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BELTRAMI PARAMETRIZATION AND GAUGING OF VIRASORO AND W-INFINITY ALGEBRAS

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ABSTRACT

The gauging of the Virasoro and w-infinity algebras are discussed from the point of view of BRST symmetry. Both algebras are realised as "Russian formulas" for the curvatures built from the generators of the Lie algebras and the corresponding gauge fields. The generalized curvatures are used to determine the gauge invariant Lagrangians as well as the anomaly structures of the conformal two dimensional theory and the w-gravity.

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1. INTRODUCTION

The two dimensional conformal field theory has the Virasoro algebra as the underlying symmetry. The classical string action is a typical example of a theory invariant under the Virasoro algebra. Its invariance and its conformal proprerties are most clearly exhibited in terms of Beltrami differential /1-4/. In this parametrization the BRST algebra factorizes in two independent and separate structures, which implies that the ghost Lagrangian is a sum of a holomorphic and an antiholomorphic terms and the action for the string can be expressed only in terms of Weyl invariant quantities.

However, any attempt to treat the spin two gauge field , i.e. the Beltrami differential, on the same footing as the higher spin fields, which occur in the w-gravity, does not have any future,since there are no higher -spin zweibein fields and any higher-spin Beltami differential, with a similar geometric interpretation as the spin-2 zweibein field.

On the other hand, in two dimension there is an alternative formulation /5,6/ to describe the coupling of gravity to matter which includes the auxiliary fields J and $\bar{J}$. This alternative formulation has two advantages: on the one hand , it can be quite naturally connected with the gauging of the Virasoro algebra and , on the other hand ,it does allow a natural extension for the higher-spin gauge fields, Which can be treated on the same footing as the spin 2 gauge field.

In this paper we shall adopt a very nice point of view, advocated in some recent papers by Baulieu, Bellon and Grimm

/7-11/. We shall consider the Lie algebras as the starting point in our investigation, rather than considering them as special invariance properties of a given Lagrangian. For a given Lie algebra we associate a gauge field and a ghost to each generator of it and we build the corresponding BRST symmetry from a geometrical constrain on the curvature called the "Russian formula". This can be done very efficient if we use a Poisson bracket algebra realisation of the Lie algebra, which is possible not only for the Virasoro algebra but also for the w-infinity algebra. For using the Poisson bracket, in addition to the space dependence is is convenient to introduce one (t) or two (t,u) additional variables. It seems that a modification of the Moyal bracket /12/ could be used instead of the Poisson bracket to obtain a realization of W-infinity algebra /13/.

The generalized one-form connection, constructed in the standard way /11/, will contain the Beltrami differential and the corresponding ghost for the Virasoro algebra and the high spin gauge fields and their ghosts for w-infinity algebra. This connection is the fundamental object of our theory and it can be used to build up the gauge invariant action, the possible anomalies and the Wess-Zumino action. For accomplishing these tasks it is necessary to introduce the matter fields. Furthemore, the gauging of left-moving and right-moving of w-infinity algebra, cannot be achieved by simply adding gauge field times current terms. The action in this case could be most conveniently written by introducing ,once again, the auxiliary fields J and J, which must be eliminated at the end of the calculation.

2. THE BELTRAMI DIFFERENTIAL AND THE VIRASORO ALGEBRA

2.1. THE GAUGE FIELDS

The Virasoro algebra, without the central charge, contains an infinite number of generators L_{-1} ,L_0 ,L_1,... which satisfy the following commutation relations:

$$\left[L_n , L_m\right] = (m-n)\, L_{m+n} \quad ; \qquad -1 \leq n,m < \infty \qquad (1)$$

This Lie algebra can be realised very simple as a Poisson-bracket algebra of functions an a <u>one-dimensional</u> phase space, with the Poisson bracket defined as:

$$\left\{f , g\right\}_t = f \overleftrightarrow{\partial_t}\, g - (\partial_t f)\, g \ , \qquad (2)$$

for two functions f(t) and g(t). Taking a basic set of functions

$$\ell_m = t^{m+1} \ , \qquad (3)$$

we obtain the Virasoro algebra

$$\left\{\ell_n, \ell_m\right\}_t = (m-n)\, \ell_{m+n} \ . \qquad (4)$$

The BRST symmetry is realised , in the ghost sector, by associating a ghost c_n ,$n \geqslant -1$ to each generator L^n . In the general matrix representation (1) the ghost fields are gathered together in a Lie-valued ghost:

$$c = \sum_{n=-1}^{\infty} L^n c_n \ , \qquad (5)$$

which is not very covenient for the following discution since (5) contains the unknown generators L^n . However if we use the basis (3) then the ghost c(t) has a simpler form:

$$c(t) = \sum_{n=-1}^{\infty} c_n\, t^{n+1} \ . \qquad (6)$$

For a general Lie algebra, the BRST transformation of the ghosts c_a associated to it are given by:

$$s\, c^a = -\frac{1}{2} f_{bc}{}^a\, c^b\, c^c \quad ,$$

where the coefficients $f_{ab}{}^c$ are defined by

$$[T_a , T_b] = f_{ab}{}^c\, T_c$$

with T_a the generators of the Lie algebra. This BRST transformation can be rewritten in a simpler form if one introduces the ghost $c = T_a\, c^a$

$$sc = -\frac{1}{2} [c , c] \qquad (7)$$

For the Virasoro algebra (4) the BRST symmetry takes the following compact form

$$sc + \frac{1}{2} \{c , c\}_t = 0 \ . \qquad (8)$$

This BRST equation can be extended to include the gauge fields associated to the generators L^n , For the Virasoro algebra we have found convenient to associate a one-form A^n . Furthermore, following Stora we add the ghost number to the form degree and assume all commutators to be graded by this total degree. Therefore , we can combine the ordinary one-forms with ghost number zero and the zero-form with ghost number one i.e. $\widetilde{A}^n = A^n + c^n$. For the Yang-Mills fields associated with a given algebra with the ghosts satisfying Eq.(7) we can write "the Russian formula" :

$$\widetilde{F} = \widetilde{d}\widetilde{A} + \frac{1}{2}[\widetilde{A} , \widetilde{A}] = F = dA + \frac{1}{2} [A , A] \quad ,$$

where $\widetilde{d} = d + s$ and $\widetilde{A} = A + c$ with $A = A^a\, T_a$ is the Lie- valued connection form. For the Virasoro algebra we claim that a similar

formula takes place i.e.

$$\widetilde{d}\widetilde{A} + \frac{1}{2}\{\widetilde{A} , \widetilde{A}\}_t = \hat{d}\widetilde{A} + \widetilde{A}\,\partial_t \widetilde{A} = 0 \ . \qquad (9)$$

with

$$\widetilde{A} = \sum_{n=1}^{\infty} (A^n + c^n)\, t^{n+1} \quad , \qquad (10)$$

and $\widetilde{d} = d + s$ with d the usual differential.

Since the Virasoro algebra is deeply related to the two dimensional conformal symmetry, it is natural to try to connect the one-form A^n with the complex structure of a Riemanniann surface. Conformal classes of metrics on a Riemann surface can be parametrized by Beltrami coefficients $\mu(z , \bar{z})$ which are smooth complex-valued funtion of the complex coordinates $(z , \bar{z})$ of the surface, with specific transformation properties. The complex coordinate $(Z , \bar{Z})$ corresponding to the complex stucture parametrized by the Beltrami differential are given by the relations

$$d\, Z = \lambda [dz + \mu d\bar{z}] \text{ and c.c.} \qquad (11)$$

Here λ and μ are smooth complex-valued functions of $(z , \bar{z})$ which satisfy:

$$(\bar{\partial} - \mu \partial) Z = 0 \text{ and c.c.} \qquad (12)$$

$$(\bar{\partial} - \mu \partial) \lambda = (\partial \mu) \lambda \quad \text{and c.c.} \qquad (13)$$

The infinitesimal diffeormorphism generated by the vector field $\xi\cdot\partial = \xi(z , \bar{z}).\partial + \bar{\xi}(z , \bar{z})\bar{\partial}$ can be obtained with the Lie derivative $L_{\xi\cdot\partial} = i_{\xi\cdot\partial}\, d + d\, i_{\xi\cdot\partial}$ acting on Z

$$\delta Z = L_{\xi\cdot\partial} Z = i_{\xi\cdot\partial}\, dZ = [\lambda (d z + \mu d \bar{z})]\ (\xi\cdot\partial) =$$

$$= \lambda(\xi + \mu \bar{\xi}) = \lambda\, c \quad , \qquad (14)$$

with $c = \xi + \mu\bar{\xi}$. By evaluation the variation of dZ in two ways $\delta (d Z) = d (\delta Z)$ we can get the induced variation of μ

$$\delta\mu = [\bar{\partial} - \mu\partial + (\partial\mu)]\, c \,. \tag{15}$$

If we identify c in (14) and (15) with the ghost vector field of two dimensionsal diffeomorphism, we can identify Eqs. (14) and (15) with the definition of the BRST differential s

$$sZ = \lambda c \quad ; \quad s\mu = [\bar{\partial} - \mu\partial + (\partial\mu)]\, c \,. \tag{16}$$

The nilpotency of s requires

$$s\, c = c\,\partial c \,. \tag{17}$$

Now the equation (9) for ghost number zero and one and for t = 0 gives:

$$d A^{-1} + A^{-1} A^{0} = 0 \,,$$

$$s A^{-1} + d c^{-1} + A^{-1} c^{0} + c^{-1} A^{0} = 0 \tag{18}$$

Comparing Eqs. (18) and (17) we can easily see that a possible solution of these equations is :

$$A^{-1} = dz + \mu d\bar{z} \quad ; \quad c^{-1} = c \quad ;$$

$$A^{0} = (\partial\mu)\, d\bar{z} \quad ; \quad c^{0} = \partial c \,. \tag{19}$$

The rest of the one-forms A^n and the ghosts c^n can be can be found out by imposing the validity of Eq. (9) for all values of t /11/ . On the other hand , we can solve Eq.(9) by making a gauge choice

$$\tilde{A} = dz + d\bar{z}\, \tilde{A}_{\bar{z}} (z , \bar{z} , t) + \tilde{c}(z, \bar{z}, t) \tag{20}$$

With this choice Eq. (9) yields

$$\frac{\partial}{\partial z}\tilde{A}_{\bar{z}} = \frac{\partial}{\partial t}\tilde{A}_{\bar{z}} \quad ; \quad \frac{\partial}{\partial z}\tilde{c} = \frac{\partial}{\partial t}\tilde{c} \quad ,$$

equations which have the obvious solution

$$\tilde{A} = dz + d\bar{z}\, A_{\bar{z}} (z+t, \bar{z}) + c\,(z+t, \bar{z}) = e^{t\partial_z}[\, dz + d\bar{z}\mu + c] \tag{21}$$

2.2 VIRASORO INVARIANT LAGRANGIAN

From the field $\tilde{A}$ one could construct an invariant Lagrangian if one looks for a two-form $\tilde{\mathcal{L}}$, which is $\tilde{d}$-closed and it is defined up to $\tilde{d}$-exact terms. The ghost zero part of $\tilde{\mathcal{L}}$ is a possible BRST-invariant Lagrangian. The only possible candidate built only from A is $\tilde{A}\,\tilde{\bar{A}}$, which nevertheless is not $\tilde{d}$-closed since $\tilde{A}$ satisfies Eq.(9). Here $\tilde{\bar{A}}$ is the complex conjugate of $\tilde{A}$. Therefore, in order to build up an invariant Lagrangian we must couple $\tilde{A}$ to a new field , the matter fields.

The matter fields are zero-forms,which cannot contain ghosts. For our purpose the starting point is the equation (11) In two-dimension, there is a possibility to describe the coupling of gravity to matter field , which includes two auxiliary fields J and $\bar{J}$ /5/ . The matter field in this approach is described by a scalar field, which we will take to be a single real scalar φ . We will suppose that the realfield φ and $(J , \bar{J})$ are connected by the equation

$$d\varphi = J A^{-1} + \bar{J} \bar{A}^{-1} \tag{22}$$

i.e. the field φ, J and μ are related by :

$$J = \partial\varphi - \bar{\mu}\bar{J}$$
$$\bar{J} = \bar{\partial}\varphi - \mu J \qquad (22')$$

The auxiliary field J could be considered as the first term in a set of zero forms $J^{(n)}$ whth $n \geq -1$, which we assemble into:

$$\tilde{J} = \sum_{n=-1}^{\infty} t^{n+1} J^{(n)} \qquad (23)$$

and the equation (22) can be extended for the tilde fields as:

$$\tilde{d}\tilde{\varphi} = \tilde{J}\tilde{A} + \tilde{\bar{J}}\tilde{\bar{A}} \qquad (24)$$

Applying $\tilde{d}$ to this equation and using $\tilde{d}^2 = 0$ we get

$$(\tilde{d}\tilde{J})\tilde{A} + \tilde{J}(\tilde{d}\tilde{A}) + c.c. = 0 \quad . \qquad (25)$$

The action of the BRST symmetry on $\tilde{\varphi}$ and $\tilde{J}$ can be read off from Eqs.(24) and (25) . The equation (24) can be fulfilled whether one imposes the condition for vanishing of the curvature of $\tilde{J}$:

$$\tilde{d}\tilde{J} + \{\tilde{A}, \tilde{J}\}_t = 0 \ . \qquad (26)$$

With the gauge choice (21) eq. (26) yields

$$\tilde{J} = J(z+t, \bar{z}) \qquad (27a)$$

and

$$\left(\frac{\partial}{\partial\bar{z}} - A_{\bar{z}}\frac{\partial}{\partial t}\right)\tilde{J} = \tilde{J}.(\partial_z A_{\bar{z}}) \ . \qquad (27b)$$

For t=0 eq. (27b) coincides with eq. (13) i.e. we can identify J with λ and φ with Z + $\bar{Z}$.

With $\bar{J}$ and $\bar{A}$, it is quite easy to construct a BRST invariant action as the real two-form

$$\tilde{\mathcal{L}} = (\tilde{J}\tilde{A})(\tilde{\bar{J}}\tilde{\bar{A}}) \qquad (28)$$

One can indeed verify that

$$\tilde{d}\tilde{\mathcal{L}} = 0 \quad .$$

which proves that the ghost zero part of $\tilde{\mathcal{L}}$ is a BRST-invariant two-form. Now whether we take into consideration eq. (22) the classical Lagrangian,obtained from (28) for t=0 has the usual form:

$$\mathcal{L}_{cl} = \frac{1}{2}\frac{1}{1-\mu\bar{\mu}}(\partial\varphi - \bar{\mu}\bar{\partial}\varphi)(\bar{\partial}\varphi - \mu\partial\varphi) \qquad (29)$$

In fact, in the gauge we have considered, t occurs only through z+t and after integration, the action does not depend on it. Therefore, the Virasoro gauge theory reduces rather naturally to the two dimensional conformal field theory.

2.3. VIRASORO COVARIANT ANOMALIES. THE WESS-ZUMINO ACTION

In this formulation of the Virasoro gauge field theory the general forms of the consistent and covariant anomalies can be determined rather straightforwardly. Besides, the Wess-Zumino action has a simple form and can be calculated very easily. As it is well known, in the BRST formalism, an anomaly for the Virasoro algebra is a two-form with ghost number one. A covariant anomaly is an anomaly which has a covariant form and therefore it is well defined on the whole Riemann surface.

Thus, in order to find an anomaly, one must look for a general (i.e. including the ghosts) three form $\tilde{\Delta}_3$ satisfying $\tilde{d}\tilde{\Delta}_3=0$. A solution of this equation was proposed by Baulieu, Bellon and Grimm /11/ and it has the form

$$\tilde{\Delta}_3 = \tilde{A}\,\dot{\tilde{A}}\,\ddot{\tilde{A}} \tag{30}$$

where a dot means the derivative with respect to t. In the gauge (21) $\tilde{\Delta}_3$ has its ghost one part given by

$$\mathcal{A}(c\;,\mu)=-\partial c\,\partial^2\mu\;dz\wedge d\bar{z} \tag{31}$$

for t=0, which is the diffeomorphism anomaly obtained in a factorized form /2,3/.

The form of $\mathcal{A}$ is not well defined on the whole Riemann surface since it does not have an covariant form under a conformal charge of coordinate $z \dashrightarrow z'(z)$.

To obtain the covariant form of the anomaly, we might follow the algebraic approach proposed by Abud, Ader, Gieres and Noirot /14, 15/. However, we have found rather difficult the implementation of these ideas for the Virasoro algebra. So, at this point we will just follow the general prescription for the covariantisation on a generi Riemann surface. In fact, the anomaly (31) is equivalent to

$$\mathcal{A} = c\;\partial^3\mu\;\;dz\wedge dz \tag{31'}$$

and it involves the third order differential operator ∂^3. This expression is not well defined on a generic Riemann surface since the integrand does not transform with the Jacobian upon passage

from one coordinate chart to another. In fact the modified expression

$$\tilde{\mathcal{A}} = c[\,\partial^3 + (\;R\partial+\partial R\;)\;]\,\mu \tag{32}$$

with R, a projective connection, given by

$$R = \partial^2\,\ln\lambda - \frac{1}{2}\,(\partial\ln\lambda)^2\;, \tag{33}$$

transforms with the the Jacobian and represents the covariant anomaly. We believe that this form of the anomaly can be obtained by using the general algebraic methods for the covariant anomaly.

Since the Virasoro algebra is closed connected to the general coordinate transformations, which define a non-commutative group, the construction of the associated Wess- Zumino action represents a serious problem. However, for the factorized anomaly, the problem is simpler. This factorized anomaly could be obtained from $\tilde{\Delta}_3$ by using the standard procedure /17/. In fact we have to "kill" the anomaly $\mathcal{A}$ by enlarging the space of fields. We shall lift the whole construction from the Riemann surface M to M $\times$ [0,1] by considering a family of Beltrami differentials μ_u such that $\mu_0=0$ and $\mu_1=\mu$ and a family of the "Goldstone field" φ_u which takes its values in the group of diffeomorphisms and $\varphi_0=$ identity and $\varphi_1=\varphi$. The field $\tilde{A}$ and the differential are replaced in this case by

$$\check{A} = \tilde{A} + a\;du\;;\quad d_{tot} = \tilde{d} + d_u\;\;.$$

The function a is determined from Eq.(9) written in terms of the new fields and differentials. The Wess- Zumino action for the Virasoro algebra (30) is the ghost zero part of the thre-form $\check{\Delta}_3\,(\check{A}^{\varphi})$ with $\check{A}^{\varphi_u}$, the field obtained from $\check{A}$ by the action of the diffeomorphism φ. If we integrate out the auxiliary variable u, one finds the following form of The Wess-Zumino action:

$$\mathcal{L} = \frac{1}{2} dz \wedge d\bar{z}\, (\mu^{\varphi} \partial^2 \ln \lambda^{\varphi} - \mu \partial^2 \ln \lambda) \quad ,$$

which takes the form given by Polyakov /18/

$$\mathcal{L} = -\frac{1}{2} dz \wedge d\bar{z}\, \mu\, \partial^2 \ln \lambda \tag{34}$$

if φ is restricted by the condition $\mu^{\varphi} = 0$.

It is worth pointing out that the form (34) of the WZ can be written with the one-form $\tilde{A}$ and so called "half Liouville field" L /13/ .The field L is a matter field ,which has the first term in the t expansion just $\ln \lambda$, with λ define in(13) and which fulfils the equation

$$dL + A\partial_t L - \partial_t A = 0 .$$

With this definition we can find out, by a simple inspection , that the two-form

$$\mathcal{L} = -L\, \tilde{A}\, \tilde{\dot{A}} \quad , \tag{35}$$

satisfies the equation $\tilde{d}\mathcal{L} = \tilde{\Delta}_3$ i.e. its zero ghost part is the WZ action for the Virasoro algebra.In fact , it is easy to see that that the ghost number zero of $\mathcal{L}$ coincides with (34).

3. W-INFINITY ALGEBRA

3.1. The fields

The $w_{1+\infty}$ algebra is an extension of the Virasoro algebra on the one hand , and a limiting case of the W-infinity algebra, on the other hand /18/ . It can be written in the following simple form:

$$[\, L_n^{\,i} , L_m^{\,j} \,] = [\, (i+1)m - (j+1)n \,]\, L_{m+n}^{\,i+j} \tag{36}$$

This algebra admits an algebraic interpretation, as the algebra of smooth symplectic , area -preserving , diffeomorphisms of a cylinder. This can be easily be seen considering a set of functions/19/ :

$$u_m^{\ell} = i\, e^{mx}\, y^{\ell+1}$$

on a cylinder $S^1 \times \mathbb{R}$, with $0 \le x \le 2\pi$, $-\infty < y < +\infty$. These functions form a complete set if $-\infty < m < +\infty$ and $l \geq -1$ .The symplectic structure is generated by the Poisson bracket

$$\{f\, ,g\}_{x,y} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \tag{37}$$

and the area preserving transformations are generated by $\delta x^{\mu} = \{\Lambda, x^{\mu}\}$ ($\mu = 1,2$) where Λ is an arbitrary function . One can see that the basis $\{u_m^{\ell}\}$ satisfies the $w_{1+\infty}$ algebra:

$$\{u_m^{\,i} , u_n^{\,j}\} = [\, (i+1)m - (j+1)n \,]\, u_{m+n}^{\,i+j} \tag{38}$$

The ghost sector of the BRST symmetry for this algebra can be constructed in a similar manner with the Virasoro case. Here we shall use the basis

$$\ell_n^{\,i} = t^{n+1}\, u^{i+1}$$

instead of $u_n^{\,i}$ and we define the ghost

$$c = \sum_{n,i=-1}^{\infty} t^{n+1}\, u^{i+1}\, c_i^{\,n-i}$$

with $c_n^{\,i}$ the ghost associated to the generater $L_n^{\,i}$.

The Virasoro case (8) can be extended for the w-infinity algebra in a straightforward way. The BRST symmetry of the ghosts has now the form

$$sc + \frac{1}{2}\{c\, ,\, c\}_{t,u} = 0 \, . \tag{39}$$

This equation can be extended to include the gauge fields associated to the generators $L_n^{\,i}$. We can assemble all these fields into a power series

$$\tilde{A}_0 = \sum_{n,i=-1}^{\infty} T^{n+1}\, u^{i+1}\, A_i^{\,n-i}$$

where $A_n^{\,i}$ is a one-form attached to the generator $L_n^{\,i}$, which contains the gauge fields. Moreover, for the complete one-form

$$\tilde{A} = \tilde{A}_0 + c \; .$$

it has been proposed /11/ the equation

$$\tilde{d}\tilde{A} + \frac{1}{2}\left\{\tilde{A}\, ,\, \tilde{A}\right\}_{t,u} = 0, \qquad (40)$$

where $\tilde{d} = d + s$. This equation contains eq.(39) for the ghost number two.

As in the Virasoro case, we can chose a special gauge and identify the physical gauge fields. If one identifies A_0^{-1} as the Beltrami differential, then the equation (40), which is equivalent to the BRST symmetry for w-infinity algebra, has the solution/11/ :

$$\tilde{A} = udz + \sum_{\ell=-1}^{\infty} u^{\ell+1} \left(A_\ell(z+t)\, d\bar{z} + c_\ell(z+t) \right) \qquad (41)$$

where A_ℓ is the complex gauge field, coupled to the spin-(l+2) conserved current in the w-gravity, and c_ℓ is the corresponding ghost. The BRST transformations for these fields can be obtained from eq. (40) and are given by

$$sA_\ell = \sum_{j=0}^{\ell=1} \left[(j+1)\, A_j\, \partial c_{\ell-j} - (\ell - j + 1)\, c_{\ell-j}\, \partial A_j \right]$$

$$sc_\ell = \sum_{j=0}^{\ell+1} (j+1)\, c_j\, \partial c_{\ell-j}$$

3.2 Action for W-gravity.

There are a relative small number of realisations for w-infinity algebra by gauging it, in comparison to the more known classes realisations of Virasoro algebra, despite the kindship between the two. Gauging this algebra we obtain W-gravity. As in the Virasoro case, a BRST -invariant action cannot be constructed only with the field $\tilde{A}$. However, the auxiliary fields J and $\bar{J}$ and the scalar field φ are introduced here in a different manner. For w-gravity ,we replace eq. (22) by

$$\tilde{d}\varphi = \tilde{A}\,(\,J\,) + \tilde{\bar{A}}\,(\,\bar{J}\,) \qquad (43)$$

where

$$\tilde{A}\,(\,J\,) = \tilde{A}(u{=}j,t,z,\bar{z}\,) = Jdz + \sum_{\ell=-1}^{\infty} (d\bar{z}A_\ell + c_\ell)J^{\ell+1} \; ,$$

From this equation we can obtain the BRST transformations of φ and J and ,furthermore, the relation between these fields since this equation is equivalent to the following ones:

$$s\varphi = \sum_{\ell=-1}^{\infty} (J^{\ell+1}\, c_1 + \bar{J}^{\ell+1}\, \bar{c}_\ell\,) \qquad (44)$$

$$J = \partial\varphi - \sum_{\ell=-1}^{\infty} \bar{A}_1\, \bar{J}^{\ell+1}$$
$$\bar{J} = \bar{\partial}\varphi - \sum_{\ell=-1}^{\infty} A_1\, J^{\ell+1} \; . \qquad (45)$$

The BRST transformations of the auxiliary fields J and $\bar{J}$ and the compatibility of eqs.(45) can be obtained from eq. (43) by using the nilpotence of $\tilde{d}$ i.e. $\tilde{d}^2 = 0$. In this way we obtain

$$\tilde{d}J = \partial\tilde{A}(\,J\,) \text{ and c.c.}$$

i.e.

$$sJ = \sum_{\ell=-1}^{\infty} \partial(c_\ell J^{\ell+1}) \; ,$$

and

$$\bar{\partial} J = \sum_{\ell=-1}^{\infty} [\,(\ell+1)\, A_\ell J^\ell + (\partial A_\ell)\, J^{\ell+1}\,] \; .$$

It is worth while to point out that eqs.(45) could be considered the equation of motion for the auxiliary fields J and $\bar{J}$, given by the action

$$\mathcal{L} = -\frac{1}{2}(\partial\varphi)(\bar{\partial}\varphi) - J\,\bar{J} + (\bar{\partial}\varphi)\, J + (\partial\varphi)\bar{J} - \sum_{\ell=-1}^{\infty} \frac{1}{\ell+2}(\,A_\ell\, J^{\ell+2} + \bar{A}_\ell\, \bar{J}^{\ell+2}\,) \; , \qquad (46)$$

which describes the coupling of the gauge fields A_1 to the spin-((l+2) conserved current $(\partial\varphi)^{l+2}$.

The Lagrangian (46) can be extended to describe W_N gravity /22/ . If one replaces in $\mathcal{L}$ the scalar field φ and the auxiliary fields J and $\bar{\bar{J}}$ with a set of scalar fields that take their values in the Lie algebra of SU(N), then ,although the entire w-algebra is realised as a symmetry, it is really only the gauge fields A_ℓ , $\ell \leq N-1$ that play an essential role . The rest of the gauge fields can be set to zero by means of additional symmetries of the Lagrangian, that are of the Stueckelberg type. Therefore, in this case the remaining fields give rise to a non-trivial gauging of the W_N algebra.

The BRST invariance of $\mathcal{L}$ given by (46) can be checked by using the BRST transformations of the fields A_1, φ and J . Nevertheless, it is desirable to obtain an action which is d-closed and the ghost zero part just $\mathcal{L}$. For this we will introduce

a new one-form

$$\tilde{B}(J) = \frac{1}{2} J dz + \sum_{\ell=-1}^{\infty} (A_\ell \, d\bar{z} + c_\ell) \frac{1}{\ell+2} J^{\ell+1} \qquad (47)$$

which seems to be the "integral" of $\hat{A}(J)$. The action which is d-closed and has the ghost zero part just L has the form

$$\tilde{\mathcal{L}} = \frac{1}{2} \tilde{A}(J) \,\tilde{\bar{A}}(J) - [\, \tilde{A}(J)\tilde{B}(J) + \tilde{\bar{A}}(J)\tilde{\bar{B}}(J) \,] \qquad (48)$$

Indeed ,on the one hand , the ghost zero part of $\tilde{\mathcal{L}}$ is

$$\mathcal{L} = \frac{1}{2} [\, J \bar{J} (1 - |\sum_{\ell=1}^{\infty} A_1 J^{\ell+1}|^2) + \sum_{\ell=-1}^{\infty} \frac{\ell}{\ell+2} (A_\ell J^{\ell+2} + \bar{A}_\ell \bar{J}^{\ell+2}) \,]$$

If we take into account the relations (45) ,this Lagrangian boils down to (46). On the other hand L is d-closed , fact which can be verified by a direct computation and the use of the form of dJ.

3.3. W-anomaly

As for the Virasoro algebra, we shall find , by inspection a $\tilde{d}$-closed form , which depends on A_1 and c_1 . It is easy to verify that the looking for three-form can be chosen in this case as /11/ :

$$\tilde{\Delta}_3 = \tilde{A} \, \tilde{d} \, \tilde{A} \qquad (49)$$

The closeness of $\tilde{\Delta}_3$ can be verified by using eq.(40) In the gauge (41) the ghost part of $\tilde{\Delta}_3$ for t=u=0 takes the simple form

$$\Delta_2^1 = (\, A_{-1} \partial c_{-1} - c_{-1} \partial A_{-1} \,)\, dz \wedge d\bar{z} \qquad (50)$$

which is invariant under holomorphic coordinate transformations. However this part of the anomaly is just the first term in a much more complicated expression obtained by Hull /20/ and K. Li and Pope /21/.

A possible solution of this problem seems to be connected to the definition of trace for the auxiliary variables t and u .

In order to get rid of these variables we must add a "trace" in front of the anomaly,which means either putting t=u=0 after doing all differentiations with respect to them, or integration in a special way over t and u. If one wants to follow , as close as possible , the Yang-Mills case, we shall try to write the anomaly in a w-infinity basis . Since in the YM case the anomaly Δ(c ,A) is written as:

$$\Delta (c, A) = \mathrm{Tr}\,(c\, G(A)) = c_a G_b \,\mathrm{Tr}(T^a T^b)$$

it seems natural to try to write our anomaly in the same form. With a suitable definition of the trace ,we can suppose that

$$\mathrm{Tr}(\, u_m^i \, u_n^j \,) = \delta^{ij} \, \delta_{mn} \qquad (51)$$

Therefore the anomaly Δ_2^1 takes the form

$$\Delta_2^1 = \sum_{\ell=-1}^{\infty} a_\ell \, (A_\ell \, \partial^{2\ell+1} c_\ell - c_\ell \, \partial^{2\ell+1} A_\ell) \, dz \wedge d\bar{z}$$

with a_ℓ are certain coefficients. This form of the anomaly has the same form as the one given by C. Hull /20/ . However the form and the interpretation of these anomaly structures deserve further study.

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REFERENCES

1 L. Baulieu,C. Becchi and R. Stora, Phys,Lett., B180(1986) 55 .

2.C. Becchi, Nucl. Phys. B304 (1988) 513 ,

3. S. Lazzarini, These de doctorat, LAPP/ Annecy,1990.

4. F.Delduc and F.Gieres, Class.Quant. GRavity 7(1990) 1907,

5. E.Bergshoeff, A. Bilal and K.S. Stelle, W-Symmetries: Gauging and GEometry, CERN-TH 5924/90 Preprint,

6.X. Shen: W-infinity and string theory, CERN-TH,6404/92 Preprint

7. L.Baulieu, M.Bellon and R. Grimm, Phys. Lett. B198(1987) 343,

8. L.Baulieu, M.Bellon and R. Grimm, Nucl.Phys. B321(1989)697,

9. R. Grimm, Ann.of Physics 200(1990) 49,

10. L.Baulieu, M.Bellon and M.Grimm, Phys. Lett B251(1990) 105,

11. L.Baulieu, M.Bellon and M.Grimm, Phys. Lett,260(1991) 63,

12. J.E. MOYAL, Proc. Camb. Phil.Soc. 45(1949)99,

13.C.N.Pope, L.J.Romans and X.Shen, Phys. Lett. B236(1990) 173,

14. M.Abud, J.P. Ader and F.Gieres, Nucl.Phys.B339(1990) 687,

15.J.P. Ader F.Gieres and Y.Noirot, Covariant Gravitational Anomalies, CERN-TH 5952/90 Preprint ,

16. J.P.Ader and J.C. Wallet, Phys. Lett. B192(1987) 103,

17. J.Manes, R. Stora and B.Zumino, Comm. Math. Phys.102(1985) 157,

18. F.Delduc and F. Gieres,Anomalies and EFfective Actions in Two-dimensional Superspace CERN- TH 5953/90, Preprint,

19.E Bergshoeff, C.N.Pope, L.J. Romans,E.Sezgin, X.Shen and K.S.Stelle, Phys.Lett. 243B (1990) 350,

20.C.Hull, Phys. Lett. 259B (1991) 68,

21. K.Li and C.N.Pope Aspects of Anomaly Structures in W Gravity TAMU-105/90 CTP preprint.