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ALGEBRAIC STRUCTURE OF THE BRST SYMMETRY

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ABSTRACT

An explicit construction of the BRST symmetry is presented in the Hamiltonian approach. The construction is based on the splitting homotopy and the transference problem.

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1. INTRODUCTION

Recently, a lot of work has been done for understanding the algebraic and geometrical structure of the BRST symmetry/1-9/. In the Hamiltonian approach the BRST construction is basically intended to accomplish the symplectic reduction, without modifying the initial structure of the theory. In other words, in the BRST construction we start with a symplectic manifold M, the phase space , and a set of irreducible first class constraints G_a(a=1, ...,N) and if the constraints are sufficiently nice, we can reduce M to lower-dimensional symplectic manifold $\tilde{M}$, by using a standard construction, called the symplectic reduction. The classical BRST construction provides an algebraic homological realisation of the symplectic reduction.

The present paper deals with the BRST construction, considered as a transference problem in the general framework of the homological perturbation theory/10,11/. In the informal terms the basic idea is this: if a module P is a perturbation of a module A, then a resolution of A may be perturbed to obtain a resolution of P. We shall identify P to $C^\infty(M)$, the class of all smooth functions defined on M, and A to $C^\infty(M_0)$, M_0 being the zero locus of the set of the first class constraints. The perturbation of the resolution of $C^\infty(M_0)$ is given by the vertical derivative and the homological pertubation theory(HPT) will give us the BRST differential and the BRST cohomology of the dynamical system.

We shall give, in this paper, a very nice and compact form for the BRST differential s and of the BRST charge by using a construction proper to the transference problem/12/, construction formulated by Barnes and Lambe. The transference problem is a part of HPT/10,11/. This costruction allows us to build up all BRST observables as well as to give a short proof of the isomorphism between the BRST cohomology and the cohomology of the vertical derivation modulo the Koszul exact forms.

2. BRST CONSTRUCTION

Let M be the phase space of a dynamical system and $\{G_a\}$ (a=1,...,N) be a set of first class constraints i.e. a set of functions on M satisfying:

$$[G_a, G_b] = f_{ab}{}^c G_c \qquad (1)$$

where $f_{ab}{}^c$ are some functions, called the structure functions. The zero locus of $G_a=0$ is a submanifold M_0 of M. For the first class constraints $\{G_a\}$ the Hamiltonian vector fields $\{X_a\}$ associated to these constraints form an involutive distribution and they determine o foliation of M, the leaves of this foliation being the "gauge orbits". The space of all leaves of M forms the so called reduced phase space $\tilde{M}$.

In order to go from M to $\tilde{M}$ we shall introduce two derivations: the Koszul-Tate differential and the vertical derivation δ. They are defined by:

$$\delta f(q,p)=0 \ , \quad \delta\eta^a=0 \ , \quad \delta\mathcal{P}_a=G_a \qquad (2)$$

$$df=[G_a, f]\eta^a \ , \quad d\eta^a=-\tfrac{1}{2}f_{bc}{}^a\eta^b\eta^c \ ,$$
$$d\mathcal{P}_a=-f_{ab}{}^c\eta^b\mathcal{P}_c \ . \qquad (3)$$

for an irreducible theory, with$(\eta^a, \mathcal{P}_a)$ a pair of ghost and ghost momentum . It is easy to verify that δ is nilpotent but d is not, in the general case. Besides δ and d anticommutes i.e. $d.\delta + \delta.d=0$. The Koszul-Tate differential has the remarkable property to be <u>acyclic</u> i.e. any solution of the equation

$\delta\omega=0$, with $gh(\omega)\neq 0$ has the form $\omega=\delta\eta$. The ghost number gh is defined to be +1 for η^a, and -1 for $\mathcal{P}_a$, and zero for f(q,p).

Thanks to this acyclicity of δ one can introduce a <u>contracting homotopy</u> Γ defined as

$$\Gamma.\delta+\delta.\Gamma=1-\pi \qquad (4)$$

where π is the projector on the space $C^\infty(M_0)\otimes\{\eta^a\}=B$. A concrete form of the homotopy Γ is given in /13/ and this one is in addition nilpotent.

The passage from M to $\tilde{M}$ is accomplished in two steps: in the first we pass from M to M_0 by using the Koszul- Tate differential and its acyclicity and we show that the cohomology of δ is a resolution of $C^\infty(M_0)$ i.e.

$$H(\delta)=C^\infty(M_0)$$

and in the second step we pass from M_0 to $\tilde{M}$ by using the vertical derivation d and we show that

$$H_V{}^0(M_0)=C^\infty(\tilde{M})$$

where vertical cohomology is denoted by $H_V(M_0)$.

In order to complete the BRST construction we must integrate the two cohomologies theories into one. In fact the purpose of the BRST construction is to lift the vertical differential d from $C^\infty(M_0)$ to the whole complex.

3. THE TRANSFERENCE PROBLEM

The BRST construction ,formulated in this way can be developed by using HPT in a version known as <u>the transference problem</u> /12/. In this version, the classical HPT is formulated as a fixed point problem leading to new insights into the nature of its solutions.

The main purpose of HPT is to offer when, a chain subcomplex B of a given chain complex A, can be changed in a way that reflects a change in A and preserves the inclusion. If the subcomplex B is a retract of A, it is sometimes useful to be able to transfer a change in the differential of A in a way that preserves the retraction condition. The complex maps

$$f: B \to A\ ,\ g: A \to B\ ,\ \phi: A \to A \qquad (5)$$

such that they satisfy

$$g.f=1\ ,\qquad f.g= 1-(d.\phi + \phi.d\ ) \qquad (6)$$

where d is the differential of A, form the Strong Deformation Retraction data (SDR-data) /10, 11, 12 /. The chain homotopy ϕ can be chosen (or modified) to satisfy the additional hypothesis called the "side conditions"

$$g.\phi = \phi.f = \phi.\phi = 0 \qquad (7)$$

With these SDR-data we can state the fundamental problem of HPT the transference problem. Given SDR-data (5)-(7) and a change in the differential from A or B, find new SDR-data i.e new f', g', ϕ'. The conditions (5)-(7) can be replaced by the following equivalent conditions /12/ :

$$\phi: A \to A \text{ is a morphism of degree 1,}$$
$$\phi^2=0\ ,\quad \phi.d.\phi = \phi. \qquad (8)$$

With Eqs.(7) satisfied, it is easy to show that $\pi = 1-(d.\phi + \phi.d)$ is a projection and hence we have the splitting $A = \mathrm{Im}\,\pi + \ker \pi$ and with $B = \mathrm{Im}\,\pi$ we obtain again the SDR-data (5)-(7).

In these new terms, very convenient for the BRST construction, the transference problem becomes: Given a homotopy

$\phi: (A, d) \to (A, d)$ and a new differential ξ, find a splitting homotopy $\phi': (A, \xi) \to (A, \xi)$ such that $\mathrm{Im}\,\pi = \mathrm{Im}\,\pi'$, where $\pi = 1-(d.\phi + \phi.d)$ and $\pi' = 1-(\xi.\phi' + \phi'.\xi)$.

Reformulated in this way the transference problem has been solved by Barnes and Lambe /12/ and the solution has a remarkable simple form

$$\phi' = \sum_{n=0}^{\infty} (-1)^n (\phi.t)^n \phi = (1+\phi.t)^{-1}.\phi \qquad (9)$$

with $t = \xi - d$.

All these results can be applied for solving the initial transference problem. We shall not enter in into details, which are not relevant for our construction. They can be found in Ref./12/ where the transference problem has been treated completely from the HPT point of view.

4. BRST CONSTRUCTION AS A TRANSFERENCE PROBLEM

It is very interesting to point out that the BRST construction can be reformulated to become a transference problem. First of all we have to identify the complexes A and B with the Koszul complex

$$A = C^{\infty}(M) \otimes \{\eta^a, \mathcal{P}_a\}$$

and the subcomplex B with

$$B = C^{\infty}(M_0) \otimes \{\eta^a\}.$$

the differential of A with the Koszul-Tate differential and the differential of B with the multiplication by zero. The transference problem now means to change the Koszul- Tate differential δ in the BRST differential s, such that the new differential on B is just the restriction of the vertical derivation d. It is worth pointing out that despite the fact that the vertical derivation d is not nilpotent on A, it is on B and therefore it is a differential there.

The chain maps f and g are, in this case, the inclusion f and the restriction, respectively. The chain homotopy ϕ is for the BRST construction just the contracting homotopy Γ and Eq. (6) is very similar to Eq.(4). Moreover, it is easy to see that the contracting homotopy fulfils the conditions (7).

In the following we will try to simplify the formulation of the BRST construction, considered as a transference problem, and we will omit all inclusions and restrictions, supposing that we work only on the Koszul complex. Thus, the first problem in the BRST costruction is to find a BRST differential s, which is an extension of the Koszul-Tate differential and which, restricted at B, coincides to the vertical derivative d. n other words we are looking for a differential s such that

$$s=\delta+d+\ldots=s_0+s_1+s_2+\ldots \qquad (10)$$

and

$$s^2=0 \qquad (11)$$

We want to emphasize at this point that s must be a linear and nilpotent operator, on the one hand and a derivation on the other hand. In other words s must act as a derivation on the product of two elements of the algebra A.

Using the general procedure, developed by Barnes and Lamde/12/, we have found that a solution of our transference problem is:

$$s=\delta+(1+\Gamma.d)^{-1}d=\delta+d+(-\Gamma.d)d+\ldots \qquad (12)$$

i.e. s_n in Eq. (10) can be taken as

$$s_0=\delta, \qquad s_n=(-\Gamma d)^{n-1}.d. \qquad (12')$$

Besides the new contracting homotopy Γ' is given by:

$$\Gamma'=(1+\Gamma d)^{-1}\Gamma. \qquad (13)$$

which coincides to Eq.(9). The equation satisfied by Γ should be modified and it becomes

$$s.\Gamma'+\Gamma'.s=1-\Pi', \qquad (14)$$

where

$$\Pi'=(1+\Gamma.d)^{-1}\Pi(1+\Gamma.d). \qquad (15)$$

Now, it is relatively easy to show that the linear operator s given by Eq. (12) is nilpotent. For this we shall use Eq.(4) to calculate the commutator

$$\left[\delta,(1+\Gamma.d)^{-1}\right]=-(1+\Gamma.d)^{-1}\left[\delta,\Gamma d\right](1+\Gamma d)^{-1}=$$

$$=-(1+\Gamma d)^{-1}d(1+\Gamma d)^{-1}+(1+\Gamma d)^{-1}\tilde{\Pi}d(1+\Gamma d)^{-1}$$

The last term vanishes when one calculates s^2 since $\tilde{\Pi}$ is a projection on B where d is nilpotent. Thus we we obtain eventually Eq.(11).

Now we have to show that the solution of our transference i.e. the BRST differential s preserves algebra structure. In other words we must show that s is a derivation. This result is far from trivial, since in Eq. (12) the contracting homotopy Γ which is not a derivation. However, we shall show inductivelly that s_n is a derivation for any n. This result has been proved by Gugenheim, Lambe and Stasheff/15/, who have shown that the basic perturbation lemma /10,11/ preserves algebra or coalgebra structure in a general framework. Our proof has been inspired by Stasheff's paper/4/ . For n=2 we can write

$$s_2.\delta+\delta.s_2 = -(\delta.\Gamma+\Gamma.\delta).\ d^2 = -s_1.s_1 \quad .$$

The right side of this equation is a derivation ans so is the Koszul differential. Thus s_2 must be a derivation, too. For a general n+1 we can verify that

$$s_{n+1}.\delta+\delta.s_{n+1} = -\sum_{p+q=n+1} s_p.s_q \quad . \qquad (16)$$

The right-hand side of the last equation contains two kinds of terms: terms of the form $s_p.s_q+s_q.s_p$ and of the form s_p^2. If s_j ($j\le n$) are all derivations so are both terms. Thus the right-hand side of (16) and δ are derivations and so must be s_{n+1}.

The form of the BRST differential s can be simplified further if we work only on the kernel of π. In this case s can be written in a very elegant form

$$s=(1+\Gamma.d)^{-1}.\delta.(1+\Gamma.d) \quad , \qquad (17)$$

and the nilpotence of s is a direct consequence of the nilpotence of δ .It is amusing to remark that in the <u>general case</u> one can write s in a similar form:

$$s=(1+\Gamma.d)^{-1}.\delta(1+\Gamma.d)+(1+\Gamma.d)^{-1}\pi.d \qquad (18)$$

The last form of s can be used to prove the isomorphism between cohomology of s and the cohomology of d modulo δ . For this purpose we shall introduce a grading given by the anti-ghost number $r(\eta^a)=0$, $r(\mathcal{P}_a)=+1$, $r(f(q,p))=0$. If $sA=0$ then A can be expanded as $A=A_0+A_1+\ldots$ with $r(A_n)=n$ and we obtain an equation for A_1 and A_0 :

$$\delta A_1+dA_0=0,$$

i.e. we can define a map $H(s)\to H(d \bmod \delta)$ by

$$A\to A_0 \ .$$

This map is in fact a bijection , a fact that can be shown by using the expression (18) for s. Given A_0 as a solution of the equation $dA_0+\delta A_1=0$ one can improve A_0 by higher order terms $A_0\to A=A_0+$ "more" such that $sA=0$. In fact we can find a compact form of the solution of this problem

$$A=(1+\Gamma.d)^{-1}.A_0 \ , \qquad (19)$$

and it is easy to verify that

$$sA=\delta A_0+(1+\Gamma.d)^{-1}\pi d\,(1+\Gamma.d)^{-1}A_0=0 \ ,$$

since A_0 does not contain any $\mathcal{P}_a$ and $dA_0=-\delta A_1$ does contain at least one $\mathcal{P}_a$ or G_a .

It is worth pointing out that this isomorphism can be obtained also from (14). In fact Eqs.(14) and (15) yield

$$H(s)=\mathrm{Im}\,\pi' \cong H(d \bmod \delta) \ .$$

It should be noted that the BRST differential is not unique since one has the possibilities of ambiguities . The form of these ambiguities has been given , in the usual formulation (i.e. not as a transference problem) by Browning and Mc Mullen in /9/. However, in our opinion it is worth while to find out where these ambiguities come from in the HPT. A partial answer has been given in /13/ ,where it has been shown that a possible source of ambiguity is the definition of the contracting homotopy.

5. BRST CHARGE

The BRST differential s can be realised, in the extended phase space by a generator, the BRST charge Q and by the Poisson brackets. The BRST charge is defined by the equation:

$$sF=[Q, F] \tag{20}$$

where $[,]$ means the Poisson bracket in the extended phase space $(q ,p; \eta, \mathcal{P})$. The nilpotence of s implies

$$[Q,Q] =0 . \tag{21}$$

We will argue below that s defined by (20) and (21) is indeed the BRST differential, provided Q satisfies some initial conditions. A solution of Eq.(21) has been given by Stasheff/ in/4,5/ and it has the form

$$Q = \sum_{j=0}^{\infty} Q_j \tag{22}$$

with $Q_0 = \eta^a G_a$ and Q_{n+1} is constructed inductively as

$$Q_{n+1} = -\frac{1}{2}\Gamma([R_n, R_n]) \tag{23}$$

where $R_n = Q_0 + Q_1 +...+Q_n$. The grading which has been used in (22) is the anti-ghost number. A slightly complicated computation /5/ shows that the anti-ghost number of $[R_n ,R_n]$ is bigger than n+2,so for a finite dimensional phase space it eventually vanishes.

On the other hand, we can use the previous construction to build up a BRST charge, which generates the same BRST transformation as (22) . We will now redefine Q by

$$Q = (1+\Gamma.d)^{-1} .Q_0 = Q_0 +Q_1 +... \tag{24}$$

This new BRST charge has the first two terms identical to the one given by (22) it is s-closed. Indeed, one can verify that

$$sQ =0$$

since s has the form (17) and $Q_0 = \delta(\eta^a \mathcal{P}_a) \in \ker \pi$. Therefore, up to the well known ambiguities, which can be expressed as canonical transformations/3/ , the BRST charge defined by (22)-(24) coincides.

REMARKS. 1.We can use the same construction to build up The BRST invariant observables. Thus, if A_0 is a classical observable defined in the resticted phase space (q,p) such that $[A_0,G_a] = V_a{}^b G_b$, then the corresponding BRST invariant observable can be chosen as:

$$A= (1+\Gamma.d)^{-1} .A_0 = A_0 +... \tag{25}$$

The BRST observable A is s-closed

$$sA=0 \qquad (26)$$

This can be verified by a straightforward calculation, if one uses the eq. (18) for s and the equations fulfilled by A_0, $\delta A_0 = 0$ and $dA_0 + \delta A_1 = 0$ with $A_1 = V_a{}^b \chi^a \mathcal{P}_b$.

In particular, the unitarizing Hamiltonian H used in the path integral formulation of the theory has the form:

$$H = (1+\Gamma .d)^{-1} . H_0 + s\Psi \qquad (27)$$

where $H_0 = H_0 (q,p)$ is the Hamiltonian of the classical system and Ψ is the fermionic gauge fixing function /1/ .

2. The whole construction can be extended for the reducible theories, where the constraints G_a are not all independent. In this case we have to modify the definition of the Koszul-Tate differential in order to assure its acyclicity. The rest of the construction is the same . On the other hand this general construction can be applied also in the Lagrangian formulation for the systems with local irreducible or reducible symmetries/14/.

3. The same construction has been applied for the Hamiltonian formulation of the anti-BRST symmetry for an arbitrary gauge system with open gauge algebra/14 ,15 ,16 / . The slitting of the BRST generator and of the BRST differential in a BRST generator (differential) and an anti-BRST generator (differential) occurs quite naturally in our construction, and it is closely connected to Eqs.(13) and (20).

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