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Abstract.

We analyse the effect of the separatrix shape on stochasticity onset and on diffusion in the two waves system, and in particular the role of the separatrix angle at the X-point. Introducing anharmonicity in the expression of the perturbation, we can adjust this angle, and eventually have it go to zero. We show that, in this latter case, stochasticity appears much sooner and that the diffusion coefficient is strongly enhanced. Applications to specific physical problems are discussed.

1. Introduction

A large number of physical problems, among which magnetic turbulence in plasmas [1], is concerned with the destruction of isolating torii due to the iteration of resonant perturbations: islands can appear on rational surfaces, and their interaction leads to the appearance of a stochastic layer in the vicinity of the separatrix [2]. However, isolating torii can subsist for finite sizes of the islands [3, 4]. The destruction of the last isolating torus, i.e. the onset of large scale stochasticity, has been the subject of many studies, both analytical and numerical. All these studies concern the interaction of islands represented by pure harmonic Fourier components. This representation (pendula type motion) yields for an isolated (unperturbed) component, an island which intrinsically exhibits a finite angle at the X-point of the separatrix.

However, some magnetic plasma configurations cannot be adequately represented by such harmonic perturbation expansions. Indeed there are situations where the angle at the X-point of the separatrix might tend to zero: this is the case when a singular current layer appears in the vicinity of the separatrix. Two specific physical examples can be given: two-ribbons flare configurations, and the $q = 1$ island in tokamaks responsible for the so-called internal disruptions. This geometrical flattening of the separatrix can be intuitively understood as a squeezing of the island by i.) forced flows of the external plasma in the case of solar flares ii.) energetically favorable motion of the inner plasma core in the case of the tokamak problem [5]. Both situations are known to undergo unexplained catastrophic evolution, namely flares and disruptions. We propose to investigate whether the flattening of the angle of the separatrix during the island evolution changes the threshold to large scale stochasticity. We think it is likely that a wider class of applications does exist for these results, and we will therefore study the influence of the separatrix shape in a very general case.

2. Formalism and numerical results

It is well known that magnetic structures are hamiltonian flows. Interaction of two magnetic islands in a slab plasma is conveniently described by the two waves Hamiltonian which fundamentally represents the motion of an effective particle in two electrostatic waves propagating in the same direction with different phase velocities. In the two waves problem, the two perturbations are harmonic components depending on the phase in the form of $\cos(\theta)$ and $\cos(\theta - \omega t)$ assuming identical wave numbers. A simple and natural way to control the angle at the hyperbolic point (i.e. at the X-point of the separatrix) is to introduce anharmonicity in the argument of the cosine. It can be shown that in order to fulfill this requirement the argument has to be an odd and nonlinear function of the phase. A convenient form is the following perturbation: $\cos(\theta - \frac{\alpha}{m}\sin(m\theta))$ where α is the control parameter for the angle and m is a positive integer. For $\alpha = 0$ we retrieve the classical harmonic perturbation, while for $\alpha = 1$, the angle at the X- point of the separatrix goes to zero. The two waves Hamiltonian then becomes:

$$H(x, y, z) = \frac{y^2}{2} - E_1\cos(\theta_1) - E_2\cos(\theta_2) \tag{1}$$

$$\text{with } \theta_1 = (x + z) - \frac{\alpha}{m}\sin(m(x + z)) \tag{2}$$

$$\theta_2 = (x - z) - \frac{\alpha}{m}\sin(m(x - z)) \tag{3}$$

where x and y are conjugated canonical variables and z plays the role of "time".

The threshold for large scale stochasticity , ie. destruction of the last isolating KAM torus, of the two waves Hamiltonian [6] has been found by numerical [7] and analytical [8] studies to be given by $s \geq 0.68$ where s is, as usual, the sum of the individual islands half-widths divided by the distance between the two resonant surfaces (fig. 1.a).

By numerical integration of the modified Hamiltonian of equation 1, we find that for $m = 1$ and $\alpha = 1$, the threshold for large scale stochasticity is below $s = 0.60$ (see fig. 1.b), while for the same s, with $\alpha = 0$, a large number of isolating KAM torii are still present (fig. 1.c): the effect of anharmonicity is to lower the stochasticity threshold. This is true also for $m > 1$: Figure 1.d shows the case with $m = 2$, $\alpha = 0$, $s = 0.60$: although an isolating region is still present, the stochastic region is clearly more important than in the harmonic case (the stochastic theshold for $m = 2$, $\alpha = 1$, is found to be $s \approx 0.65$. The case of the $m = 2$ anharmonic perturbation is interesting as analytical results can be obtained.

3. Analytical approach

Let us consider the motion of the anharmonic pendulum at the separatrix. When $E_2 = 0$, the motion at the separatrix corresponds to the value $H_0 = 1$ of the constant unperturbed hamiltonian. In this case the oscillations are governed by the equation :

$$\begin{aligned} \frac{1}{2}\left(\frac{d\theta}{dz}\right)^2 &= 1 - \cos\left(\theta - \frac{\alpha}{m}\sin(m\theta)\right) \\ &= 2\sin^2\left(\frac{1}{2}\left(\theta - \frac{\alpha}{m}\sin(m\theta)\right)\right) \end{aligned} \tag{4}$$

The integration of equation (4) seems to be impossible in a closed analytical form in the general case. However, for $m = 2$, with $\alpha = 1$, one can obtain an exact integration for an approximate expansion of

$$\begin{aligned}\frac{d\theta}{dz} &= \pm 2\sin(\frac{1}{2}(\theta - \frac{1}{2}\sin(2\theta)) \\ &\simeq \pm 2\sin^3(\frac{\theta}{2})(1 + 2\cos^2(\frac{\theta}{2}))\end{aligned} \qquad (5)$$

(See Figure 2 to compare the two curves representing respectively the exact function $\sin(\frac{1}{2}(\theta - \frac{1}{2}\sin(2\theta))$ and its approximation on the interval $\{0, 2\pi\}$). The integration of the equation $\frac{d\theta}{dz} \simeq \pm 2\sin^3(\frac{\theta}{2})(1 + 2\cos^2(\frac{\theta}{2}))$ leads to the solution:

$$-\frac{1}{3}\frac{\cos(\frac{\theta}{2})}{\sin^2(\frac{\theta}{2})} + \frac{7}{9}\log(\tan(\frac{\theta}{4})) - \frac{8}{9\sqrt{2}}\arctan[\sqrt{2}\cos(\frac{\theta}{2})] = \pm 2(z - z_0)$$

where z_0 is a constant of integration which is equal to zero if the maximum of $\frac{d\theta}{dz}$ is obtained for $z = 0$. Equation (6) gives the "time" z as a function of the angle θ. As it is clear from this expression for $\theta \rightarrow 0$, $z \rightarrow -\infty$, while for $\theta \rightarrow 2\pi$, $z \rightarrow +\infty$ due to the contribution of the first two terms of the left hand side of equation (6). The second term is the usual logarithm term (with a coefficient $\frac{7}{9}$); the first term has a different behaviour, as it tends to infinity as a power law in both directions. This strong modification of the infinite period of the oscillation at the separatrix when $\alpha = 1$ might be at the origin of the lower stochasticity threshold and possibly of the local increase of diffusion, as will be developped in a forthcoming paper.

4. Further numerical results

This approach yields some insight on the effect on diffusion in the immediate vicinity of the separatrix. To obtain a crude estimate of the effect on the global diffusion coefficient, it is necessary to use numerical integration again. Fig.3 shows the average diffusion coefficient

D for n_{ci} initial conditions, over N periods in the z direction. D is defined here as usual as [9]:

$$D = \frac{1}{n_{ci}} \sum_{i=1}^{n_{ci}} \frac{1}{2\pi N} \sum_{j=1}^{N} (y_j - y_o)^2$$

D is plotted for different values of s as a function of α.

A very clear increase of D with α is observed near the stochastic threshold, although this tendency disappears at high s values. Indeed, for such large amplitude perturbations, the resulting extended stochastic sea is no longer sensitive to the detailed shape of the separatrix.

5. Conclusion

The effects of the shape of the separatrix on the stochasticity threshold and on the associated diffusion coefficient for this chaotic hamiltonian system have been studied both analytically and numerically. It has been shown that the flattening of the angle at the X-point of the separatrix has a very significant effect which consists i.) in the reduction of the stochasticity threshold by more than 15% , and ii.) in a large increase of the diffusion coefficient. During the evolution of magnetic configurations (preflare structures and $q = 1$ tokamak islands for instance) , the separatrix can be squeezed by the growth of the island(s): thus there is an increase of the stochasticity parameter s , while simultaneously, the stochastic threshold is lowered. As a consequence, a small evolution yields a dramatic increase in the diffusion coefficient, therefore giving a possible explanation of the catastrophic behaviour observed during flares and disruptions.

Although this study has been initiated mainly because problems of catastrophic evolution of magnetohydrodynamical equilibria, the very general form of analysis, ie. the two waves two waves Hamiltonian, suggests the possibility of a wider class of applications in physics.

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Figure captions

Figure 1: Poincaré map computed from equations 1, for 10 trajectories ($x_i = 0$, $y_i \in [0, 1]$, 2500 crossings for each initial condition). The figure is in fact symetric with respect to the x-axis. The solid line shows the unperturbed separatrix as if there was only one perturbation around $y = -1.$:

a: $s = 0.68, \alpha = 0$ (ie. standard two waves Hamiltonian) .

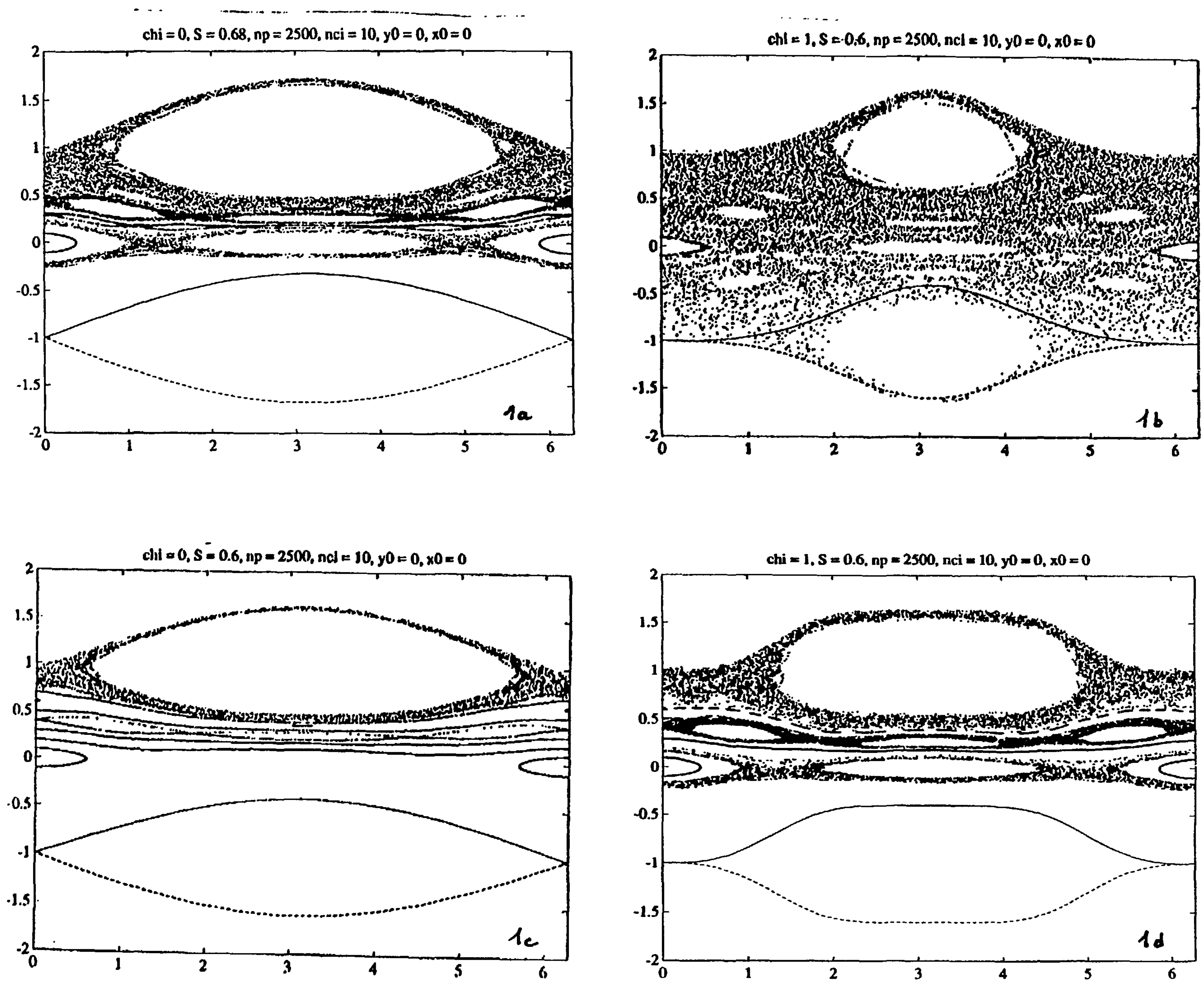
b: $s = 0.6, \alpha = 1$, $m = 1$.

c: $s = 0.6, \alpha = 0$, $m = 1$.

d: $s = 0.6, \alpha = 0$, $m = 2$.

Figure 2: The solid line represents the exact function and the dotted line its approximation (equations 5) on the interval $\{0, 2\pi\}$.

Figure 3: Evolution of the average diffusion coefficient D versus α, for $s = 0.6$, 0.63, 0.65 0.68, for $n_{ci} = 10$ initial conditions between $x = 0$ and $x = \pi/2$, $y = 0.9$, $N = 2500$.


chi = 0, S = 0.68, np = 2500, nci = 10, y0 = 0, x0 = 0
1a
chi = 1, S = 0.6, np = 2500, nci = 10, y0 = 0, x0 = 0
1b
chi = 0, S = 0.6, np = 2500, nci = 10, y0 = 0, x0 = 0
1c
chi = 1, S = 0.6, np = 2500, nci = 10, y0 = 0, x0 = 0
1d

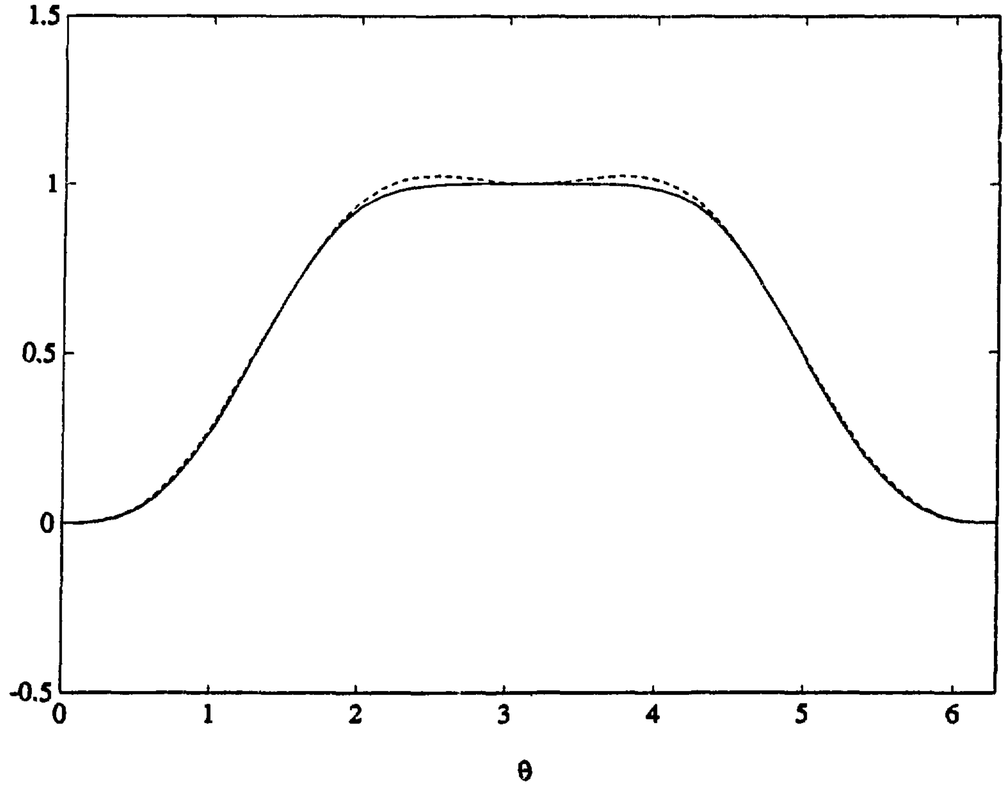


Fig. 2

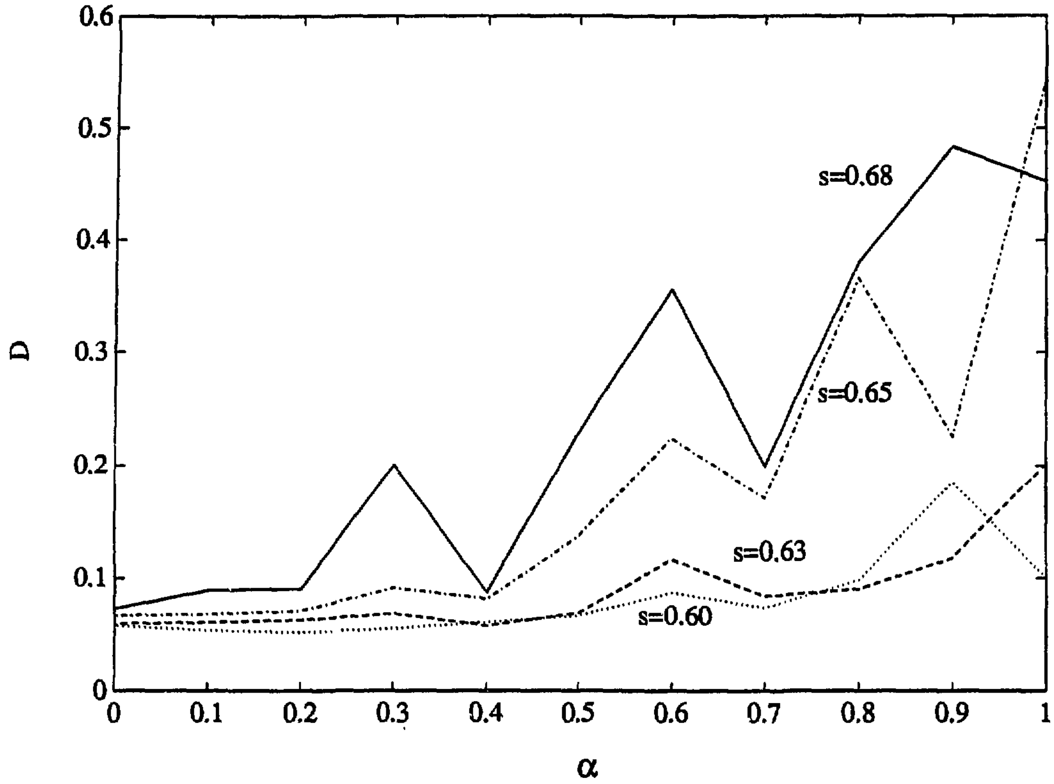

0.6
0.5
0.4
0.3
0.2
0.1
0
D
s=0.68
s=0.65
s=0.63
s=0.60
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
α

fig. 3