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ДУБНА**

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**LINEAR SYNCHRO-BETATRON RESONANCES,  
DRIVEN BY BEAM-BEAM INTERACTION**

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## I. INTRODUCTION

When two bunches pass through one another at the interaction point in a colliding beam storage device the particles of each of them experience a localized force. In the linear approximation when the dispersion at the interaction point is zero this force produces a small tune shift from the unperturbed tune and causes parametric beam-beam resonances. In the case of non-zero dispersion the beam-beam force excites nonlinear resonances of betatron motion (this takes place in the case of zero dispersion as well), as well as synchro-betatron resonances.

In the present paper we analyse the linear synchro-betatron resonance due to a non-zero dispersion at the interaction point, acting together with the standard beam-beam parametric resonance. Using further the single linear synchro-betatron resonance model we calculate the amplitude beating in the stability region. Finally we show that the square of projected emittance on the horizontal phase subspace oscillates around a mean value, proportional to the square of the synchro-betatron coupling strength.

## II. HAMILTON FORMULATION OF THE SYNCHROTRON AND BETATRON MOTION, TREATED SIMULTANEOUSLY

In order to be self-explanatory we begin this Section by summarizing some well-known facts. The motion of a particle with rest mass  $m_0$  and charge  $e$  is governed by the Hamiltonian

$$\mathcal{H} = c \left\{ m_0^2 c^2 + (p_x - eA_x)^2 + (p_z - eA_z)^2 + \left( \frac{p_s}{1 + \kappa K_x + 2\kappa K_z} - eA_s \right)^2 \right\}^{1/2} + e\phi, \quad (2.1)$$

written in the natural coordinate system, defined by the triple  $(\vec{n}, \vec{b}, \vec{z})$  along the design orbit. The quantities  $\phi$  and  $\vec{A} = (A_x, A_z, A_s)$  are as usual the scalar and the vector potentials of electromagnetic

field and  $K_x, K_z$  being the curvatures of the design orbit in horizontal and vertical plane respectively with  $K_x K_z = 0$  ( $(K_x, K_z) \neq (0, 0)$ ). A new Hamiltonian can be constructed<sup>/1/</sup> from (2.1) in a new independent variable  $s$  - curve length of design orbit, instead of time  $t$

$$H = -p_s = -(1 + xK_x + zK_z) \left\{ \frac{(\mathcal{K} - e\varphi)^2}{c^2} - m_0^2 c^2 - (p_x - eA_x)^2 - (p_z - eA_z)^2 \right\}^{1/2} - (1 + xK_x + zK_z) eA_s \quad (2.2)$$

Introduction of new variables

$$x \Rightarrow \tilde{p}_x = \frac{p_x}{p_{os}} \quad ; \quad z \Rightarrow \tilde{p}_z = \frac{p_z}{p_{os}} \quad ; \quad -\beta_s ct = \tau \Rightarrow h = \frac{\mathcal{K}}{\beta_s^2 E_s} \quad (2.3)$$

yields the new Hamiltonian

$$\tilde{H} = \frac{H}{p_{os}} = -(1 + xK_x + zK_z) \left\{ \left( \beta_s h - \frac{e\varphi}{\beta_s E_s} \right)^2 - \frac{1}{\beta_s^2 \gamma_s^2} - \left( \tilde{p}_x - \frac{e}{p_{os}} A_x \right)^2 - \left( \tilde{p}_z - \frac{e}{p_{os}} A_z \right)^2 \right\}^{1/2} - (1 + xK_x + zK_z) \frac{e}{p_{os}} A_s \quad (2.4)$$

where  $p_{os}$  and  $E_s = m_0 \gamma_s^2 c^2$  are the momentum and energy of synchronous particle, and  $\beta_s = v_s/c$ ,  $\gamma_s = (1 - \beta_s^2)^{-1/2}$ .

The electric  $\vec{\mathcal{E}}$  and magnetic  $\vec{\mathcal{B}}$  fields are given by<sup>/2/</sup>

$$\vec{\mathcal{E}} = -\frac{\partial \vec{A}}{\partial t} - \frac{\partial \vec{A}}{\partial \tau} (-\beta_s c) = \beta_s c \frac{\partial \vec{A}}{\partial \tau}$$

$$B_x = \frac{1}{h_0} \left[ \frac{\partial (h_0 A_s)}{\partial z} - \frac{\partial A_z}{\partial s} \right],$$

$$B_z = \frac{1}{h_0} \left[ \frac{\partial A_x}{\partial s} - \frac{\partial (h_0 A_s)}{\partial x} \right],$$

$$B_s = \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z},$$

where  $h_0 = 1 + xK_x + zK_z$ . For various lattice structure elements in particular we have:

1. RF-cavity:

$$\mathcal{E}_s = \hat{\mathcal{E}}_0(s) \sin\left(\frac{\omega\tau}{\beta_s c} + \phi_0\right) \quad ; \quad \mathcal{E}_x = \mathcal{E}_z = 0$$

and therefore

$$R \frac{e}{p_{os}} A_s = -\frac{e \hat{\mathcal{E}}_0(s)}{\beta_s E_s} \frac{cR}{\omega} \cos\left(\frac{\omega\tau}{\beta_s c} + \phi_0\right) \quad ; \quad A_x = A_z = 0 \quad (2.5)$$

where  $R$  is the mean radius of the ring,  $\hat{E}_0$  is the peak value of electric field,  $\omega = k\omega_0$  is the frequency of RF-generator ( $k$  being the harmonic acceleration mode,  $\omega_0$  being the angular frequency of particle revolution) and  $\phi_0$  is the initial phase of RF-field.

2. Quadrupole:

$$A_s = \left( \frac{\partial B_z}{\partial x} \right)_{x=z=0} \frac{z^2 - x^2}{2} ; \quad A_x = A_z = 0 ,$$

so that

$$R \frac{e}{p_{os}} A_s = \frac{g_0}{2R} (z^2 - x^2) ; \quad g_0 = R^2 \frac{e}{p_{os}} \left( \frac{\partial B_z}{\partial x} \right)_{x=z=0} . \quad (2.6)$$

3. Skew quadrupole:

$$A_s = \left( \frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z} \right)_{x=z=0} \frac{xz}{2} ; \quad A_x = A_z = 0 ,$$

$$R \frac{e}{p_{os}} A_s = \frac{N_0}{R} xz ; \quad N_0 = R^2 \frac{e}{2p_{os}} \left( \frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z} \right)_{x=z=0} . \quad (2.7)$$

4. Sextupole:

$$R \frac{e}{p_{os}} A_s = -\frac{\lambda_0}{6R} (x^3 - 3xz^2) ; \quad \lambda_0 = R^3 \frac{e}{p_{os}} \left( \frac{\partial^2 B_z}{\partial x^2} \right)_{x=z=0} . \quad (2.8)$$

5. Octupole:

$$R \frac{e}{p_{os}} A_s = \frac{\mu_0}{24R^3} (x^4 - 6x^2z^2 + z^4) ; \quad \mu_0 = R^4 \frac{e}{p_{os}} \left( \frac{\partial^3 B_z}{\partial x^3} \right)_{x=z=0} . \quad (2.9)$$

6. Synchrotron magnet:

$$R \frac{e}{p_{os}} A_s = -\frac{R}{2} (1 + xK_x + zK_z) + \frac{g_0}{2R} (z^2 - x^2) + \dots , \quad (2.10)$$

where

$$K_x = \frac{e}{p_{os}} B_z^{(0)} ; \quad K_z = -\frac{e}{p_{os}} B_x^{(0)} \quad \text{and} \quad K_x K_z = 0 . \quad (2.11)$$

Next we substitute expressions (2.5)-(2.10) into the Hamiltonian (2.4) and expand the result in a power series in the variables  $\tilde{p}_x/\Gamma$ ,  $\tilde{p}_z/\Gamma$  and  $e\Phi/E_s$  to obtain

$$\tilde{H} = \tilde{H}_0 + \tilde{H}_1 + \tilde{H}_2 + \tilde{H}_3 + \dots + \tilde{H}_{bb} , \quad (2.12)$$

where

$$\tilde{H}_0 = -\Gamma + \frac{e\hat{E}_0(s)}{\beta_s E_s} \frac{c}{\omega} \cos \left( \frac{\omega\tau}{\beta_s c} + \phi_0 \right) , \quad (2.13a)$$

$$\tilde{H} = -(\Gamma - 1) (xK_x + zK_z) , \quad (2.13b)$$

$$\tilde{H}_2 = \frac{\tilde{p}_x^2 + \tilde{p}_z^2}{2\Gamma} + \frac{G_x x^2 + G_z z^2}{2R^2}, \quad (2.13c)$$

$$\tilde{H}_3 = (xK_x + zK_z) \left[ \frac{\tilde{p}_x^2 + \tilde{p}_z^2}{2\Gamma} + \frac{g_0}{2R^2} (x^2 - z^2) \right] + \frac{\lambda_0}{6R^3} (x^3 - 3xz^2) + \dots, \quad (2.13d)$$

$$\tilde{H}_{bb} = (1 + xK_x + zK_z) \frac{1 + \beta_s^2}{\beta_s^2} \frac{e\varphi_b}{E_s} \delta_b(s). \quad (2.13e)$$

Here we have used the notations:

$$\Gamma = \left( \beta_s^2 h^2 - \frac{1}{\beta_s^2 \gamma_s^2} \right)^{1/2}, \quad (2.14a)$$

$$G_x = g_0 + R^2 K_x^2; \quad G_z = -g_0 + R^2 K_z^2, \quad (2.14b)$$

$$\delta_b(s) = \sum_{k=1}^M \delta_p(\ddot{\varphi} - \ddot{\varphi}_k) = \frac{1}{2\pi} \sum_{k=1}^M \sum_{n=-\infty}^{\infty} e^{in(\ddot{\varphi} - \ddot{\varphi}_k)}, \quad (2.14c)$$

assuming that there are  $M$  interaction points, placed at azimuth  $\ddot{\varphi}_k$  ( $k=1, 2, \dots, M$ ). The quantity  $\varphi_b$  in eq. (2.13e) is the beam-beam interaction potential for a Gaussian bunch, which reads as<sup>3/</sup>

$$\varphi_b(x, z) = -\frac{Ne}{4\pi\epsilon_0 R} \int_0^{\infty} \frac{\exp\left(-\frac{x^2}{2\sigma_x^2 + q} - \frac{z^2}{2\sigma_z^2 + q}\right)}{\left[(2\sigma_x^2 + q)(2\sigma_z^2 + q)\right]^{1/2}} dq, \quad (2.15)$$

where  $N$  is the total number of particles contained in the partner bunch and  $\sigma_x$ ,  $\sigma_z$  are the horizontal and vertical standard deviations respectively. The definition of the classical charged particle radius

$$r_e = \frac{e^2}{4\pi\epsilon_0 m_0 c^2}$$

helps us to rewrite the Hamiltonian  $\tilde{H}_{bb}$  as

$$\tilde{H}_{bb} = -\frac{Nr_e}{R\gamma_s} \frac{1 + \beta_s^2}{\beta_s^2} (1 + xK_x + zK_z) \delta_b(s) \int_0^{\infty} \frac{\exp\left(-\frac{x^2}{2\sigma_x^2 + q} - \frac{z^2}{2\sigma_z^2 + q}\right)}{\left[(2\sigma_x^2 + q)(2\sigma_z^2 + q)\right]^{1/2}} dq. \quad (2.16)$$

The description of particle dynamics in terms of a new

independent variable azimuth  $\phi$  instead of  $s$  means a simple multiplication of the Hamiltonian (2.12) by  $R$ .

We perform a canonical transformation with a generating function

$$F_2(u, \tilde{p}_u, \tau, \eta; \phi) = x \tilde{p}_x + z \tilde{p}_z + \tau (\eta + \beta_s^{-2}) + \eta R \phi, \quad (2.17)$$

defining new canonical variables

$$\tilde{u} = u; \quad \tilde{p}_u = \tilde{p}_u; \quad \eta = h - \beta_s^{-2}; \quad \sigma = R \phi + \tau = s + \tau \quad (2.18)$$

and the new Hamiltonian

$$\bar{H}_0 = \frac{R\eta}{2\gamma_s^2} + \frac{e \tilde{E}_0(s)}{\beta_s E_s} \frac{cR}{\omega} \cos \left( \frac{\omega \sigma}{\beta_s c} - k \phi + \phi_0 \right), \quad (2.19a)$$

$$\bar{H}_1 = -R (x K_x + z K_z) \eta, \quad (2.19b)$$

$$\bar{H}_2 = \frac{R}{2} (\tilde{p}_x^2 + \tilde{p}_z^2) + \frac{G_x \tilde{x}^2 + G_z \tilde{z}^2}{2R}, \quad (2.19c)$$

$$\bar{H}_3 = R (K_x \tilde{x} + K_z \tilde{z}) \left[ \frac{\tilde{p}_x^2 + \tilde{p}_z^2}{2} + \frac{g_0}{2R^2} (x^2 - z^2) \right] + \frac{\lambda_0}{6R^2} (x^3 - 3xz^2) + \dots, \quad (2.19d)$$

$$\bar{H}_{bb} = -\frac{Nr}{\gamma_s} \frac{1 + \beta_s^2}{\beta_s^2} \delta_b(s) \int_0^{\infty} \frac{\exp \left( -\frac{\tilde{x}^2}{2\sigma_x^2 + q} - \frac{\tilde{z}^2}{2\sigma_z^2 + q} \right)}{\left[ (2\sigma_x^2 + q)(2\sigma_z^2 + q) \right]^{1/2}} dq, \quad (2.19e)$$

where we have represented the quantity  $\Gamma$  as a power series in  $\eta$ , and assumed that  $K_x = K_z = 0$  at the interaction point.

The next step is to define the dispersions  $\psi_x$  and  $\psi_z$  by the canonical transformation

$$G_2(\tilde{u}, \hat{p}_u, \sigma, \hat{\eta}; \phi) = \sum_{u=(x,z)} \left[ \hat{p}_u (\tilde{u} - \hat{\eta} \psi_u) + \frac{\hat{u} \hat{\eta} \psi_u}{R} - \frac{\psi_u \psi_u}{2R} \hat{\eta}^2 \right] + \sigma \hat{\eta}, \quad (2.20)$$

which cancels  $\bar{H}_1$ . Furthermore we have

$$\tilde{u} = \hat{u} + \hat{\eta} \psi_u; \quad \tilde{p}_u = \hat{p}_u + \frac{\hat{u}}{R} \hat{\eta} \psi_u; \quad u = (x, z), \quad (2.20a)$$

$$\sigma = \sum_{u=(x,z)} \left[ \hat{p}_u \psi_u - \frac{\hat{u} \psi_u}{R} \right] + \hat{\sigma}; \quad \hat{\eta} = \hat{\eta}, \quad (2.20b)$$

where the dot means differentiation with respect to  $\dot{\phi}$ .

The new Hamiltonian

$$\hat{H}_0 = -\frac{R}{2} \left( K_x \psi_x + K_z \psi_z - \frac{1}{\gamma_s^2} \right) \eta^2 + \frac{e \hat{E}_0(s)}{\beta_s E_s} \frac{cR}{\omega} \cos \left( \frac{\omega \sigma}{\beta_s c} - k \dot{\phi} + \phi_0 \right), \quad (2.21a)$$

$$\hat{H}_2 = \frac{R}{2} (\hat{p}_x^2 + \hat{p}_z^2) + \frac{G_x^2 + G_z^2}{2R}, \quad (2.21b)$$

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 has been obtained by the utilization of the definition of dispersions  $\psi_u$ , satisfying equations

$$\frac{d^2 \psi_u}{d\dot{\phi}^2} + G_u \psi_u = R^2 K_u \quad ; \quad u=(x, z). \quad (2.22)$$

We replace now the coefficient of  $\eta^2$  in (2.21a) by its average over one revolution and note that  $\langle K_x \psi_x + K_z \psi_z \rangle = \alpha_M$ , where  $\alpha_M$  is the momentum compaction factor. Thus we have

$$\hat{H}_0 = -\frac{R K}{2} \eta^2 + \frac{\Delta E_0}{\beta_s E_s} \frac{c}{2\pi\omega} \cos \left( \frac{\omega \sigma}{\beta_s c} + \phi_0 \right), \quad (2.23)$$

where

$$K = \alpha_M - \frac{1}{\gamma_s^2} \quad ; \quad \phi_0 = \phi_0 - k \dot{\phi}_0, \quad (2.23a)$$

$$\dot{\phi}_0 = \arctg \frac{\langle eR \hat{E}_0(\dot{\phi}) \sin k \dot{\phi} \rangle}{\langle eR \hat{E}_0(\dot{\phi}) \cos k \dot{\phi} \rangle}. \quad (2.23b)$$

The quantity  $\Delta E_0 = \langle eR \hat{E}_0(\dot{\phi}) \rangle$  is the total energy gain for one revolution. The particle's phase  $\omega\sigma/\beta_s c$  is usually quite close to the synchronous phase  $\omega\sigma_s/\beta_s c$ , thus one is allowed to expand the cosine term of (2.23) in a power series in  $\Delta\sigma = \sigma - \sigma_s$ . Supposing that there is no energy uptake in the cavities (the energy gain provided by the RF compensates for the radiation loss) we drop the term proportional to  $\Delta\sigma$  and get

$$\hat{H}_0 = -\frac{R K}{2} \eta^2 - \left[ \frac{\Delta E_0}{\beta_s E_s} \frac{\omega}{2\pi c \beta_s^2} \cos \phi_s \right] \frac{(\Delta\sigma)^2}{2}, \quad (2.24)$$

where  $\phi_s = \frac{\omega\sigma_s}{\beta_s c} + \phi_0$ .



It should be mentioned here that  $\dot{\psi}_u = \dot{\psi}_u = 0$  for RF-cavities, in order to avoid synchro-betatron resonances in them. This fact has already been used in the derivation of eq. (2.23) as  $\sigma = \hat{\sigma}$  (equivalently  $\Delta\sigma = \hat{\Delta\sigma}$ ) for  $\dot{\psi}_u = \dot{\psi}_u = 0$ .

The final step of our synchro-betatron formalism is to cast the Hamiltonian  $\hat{H}$  to action-angle variables. Let us introduce the last canonical transformation, given by the generating function

$$S_1(\hat{u}, \alpha_u, \hat{\Delta\sigma}, \alpha_s; \hat{\varphi}) = \sum_{u=(x,z)} \frac{\hat{u}^2 \left[ \frac{\beta_u}{2R} - \text{tg}(\alpha_u + \chi_u - \nu_u \hat{\varphi}) \right]}{2\beta_u} - \lambda \frac{(\hat{\Delta\sigma})^2}{2} \text{tg}\alpha_s, \quad (2.25)$$

with new canonical variables  $\alpha_u$ ,  $J_u$  determined from the expressions

$$\hat{u} = \left( 2\beta_u J_u \right)^{1/2} \cos(\alpha_u + \chi_u - \nu_u \hat{\varphi}), \quad (2.26a)$$

$$\hat{p}_u = \left( \frac{2J_u}{\beta_u} \right)^{1/2} \left[ \frac{\beta_u}{2R} \cos(\alpha_u + \chi_u - \nu_u \hat{\varphi}) - \sin(\alpha_u + \chi_u - \nu_u \hat{\varphi}) \right], \quad (2.26c)$$

where the well-known  $\beta_u$ -functions and the phase advances  $\chi_u$  satisfy the equations

$$\frac{\beta_u \dot{\beta}_u}{2} - \frac{\dot{\beta}_u^2}{4} + G_u \beta_u^2 = R^2 \quad ; \quad \dot{\chi}_u = \frac{R}{\beta_u}. \quad (2.27)$$

For the longitudinal degree of freedom one is ready to obtain

$$\hat{\Delta\sigma} = \left( \frac{2J_s}{\lambda} \right)^{1/2} \cos\alpha_s \quad ; \quad \hat{\eta} = - \left( 2\lambda J_s \right)^{1/2} \sin\alpha_s, \quad (2.28)$$

$$\lambda^2 = \frac{\Delta E_o}{E_s} \frac{k}{2\pi\beta_s^2 R^2 \mathcal{K}} \cos\phi_s. \quad (2.29)$$

The unperturbed by the beam-beam interaction Hamiltonian is transformed now as follows:

$$\hat{H}_{oo} = \hat{H}_o + \hat{H}_2 = \nu_x J_x + \nu_z J_z - \nu_s J_s, \quad (2.30)$$

where

$$\nu_s^2 = \frac{\Delta E_o}{E_s} \frac{k \mathcal{K}}{2\pi\beta_s^2} \cos\phi_s. \quad (2.31)$$

In what follows we examine the case of  $\psi_z = \psi_z^* = 0$  at the interaction point and study the motion in horizontal and longitudinal directions only.

### III. LINEAR SYNCHRO-BETATRON RESONANCES

In order to proceed further with our study of synchro-betatron resonances due to beam-beam interaction we must first represent the Hamiltonian (2.19e) as a series of homogeneous polynomials in the variables  $\tilde{x}$  and  $\tilde{z}$ . Doing so up to the fourth order in  $\tilde{x}$  and  $\tilde{z}$  we have

$$\begin{aligned} H_{bb} = & \frac{1+\beta_s^2}{\beta_s^2} \sum_{k=1}^M \delta_p(\tilde{q}-\tilde{q}_k) \left\{ 2\pi \xi_x^{(k)} \frac{x^2}{\beta_y} + 2\pi \xi_z^{(k)} \frac{z^2}{\beta_z} - \frac{\pi^2 \gamma_s}{Nr_e} \left[ \xi_x^{(k)2} \frac{2\sigma_x^2 - \sigma_x \sigma_z - \sigma_z^2}{3\sigma_x(\sigma_x - \sigma_z)} x \right. \right. \\ & \left. \left. + \xi_z^{(k)2} \frac{2\sigma_z^2 - \sigma_x \sigma_z - \sigma_x^2}{3\sigma_z(\sigma_x - \sigma_z)} \frac{z}{\beta_z} \right] + \dots \right\}, \quad (3.1) \end{aligned}$$

where the index  $k$  denotes  $k$ -th interaction point and  $\xi_{x,z}^{(k)}$  are the horizontal and vertical beam-beam parameters, respectively:

$$\xi_{x,z}^{(k)} = \frac{1}{2\pi} \frac{Nr_e}{\gamma_s} \frac{1}{\sigma_x + \sigma_z} \left( \frac{\beta^{(k)}}{\sigma} \right)_{x,z}. \quad (3.2)$$

Insert now eqs.(2.20a), (2.26a) and (2.28) into the equation (3.1). After some simple algebra one can pick out the resonance Hamiltonian, which is the sum of  $\hat{H}_{00}$  and the quadratic in  $\tilde{x}$  and  $\tilde{z}$  terms of  $\hat{H}_{bb}$  in the form:

$$\begin{aligned} \hat{H} = & (v_x + \Delta v_x) J_x + (v_z + \Delta v_z) J_z - (v_s + \Delta v_s) J_s + \frac{1+\beta_s^2}{\beta_s^2} \sum_{k=1}^M \xi_x^{(k)} J_x \cos(2\alpha_x - n\tilde{q} + \phi_{kn}) + \\ & + 2V\lambda \frac{1+\beta_s^2}{\beta_s^2} \sum_{k=1}^M \xi_x^{(k)} \frac{\Psi_k}{(\beta_x^{(k)})^{1/2}} (J_x J_s)^{1/2} \cos(\alpha_x + \alpha_s - m\tilde{q} + \phi_{km}), \quad (3.3) \end{aligned}$$

where

$$\Delta v_{x,z} = \frac{1+\beta_s^2}{\beta_s^2} \sum_{k=1}^M \xi_{x,z}^{(k)}; \quad \Delta v_s = -\lambda \frac{1+\beta_s^2}{\beta_s^2} \sum_{k=1}^M \xi_x^{(k)} \frac{\Psi_k}{\beta_x^{(k)}}, \quad (3.4)$$

$$\phi_{km} = \chi_x^{(k)} + (m - v_x) \tilde{q}_k + \frac{\pi}{2}; \quad \phi_{kn} = 2\chi_x^{(k)} + (n - 2v_x) \tilde{q}_k. \quad (3.5)$$

The resonance conditions are written as

$$v_x + \Delta v_x - v_s - \Delta v_s = m + \epsilon_1 \quad ; \quad 2(v_x + \Delta v_x) = n + \epsilon_2 \quad (3.6)$$

The quantity  $\epsilon_1$  is the resonance detuning for the linear synchro-betatron resonance, while  $\epsilon_2$  refers to the standard beam-beam parametric resonance.

For the sake of simplicity we consider one interaction point and omit terms, responsible for the motion in z-direction.

Our aim now is to remove the explicit  $\phi$ -dependence in the Hamiltonian (3.3) by the canonical transformation

$$\tilde{J}_x = J_x \quad ; \quad \tilde{J}_s = J_s \quad ; \quad \tilde{\alpha}_x = \alpha_x + (v_x + \Delta v_x - \lambda_x) \phi - \zeta_x \quad ; \quad \tilde{\alpha}_s = \alpha_s - (v_s + \Delta v_s + \lambda_s) \phi - \zeta_s \quad (3.7)$$

with a generating function

$$\tilde{F}_2(\alpha_x, \alpha_s, \tilde{J}_x, \tilde{J}_s; \phi) = \left[ \alpha_x + (\lambda_x - v_x - \Delta v_x) \phi + \zeta_x \right] \tilde{J}_x + \left[ \alpha_s + (\lambda_s + v_s + \Delta v_s) \phi + \zeta_s \right] \tilde{J}_s \quad (3.8)$$

where

$$\lambda_x = \frac{\epsilon_2}{2} \quad ; \quad \lambda_s = \epsilon_1 - \frac{\epsilon_2}{2} \quad (3.9)$$

$$\zeta_x = \frac{\phi_{1n}}{2} \quad ; \quad \zeta_s = \phi_{1m} - \frac{\phi_{1n}}{2} \quad (3.10)$$

The transformed Hamiltonian reads as

$$\hat{K} = \lambda_x J_x + \lambda_s J_s + 2D_s \sqrt{J_x J_s} \cos(\tilde{\alpha}_x + \tilde{\alpha}_s) + D_p J_x \cos 2\tilde{\alpha}_x \quad (3.11)$$

with the notations

$$D_s = \xi_x \frac{1 + \beta_s^2}{\beta_s^2} \frac{V_{\lambda} \psi_1}{V_{\beta_{x1}}} \quad ; \quad D_p = \xi_x \frac{1 + \beta_s^2}{\beta_s^2} \quad (3.12)$$

Next we utilize the more suitable for investigation of the case of two resonances combined action rectangular canonical variables

$$X = \sqrt{2J_x} \cos \tilde{\alpha}_x \quad ; \quad P_x = -\sqrt{2J_x} \sin \tilde{\alpha}_x \quad (3.13a)$$

$$S = \sqrt{2J_s} \cos \tilde{\alpha}_s \quad ; \quad P_s = -\sqrt{2J_s} \sin \tilde{\alpha}_s \quad (3.13b)$$

defined by a new canonical transformation with generating function

$$\tilde{S}_1(X, \tilde{\alpha}_x, S, \tilde{\alpha}_s) = \frac{X^2}{2} \operatorname{tg} \tilde{\alpha}_x + \frac{S^2}{2} \operatorname{tg} \tilde{\alpha}_s \quad (3.14)$$

The Hamiltonian (3.11) is now cast to the form

$$\hat{K} = \frac{\lambda_x}{2} (P_x^2 + X^2) + \frac{\lambda_s}{2} (P_s^2 + S^2) + D_s (XS - P_x P_s) + \frac{D_p}{2} (X^2 - P_x^2) \quad (3.15)$$

The characteristic equation of Hamilton equations of motion

$$\dot{X} = (\lambda_x - D_p) P_x - D_s P_s \quad (3.16a)$$

$$\dot{P}_x = -(\lambda_x + D_p)X - D_s S, \quad (3.16b)$$

$$\dot{S} = -D_s P_x + \lambda_s P_s, \quad (3.16c)$$

$$\dot{P}_s = -D_s X - \lambda_s S \quad (3.16d)$$

governed by (3.15) is easily calculated to give

$$x^4 + (\lambda_x^2 + \lambda_s^2 - D_p^2 - 2D_s^2)x^2 + D_s^4 - 2\lambda_x \lambda_s D_s^2 + \lambda_s^2 (\lambda_x^2 - D_p^2) = 0. \quad (3.17)$$

The motion, described by the system (3.16) is stable, provided all the roots of eq. (3.17) are pure imaginary. This means that the following inequalities

$$D_s^4 - 2\lambda_x \lambda_s D_s^2 + \lambda_s^2 (\lambda_x^2 - D_p^2) > 0, \quad (3.18a)$$

$$\lambda_x^2 + \lambda_s^2 - D_p^2 - 2D_s^2 > 0 \quad (3.18b)$$

must hold. Note that the quantities  $\lambda_x$  and  $\lambda_s$  are linear combinations [see eqs. (3.9)] of the two resonance detunings  $\epsilon_1$  and  $\epsilon_2$ . Thus conditions (3.18) determine the stability region in the  $\epsilon_1 - \epsilon_2$  plane.

Further we assume that the horizontal betatron frequency is sufficiently far from the parametric resonance value. This means that the standard beam-beam parametric resonance term in eq. (3.3) may be neglected.

Choose now a generating function of the form<sup>/4/</sup>

$$E_2(\alpha_x, \alpha_s, \hat{J}_x, \hat{J}_s; \phi) = (\alpha_x + \alpha_s - m\phi + \phi_{1r}) \hat{J}_s - \alpha_s \hat{J}_x, \quad (3.19)$$

defining new canonical variables

$$J_x = \hat{J}_x; \quad J_s = \hat{J}_x - \hat{J}_s; \quad \hat{\alpha}_x = \alpha_x + \alpha_s - m\phi + \phi_{1r}, \quad \hat{\alpha}_s = -\alpha_s \quad (3.20)$$

and the new Hamiltonian

$$\hat{H} = \epsilon_1 \hat{J}_x + (\nu_s + \Delta\nu_s) \hat{J}_s + 2D_s \sqrt{\hat{J}_x (\hat{J}_x - \hat{J}_s)} \cos \hat{\alpha}_x. \quad (3.21)$$

Note that the Hamiltonian (3.21) does not depend on the new angle  $\hat{\alpha}_s$ . It means that  $\hat{J}_s = J_x - J_s = C$  must be constant. From the Hamilton equations for  $\hat{J}_x$  and  $\hat{\alpha}_x$  one easily obtains<sup>/5/</sup>

$$\frac{d^2 J_x}{d\phi^2} + \epsilon_1^2 J_x = \hat{D}, \quad (3.22)$$

where

$$\epsilon_1^2 = \epsilon_1^2 - 4D_s^2; \quad \hat{D} = \epsilon_1 C - 2D_s^2 C, \quad (3.22a)$$

$$\hat{K} = \hat{H} - (v_e + \Delta v_e) C. \quad (3.22b)$$

The solution of eq. (3.22) is stable, provided the following inequality

$$\epsilon^2 > 0 \quad (3.23)$$

holds. Note that eq. (3.22) follows from an effective Hamiltonian<sup>/5/</sup>

$$\hat{K}_e = \frac{p_\alpha^2}{2} + v_e(J_x), \quad (3.24)$$

where

$$v_e(J_x) = \frac{\epsilon^2 J_x^2}{2} - \hat{D} J_x. \quad (3.25)$$

Using the relation

$$p_\alpha = J_x = - \frac{\partial \hat{H}}{\partial \alpha_x} = 2 \hat{D}_s \sqrt{J_x(J_x - C)} \sin \hat{\alpha}_x \quad (3.26)$$

it is easily checked that

$$\hat{K}_e = - \frac{\hat{K}^2}{2} \leq 0. \quad (3.27)$$

The motion takes place in the potential well, described by the effective potential (3.25), provided  $\hat{D} > 0$ . Moreover the line of constant energy  $\hat{K}_e$  [see eq. (3.27)] intersects the curve  $v_e(J_x)$  if  $\hat{D}^2 - \epsilon^2 \hat{K}^2 > 0$ . One can check that these two inequalities are automatically satisfied, provided (3.23) holds. The oscillation amplitude of  $J_x$  is given by<sup>/5/</sup>

$$\text{Amp}(J_x) = \frac{J_{x2} - J_{x1}}{2} = \frac{1}{2} \sqrt{\hat{D}^2 - \epsilon^2 \hat{K}^2}, \quad (3.28)$$

where  $J_{x1}$  and  $J_{x2}$  are the minimum and the maximum values of  $J_x$  respectively, so that

$$v_e(J_{x1,2}) = - \frac{\hat{K}^2}{2}. \quad (3.29)$$

The solution of eq. (3.22) can be obtained directly from (3.24) using (3.26). We have:

$$\dot{\alpha} = \int_{J_{x0}}^{J_x} \frac{dJ_x}{\sqrt{2 [\hat{K}_e - v_e(J_x)]}}$$

or

$$J_x = \text{Amp}(J_x) \sin \left[ E(\hat{\phi} - \hat{\phi}_0) + \Delta\phi_0 \right] + \frac{\hat{D}}{E}, \quad (3.30)$$

where

$$\Delta\phi_0 = \arcsin \frac{J_{x0} - \frac{\hat{D}}{E}}{\text{Amp}(J_x)}. \quad (3.31)$$

#### IV. THE EFFECT OF SYNCHRO-BETATRON COUPLING ON PROJECTED EMITTANCES

The transport matrix of a lattice structure, defined by the equations of motion, following from a Hamiltonian  $\hat{H} = \hat{H}_2 + \hat{H}_0$  [see eqs. (2.21b) and (2.24)] is given by

$$(LS)_{ij} = \begin{pmatrix} \cos\mu_x & \beta_{x1} \sin\mu_x & 0 & 0 \\ -\frac{\sin\mu_x}{\beta_{x1}} & \cos\mu_x & 0 & 0 \\ 0 & 0 & \cos\mu_s & -\rho \sin\mu_s \\ 0 & 0 & \frac{\sin\mu_s}{\rho} & \cos\mu_s \end{pmatrix}, \quad (4.1)$$

where

$$\mu_x = R \int_{\hat{\phi}_0}^{\hat{\phi}} \frac{d\hat{\phi}'}{\beta(\hat{\phi}')} ; \quad \mu_s = \nu_s (\hat{\phi} - \hat{\phi}_0) ; \quad \rho = \frac{R\chi}{\nu_s} \quad (4.1a)$$

and  $\beta_{x1} = 0$ . The transport matrix for the interaction point ( $\beta_s = 1$ ) is [see eq. (3.1)]

$$(IP)_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{8\pi\zeta_x}{\beta_{x1}} & 1 & 0 & -\frac{8\pi\zeta_x \psi_1}{\beta_{x1}} \\ \frac{8\pi\zeta_x \psi_1}{\beta_{x1}} & 0 & 1 & \frac{8\pi\zeta_x \psi_1^2}{\beta_{x1}} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4.2)$$

The total transport matrix  $M_{ij}$  of a system consisting of a lattice structure and an interaction point will be the product of (4.2) and (4.1). Thus we have

$$M_{ij} = \sum_{k=1}^4 (IP)_{ik} (LS)_{kj} . \quad (4.3)$$

Following K.L.Brown and R.V.Servranckx<sup>/6/</sup> we write the matrix M in the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} , \quad (4.4)$$

where A, B, C and D are 2x2 matrices. Let  $\Sigma$  be a positive definite matrix called the beam envelope matrix<sup>/6/</sup>

$$\Sigma = \begin{pmatrix} \Sigma_x & T \\ \tilde{T} & \Sigma_s \end{pmatrix} , \quad (4.5)$$

and  $\tilde{T}$  is the transpose of T. The relation between initial  $\Sigma_0$  and final  $\Sigma_1$  is given by the equation

$$\Sigma_1 = M \Sigma_0 \tilde{M} . \quad (4.6)$$

If we assume that the initial beam is uncoupled  $T_0=0$  (i.e. there is no synchro-betastron coupling) a simple matrix manipulation yields the result<sup>/6/</sup>

$$\Sigma_{x1} = A \Sigma_{x0} \tilde{A} + B \Sigma_{s0} \tilde{B} , \quad (4.7a)$$

$$\Sigma_{s1} = C \Sigma_{x0} \tilde{C} + D \Sigma_{s0} \tilde{D} . \quad (4.7b)$$

Using the definition of beam emittance projections on  $\hat{x} - \hat{p}_x$  and  $\hat{\Delta\sigma} - \hat{\eta}$  subspaces

$$\epsilon_x^2 = \det \left( \Sigma_x \right) ; \quad \epsilon_s^2 = \det \left( \Sigma_s \right) \quad (4.8)$$

and the symplecticity of the transport matrix M one obtains

$$\epsilon_{x1}^2 - \epsilon_{s1}^2 = \epsilon_{x0}^2 - \epsilon_{s0}^2 , \quad (4.9a)$$

$$\epsilon_{x1}^2 = \epsilon_{x0}^2 + \frac{(8\pi\xi_x \Psi_1)^2}{\beta_{x1}} \epsilon_{x0}^2 \left( \frac{\sigma_{s0}^2}{\rho^2} \sin^2 \mu_s + \sigma_{E0}^2 \cos^2 \mu_s \right) , \quad (4.9b)$$

where  $\sigma_s$  is the bunch length and  $\sigma_E$  is the relative beam energy spread. In deriving eqs. (4.9) we have assumed that the initial beam

ellipses are upright because of  $\beta_{x1} = 0$ .

The first formula (4.9) expresses the fact that  $\epsilon_x^2 - \epsilon_s^2$  is an invariant. The most remarkable feature of formula (4.9b) is that the  $\hat{x} - \hat{p}_x$  projection of the beam emittance does not depend on the betatron phase advance. It shows that  $\epsilon_x^2$  experiences slow oscillations around the mean value

$$\epsilon_{xm}^2 = \epsilon_{xo}^2 + \frac{32 \left( \pi \xi_x \psi_1 \right)^2}{\beta_{x1}} \epsilon_{xo} \left( \frac{\sigma_{s0}^2}{\rho^2} + \sigma_{E0}^2 \right). \quad (4.10)$$

## V. CONCLUDING REMARKS

We have analysed linear synchro-betatron resonances due to a non-zero dispersion at the interaction point and obtained the stability conditions for the combined action of a synchro-betatron resonance and parametric beam-beam resonance. The beating amplitude for an isolated linear synchro-betatron resonance is calculated in invariant form.

Note that the invariant  $C = J_x - J_s$ , derived in Sec. III may be written in the form  $C^* = \epsilon_x - \epsilon_s$  for the action variable is proportional to the beam emittance. One can see the consent in principle between  $C^*$  and the exact invariant (4.9a). From eq. (4.9b) it is evident that if there are no betatron oscillations initially ( $\epsilon_{xo} = 0$ ) the projected emittance  $\epsilon_x$  will be zero all the time. This means that the projection of the 4-dimensional beam ellipsoid (see e.g. Ref.7) on  $\hat{x} - \hat{p}_x$  plane will be a rotating segment and there will be no longitudinal emittance pumping over into transversal emittance.

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Линейные синхро-бетатронные резонансы  
возбуждаемые beam-beam взаимодействием

Проанализировано совместное действие линейного синхро-бетатронного резонанса, вызванного ненулевой дисперсией в точке встречи коллайдера и обычного beam-beam параметрического резонанса. Линейный синхро-бетатронный резонанс рассмотрен при помощи модели изолированного резонанса. Вычислена проекция эмиттанса на горизонтальное фазовое подпространство.

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Tsenov S.I.

E9-91-51

Linear Synchro-Betatron Resonances,  
Driven by Beam-Beam Interaction

The combined action of a linear synchro-betatron resonance, due to a nonzero dispersion at the interaction point of a colliding beam storage device and a standard beam-beam parametric resonance has been analysed. The synchro-betatron resonance has been studied involving the single resonance model. The projected emittance on the horizontal phase subspace has been calculated.

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