Double RF System*

Jie Wei
Accelerator Development Department
Brookhaven National Laboratory
Upton, New York 11973

Abstract

Stochastic cooling for a bunched beam of hadrons stored in an accelerator with a double rf system of two different frequencies has been investigated. The double rf system broadens the spread in synchrotron-oscillation frequency of the particles when they mostly oscillate near the center of the rf bucket. Compared with the case of a single rf system, the reduction rates of the bunch dimensions are significantly increased. When the rf voltage is raised, the reduction rate, instead of decreasing linearly, now is independent of the ratio of the bunch area to the bucket area.

On the other hand, the spread in synchrotron-oscillation frequency becomes small with the double rf system, if the longitudinal oscillation amplitudes of the particles are comparable to the dimension of the rf bucket. Consequently, stochastic cooling is less effective when the bunch area is close to the bucket area.

1 INTRODUCTION

Previous studies[1] indicate that stochastic cooling for a bunched beam of hadrons in a single-frequency rf system becomes difficult when the longitudinal bunch area is small compared with the area of the stable motion. When the bunch area is small, the spread in synchrotron-oscillation frequency is small compared with the average oscillation frequency. Compared with the coasting beam of similar line density and momentum spread, the Schottky noise in the bunched beam is higher due to the relatively higher particle density in frequency domain.

Using a secondary rf system with higher frequency significantly broadens the spread in synchrotron-oscillation frequency of the particles near the center of the rf bucket. The achievable cooling rate can thus be significantly increased.

2 DOUBLE RF SYSTEM

2.1 Equations of Motion

The voltage seen per revolution by the particle of phase deviation ϕ relative to the fundamental system in a double rf system, is

$$V(\phi) = \hat{V}\sin(\phi+\phi_s) + k\hat{V}\sin(m\phi + m\phi_{2s}) \qquad (1)$$

where

$\hat{V}$ = peak voltage of the fundamental rf system

$k\hat{V}$ = peak voltage of the higher-frequency rf system

ϕ_s = the stable phase angle relative to the fundamental rf wave-form

ϕ_{2s} = the stable phase angle relative to the higher frequency wave-form.

In the case of stochastic cooling, the particle beam is typically stored in the accelerator without acceleration. Besides, the first and second derivatives of $V(\phi)$ should vanish[2] at the center of the bunch to avoid having other stability regions nearby. These conditions imply

$$mk = 1, \text{ and } \phi_s = 0,\ m\phi_{2s} = \pi, \text{ or } \phi_s = \pi,\ \phi_{2s} = 0. \qquad (2)$$

Assuming $\phi_s = 0$, and $m\phi_{2s} = \pi$, the longitudinal motion of the particle can be described by an Hamiltonian

$$\mathcal{H}(\phi, W; t) = C_W W^2 + \frac{C_\phi}{2}(1-\cos\phi) - \frac{C_\phi}{2n^2}(1-\cos m\phi). \qquad (3)$$

Here,

$C_W = \frac{h^2\omega_0^2\eta}{2E\beta^2}$, $W = \frac{\Delta E}{h\omega_0}$, $C_\phi = \frac{qe\hat{V}}{\pi h}$

qe = electric charge carried by the particle

h = harmonic number of the fundamental rf system

γ_T = transition energy

$\eta = 1/\gamma_T^2 - 1/\gamma^2$

ω_0 = synchronous revolution frequency

βc = synchronous velocity

$E = Am_0c^2\gamma$, synchronous energy.

The action variable J is a constant of motion. In the small-amplitude limit $m\phi \ll 1$, J is given by

$$J \approx \frac{8\sqrt{2}\mathrm{K}(2^{-1/2})}{3C_W^{1/2}C_\phi^{1/4}}\left(\frac{3}{m^2-1}\right)^{1/4}\alpha^{3/4}, \quad \alpha \ll C_\phi \qquad (4)$$

where α is the value of the Hamiltonian and $\mathrm{K}(2^{-1/2}) \approx 1.8541$ is the complete elliptic integral of modulus $2^{-1/2}$. The synchrotron-oscillation frequency is given by the time derivative of the angle variable Q,

$$\Omega_s = \frac{3^{1/3}\pi\Omega_0}{2^{2/3}\mathrm{K}^{4/3}(2^{-1/2})}\left(\frac{3}{m^2-1}\right)^{1/3}\left(\frac{J}{\hat{J}_0}\right)^{1/3}, \quad J \ll \hat{J}_0 \qquad (5)$$

where

$$\hat{J}_0 = 8\sqrt{\frac{C_\phi}{C_W}} \quad \text{and} \quad \Omega_0 = \sqrt{C_W C_\phi} \qquad (6)$$

are the bucket area and the zero-amplitude synchrotron-oscillation frequency, respectively, in the absence of the

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secondary rf system. ϕ and W are expressed

$$\begin{cases} \phi &= \hat{\phi}\ \mathrm{cn}\left[2\mathrm{K}(2^{-1/2})\Omega_s t/\pi\right] \\ W &= \hat{W}\sqrt{2}\ \mathrm{sn}\left[2\mathrm{K}(2^{-1/2})\Omega_s t/\pi\right]\mathrm{dn}\left[2\mathrm{K}(2^{-1/2})\Omega_s t/\pi\right] \end{cases} \quad (7)$$

where sn, cn, and dn are the Jacobian elliptic functions. Here in Eq. 7, the amplitudes of the oscillation are

$$\hat{\phi} = 2\left(\frac{3}{m^2-1}\right)^{1/4}\left(\frac{\alpha}{C_\phi}\right)^{1/4}, \quad \hat{W} = 2\left(\frac{\alpha}{C_W}\right)^{1/2} \quad (8)$$

where α is related to J by Eq. 4.

Eq. 5 indicates that a secondary rf system changes the spread in synchrotron-oscillation frequency of the particles of different oscillation amplitudes. This results a change in the structure of the synchrotron side-bands in the frequency domain. For a typical bunch of particles where the density is the highest at the center, and zero at the separatrix, the side-band splitting no longer exists. On the other hand, side-band overlapping becomes significant.

2.2 *The Transport Equation*

Consider the stochastic cooling of the transverse dimension x of a bunched beam of N particles that perform synchrotron oscillation with frequencies Ω_i and phase amplitudes $\hat{\phi}_i$. The increment $U_{x'}$ in $x' = dx/ds$, which is experienced by the particle i per unit time at the kicker, is proportional to the displacement x^P of all the particles at the pick-up,[1]

$$\begin{aligned} U_{x',i} &= \frac{f_0^2}{\sqrt{\beta_x^P\beta_x^K}}\sum_{j=1}^{N} x_j^P \sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} i^l J_l(-m\hat{\phi}_j/h) \\ &\times \exp(il\phi_j^0)G(m^\pm\omega_0 - l\Omega_j) \\ &\times \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} i^k J_k(-n\hat{\phi}_i/h) \\ &\times \exp\left[it(m\omega_0 - l\Omega_j + n\omega_0 - k\Omega_i) + ik\phi_i^0\right]. \end{aligned} \quad (9)$$

Here, $G(\omega)$ is the gain of the cooling system, β_x is the Courant-Snyder parameter, J_l is the Bessel function of lth order, and ϕ^0 is the initial phase of synchrotron oscillation. The superscripts P and K denote values at the pick-up and kicker, respectively.

Using transverse angle-action variables φ and I, the equations of motion are

$$\dot{\varphi} = \frac{1}{\beta_x} + U_\varphi, \quad \dot{I} = U_I \quad (10)$$

where

$$U_I = -\sqrt{2\beta_x I}\sin\varphi U_{x'}, \quad U_\varphi = -\sqrt{\beta_x/2I}\sin\varphi U_{x'}. \quad (11)$$

Typically, the time for stochastic cooling to produce an appreciable effect is much longer than the revolution period. The evolution of the transverse distribution function Ψ of the particles can be described by the transport equation, which is obtained by averaging the two-dimensional Fokker-Planck equation over φ

$$\frac{\partial\Psi}{\partial t} = -\frac{\partial}{\partial I}(F\Psi) + \frac{1}{2}\frac{\partial}{\partial I}\left(D\frac{\partial\Psi}{\partial I}\right). \quad (12)$$

Neglecting the thermal noise of the cooling system, the coefficients of correction F and diffusion D can be evaluated by employing the representation of the Jacobian elliptic function as a trigonometric series and keeping[3] only the leading term. Then,

$$F(I) = F^0 I, \quad D_{SH} = D^0 I\langle I\rangle. \quad (13)$$

Here,

$$\begin{aligned} F^0 &= -f_0^2\int dJ\ \rho(J)\sin(\nu_x\Delta\theta^{PK})\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \\ &\quad \times G(m^\pm\omega_0 - l\Omega_i)e^{il\Omega_i\Delta\theta^{PK}/\omega_0}J_l^2(m\omega_0\tau_i) \\ D^0 &= \pi f_0^4\int dJ\ \rho(J)\sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\frac{\rho(J')}{|l|\left|\frac{d\Omega_s(J')}{dJ'}\right|}\Bigg|_{\Omega_s(J')=k\Omega_s(J)/l}. \\ &\quad \times\left\{|G_D(-)|^2 + |G_D(+)|^2 + 2Re\left[G_D(-)G_D(+)\right]\right\} \end{aligned} \quad (14)$$

with

$$G_D(\pm) = \sum_{m=1}^{\infty} G(m^\pm\omega_0 \pm l\Omega_j)J_{\mp l}(m\hat{\phi}_j/h)J_{\mp k}(m\hat{\phi}_i/h), \quad (15)$$

and $\Delta\theta^{PK}$ is the azimuthal distance between the pick-up and the kicker. The boundary condition to this equation is

$$\begin{cases} I = 0: & -F\Psi + \dfrac{D}{2}\dfrac{\partial\Psi}{\partial I} = 0\ ; \\ I = I_{\max}: & \Psi = 0 \end{cases} \quad (16)$$

where $I_{\max}$ is the transverse aperture of the accelerator. In Eq. 14, $\rho(J)$ is the density in J, and

$$\frac{d\Omega_s(J)}{dJ} = \frac{3^{1/3}\pi C_W}{24\ 2^{2/3}\mathrm{K}^{4/3}(2^{-1/2})}\left(\frac{m^2-1}{3}\right)^{1/3}\left(\frac{\hat{J}}{J}\right)^{2/3}. \quad (17)$$

2.3 *Optimum Cooling Rate*

Because F and D are both independent of Ψ, the reduction rate of the transverse emittance can be obtained by integrating Eq. 12. The average gain G_{opt} for achieving the optimum cooling rate is

$$G_{opt}^{-1} = \frac{\Delta n f_0\langle\rho(\Omega_j)\rangle}{\pi\langle k\rangle\langle n\rangle\langle\tau\rangle}. \quad (18)$$

Here, $\langle\ \rangle$ denotes the average over the quantity J, $\langle k\rangle = \langle n\rangle\omega_0\langle\tau\rangle$ is the number of significant synchrotron sideband, $\langle n\rangle$ is the average harmonic of the cooling system,

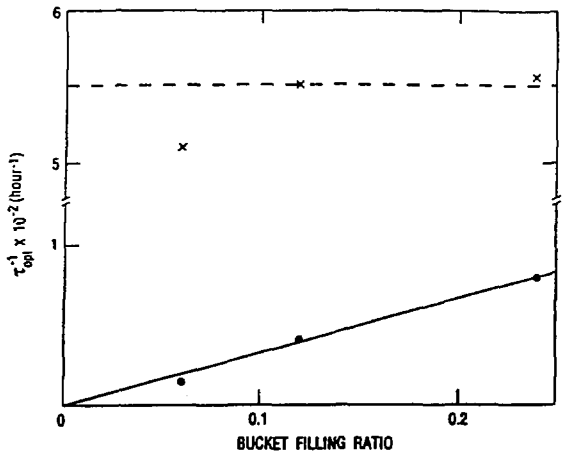

Figure 1: The reduction rates of the transverse emittance as a function of the bucket filling ratio for a bunch of $^{197}Au^{79+}$ ions stored with a single (soiid line) and a double (dashed line) rf system, respectively.

and $\Delta n f_0$ is the frequency bandwidth. The optimum rate for small-amplitude oscillation is

$$\tau_{opt}^{-1} = \frac{\langle k\rangle^2}{2\langle\rho(J)\rangle}\left\langle\left|\frac{d\Omega_s}{dJ}\right|\right\rangle \approx \frac{\pi\langle n\rangle^2 C_W}{8\mathrm{K}^2(2^{-1/2})h^2\langle\rho(J)\rangle}. \quad (19)$$

Note that in obtaining Eq. 19, synchrotron side-band overlapping has been neglected.

A computer program has been developed to calculate the cooling rate by evaluating F^0 and D^0 according to Eq. 14 considering synchrotron side-band overlapping. Fig. 1 shows the transverse cooling rates of a bunch of 10^9 $^{197}Au^{79+}$ ions during stochastic cooling in the RHIC. The dashed line indicates that with a fundamental system of 160 MHz and a secondary system of twice the frequency ($m = 2$), the cooling rate is independent of the ratio of the bunch area to the bucket area (bucket filling ratio).

3 COMPARISON

3.1 *Small Amplitude Case*

Stochastic cooling with a single rf system has been investigated previously.[1] The optimum rate of reduction of the transverse emittance for small amplitude $J < \hat{J}_0$, is

$$\tau_{opt}^{-1} \approx \frac{\langle n\rangle^2 C_W}{\pi^2 h^2\langle\rho(J)\rangle}\frac{\langle J\rangle}{\hat{J}_0}. \quad (20)$$

In the case that the particle distribution in J remains unchanged, the transverse cooling rate is linearly proportional to the bucket filling ratio. The solid line in Fig. 1 shows the same transverse cooling rate as the dashed line, except with a single rf system of 160 MHz. The cooling rate decreases linearly with the increasing bucket area.

The cooling efficiency can be significantly improved with a secondary rf system. When the secondary rf system is employed, the spread in synchrotron-oscillation frequency is broadened appreciably. Eq. 19 indicates that the optimum cooling rate is approximately independent of the bucket filling ratio.

3.2 *Large Amplitude Case*

In the case of a single rf system, the spread in synchrotron-oscillation frequency increases for particles of large synchrotron-oscillation amplitudes. Consequently, stochastic cooling becomes more effective when the particles occupy larger amount of the rf bucket.

On the contrary, cooling is less effective in the case of a double rf system when the rf bucket becomes full. For $m = 2$, the synchrotron-oscillation frequency can be derived

$$\Omega_s = \frac{\pi\Omega_0}{\sqrt{2}\mathrm{K}(\xi)}\left(\frac{\alpha}{C_\phi}\right)^{1/4}, \quad \xi = \sqrt{\frac{1}{2}\left(1+\sqrt{\frac{\alpha}{C_\phi}}\right)}. \quad (21)$$

It is found[3] that for particles with large synchrotron-oscillation amplitude (approximately with $J/\hat{J}_0$ between 0.3 and 0.8), the spread in synchrotron-oscillation frequency is small. Stochastic cooling becomes difficult if most of the particles in the bunch are in this large-amplitude region.

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