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QUALITATIVE PICTURE OF MESONS IN THE DIRAC OSCILLATOR THEORY

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ABSTRACT

A relativistic two–particle radial equation describing a particle–antiparticle system with a Dirac oscillator interaction is written and solved by means of the $1/N$ expansion. The emerging picture of *Dirac–oscillator* mesons seems to be in qualitative agreement with meson phenomenology. The spectrum of excitation energies, the strong interaction radii, and the decay widths of states in our model are calculated and compared with the corresponding experimental magnitudes for mesons with a pure quark composition and total momentum $J = L$.

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1. INTRODUCTION

At the present time we expect hadrons to be the asymptotic scattering states of Quantum Chromodynamics, the field theory of strong interactions. However, we are not able to compute the S–matrix of QCD. As a consequence, the existing approaches to obtain hadron properties are either very qualitative (for example, the $1/N_c$–expansion in QCD [1,2]), or approximate numerical (lattice calculations [..], for instance,), or phenomenological QCD–based approaches. The latter constitute in fact the basis for the current classification of hadron resonances and computation of static properties of hadrons (see, for example, Ref.[4]).

In the present paper, we study a relativistic model in which the properties of mesons may be explicitly computed. The aim is not to describe real mesons, but to gain in the qualitative understanding of meson properties. In fact, some aspects of the model are very unrealistic: the interaction is local and not field–mediated, the spin–orbit coupling is excessively strong, and there is not mixing between mesons with different quark composition.

Our starting point is a relativistically covariant two–particle equation obtained previously in Refs.[5,6] for the particle–antiparticle system with a Dirac–oscillator interaction. The conserved quantum numbers in the theory are J - the total momentum, and the parity. Besides, there is only one free parameter in the model: the oscillator frequency, ω (in units of the quark mass). For states with parity $(-1)^L = (-1)^J$ the above mentioned equation may be reduced to one radial Schrödinger–like equation with an eigenvalue–dependent potential [7].

The solutions of this radial equation exhibit many of the properties of real mesons. Indeed, all states are shown to be resonances, with decay widths that decrease with the increasing of quark mass, strong interaction radii for light mesons are proportional to the inverse quark mass and show a very soft dependence on ω, etc.

In this paper only a brief description of the model and the results is given. A more extended version will be published elsewhere [7].

2. THE MODEL

As mentioned above, our starting point is the radial equation obtained in Ref.[7] for the quark–antiquark system with a Dirac oscillator interaction. In states with total momentum $J = L$ this equation takes the form

$$\left\{(\mu^2 - 2\omega r^2)(\mu^2 - 2\omega\hat{\pi}^2) - \left(2\mu \pm \omega\sqrt{J(J+1)}\right)^2\right\}\phi = 0 \quad . \tag{1}$$

The conventions are the following. We use units in which $\hbar = m = c = 1$, ω is the oscillator frequency, and μ - the relativistic energy (eigenvalue of the mass operator). $\hat{\pi} = -\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{J(J+1)}{r^2}$, and ϕ is one of the four components of the wave function of the system.

The key ingredients for the derivation of Eq.(1) are [7]: (i) write free Dirac Hamiltonians for the particle and the antiparticle; (ii) introduce a Dirac oscillator interaction according to the replacement $\vec{p} \rightarrow \vec{p} - i\omega\vec{x}\beta$, with a frequency ω for the particle and $-\omega$ for the antiparticle; (iii) change to the centre of mass frame and use the prescirption of Refs.[8,9] to obtain an explicitly covariant relativistic equation; and (iv) use straightforward Racah algebra to reduce the obtained equation to radial equations. As the parity $(-1)^L$ is a conserved quantity, the sectors with $J = L$ or $J = L \pm 1$ may be considered separately. When $J = L$ we obtain Eq.(1), while when $J = L \pm 1$ we obtain a coupled pair of second order differential equation which are left for a further analysis.

Eq.(1) may be rewritten in the following Schrödinger–like form ($\phi = \psi/r$)

$$\left\{ 2\omega \left[-\frac{d^2}{dr^2} + \frac{J(J+1)}{r^2} \right] + \frac{\left[2\mu \pm \omega\sqrt{J(J+1)}\right]^2}{\mu^2 - 2\omega r^2} \right] \psi = \mu^2 \psi \quad . \tag{2}$$

Let us mention some interesting properties of Eq.(2). In the first place, we shall stress that there are no bound states. In Fig.1 the effective potential that enters Eq.(2) is schematically represented. This potential supports no bound states. Nevertheless, it is intuitively evident that there are metastable states confined to the region $r^2 < \mu^2/2\omega$ (the mesons). So, the second interesting property of Eq.(2) to be stressed is that it predicts strong interaction radii of mesons to be of the order of $\frac{1}{2}r = \frac{1}{2}\frac{\hbar c}{mc^2}\frac{\mu}{\sqrt{2\omega}}$ (in ordinary units). Using that $\hbar c \approx 0.2\,\mathrm{GeV} - \mathrm{fm}$ and giving m in GeV we get the radius

$$R_{str} \approx \frac{0.1}{m}\frac{\mu}{\sqrt{2\omega}}\mathrm{fm} \quad . \tag{3}$$

The spectrum of metastable levels in Eq .(2) could be obtained by requiring the wave function in $r > \frac{\mu}{\sqrt{2\omega}}$ to be an outcoming wave and looking for solutions in the complex μ–plane. This is, however, a complicated procedure. We will make use of a non- perturbative analytical method consisting in writing formally in D dimensions the Laplacian entering Eq.(2) and using $(D+2J)^{-1}$ as an expansion parameter (see, for example, Ref.[10] and references therein for applications in non–relativistic quantum mechanics, and Ref.[11] and the cited literature for the computation of energy eigenvalues from the Dirac equation in an external potential). We shall mention that using this method we get in first approximation a very localized wave function. It means that in fact we are neglecting the coupling to the disintegration channel and, consequently, we will obtain a real valued μ. The decay width may be obtained by computing the probability of tunnelling along the barrier at $r < \frac{\mu}{\sqrt{2\omega}}$.

In the present paper, we will use a refined version of this method known as the shifted $1/N$–expansion [12]. We start from the equation

$$\left\{ -\frac{1}{N^2}\frac{d^2}{dx^2} + \frac{\left(1 - \frac{a+1}{N}\right)\left(1 - \frac{a+3}{N}\right)}{4x^2} + \frac{b^2(\nu)}{4(1-x^2)} \right\} \psi = \frac{\nu^4}{4}\psi \quad , \tag{4}$$

which may be taken as an extension of Eq.(2) to D dimensions, which coincides with it at the *physical* dimension $D = 3$. The notations are as follows. $N = D + 2J + a$, where the magnitude

a will be specified below. $N_0 = N$ at $D = 3$, ν is related to μ as $\mu = \sqrt{N\omega}\nu$, $r^2 = \frac{\mu^2}{2\omega}x^2$ and $b(\nu)$ is defined as

$$b(\nu) = \frac{2\nu}{\sqrt{N_0\omega}} \pm \frac{1}{N_0}\sqrt{J(J+1)} \tag{5}$$

The solutions of Eq.(4) may be looked for as a power series in $1/N$, i.e. $\nu = \nu_0 + \nu_1/N + \ldots$, $\psi = \psi_0 + \psi_{1/2}/N^{1/2} + \ldots$. We briefly quote the results.

In the leading approximation, $\nu_0^4/4$ is determined as the minimum of a potential

$$\frac{\nu_0^4}{4} = \min\ U(x) = \min \left\{ \frac{1}{4x^2} + \frac{b^2(\nu_0)}{4(1-x^2)} \right\} \tag{6}$$

This leads to trascendental equations for ν_0 and the position of the minimum, x_0, which may be solved explicitly. One obtains the following positive solutions for μ_0 at the *physical* dimension $D = 3$ ($-\mu_0$ are also solutions of Eq.(4)),

$$\mu_0^{(+)} = 1 + \sqrt{1 + \omega\left[N_0 + \sqrt{J(J+1)}\right]} \quad , \quad \text{all values of } \omega,\ J, \tag{7}$$

$$\mu_0^{(-)} = 1 + \sqrt{1 + \omega\left[N_0 - \sqrt{J(J+1)}\right]} \quad , \omega < \frac{4N_0}{J(J+1)}, \quad \text{and} \tag{8}$$

$$\bar{\mu}_0 = -1 + \sqrt{1 + \omega\left[N_0 + \sqrt{J(J+1)}\right]} \quad , \omega > \frac{4N_0}{J(J+1)} \quad . \tag{9}$$

The square distance between q and $\bar{q}$ in this approximation is given by

$$x_0^2 = \frac{1}{\nu_0^2} = \frac{N_0\omega}{\mu_0^2} \quad , \tag{10}$$

or, in ordinary units,

$$r_0^2 = \left(\frac{0.2\,\mathrm{GeV} - \mathrm{fm}}{m}\right)^2 \frac{N_0}{2} \quad . \tag{11}$$

One shall note that r_0 defined in Eq.(11) is obtained from an improperly defined wave function. In a high energy scattering experiment we expect the whole region $r < \frac{\mu}{\sqrt{2\omega}}$ to be tested by the probe. So, Eq.(11) may be taken as an estimate of the meson diameter only if it is of the same order of magnitude as $\frac{0.2\mathrm{GeV} - \mathrm{fm}}{m}\frac{\mu}{\sqrt{2\omega}}$, i.e. if $x_0 \approx 1$.

The parameter a so far has not been determined. It is fixed by the requirement that the next–to–leading corrections give no contribution to the energy. In other words, Eqs.(7)-(9) are required to be exact up to corrections of order $1/N^2$.

The next–to–leading corrections to ν_0 are easily computed by writing $x = x_0 + \frac{y}{N^{\frac{1}{2}}}$ and considering the small (harmonic) oscillations around the equilibrium distance x_0. ψ_0 and ν_1 are determined from the equation

$$\left\{ -\frac{d^2}{dy^2} + \frac{1}{2}U''(x_0)y^2 + \frac{2a-4}{4x_0^2} + \frac{b(\nu_0)\nu_1}{\sqrt{N_0\omega}(1-X_0^2)} \right\} \psi_0 = \nu_0^3\nu_1\psi_0 \quad , \tag{12}$$

leading to

$$\left(n+\frac{1}{2}\right)\lambda+\frac{2a-4}{4x_0^2}+\frac{b(\nu_0)\nu_1}{\sqrt{N_0\omega}(1-x_0^2)}=\nu_0^3\nu_1 \tag{13}$$

where λ is the normal frequency,

$$\lambda=\sqrt{2U''(x_0)}=\frac{2}{x_0^2(1-x_0^2)^{\frac{1}{2}}} \tag{14}$$

If we require $\nu_1=0$, then after some simple algebraic manipulations we are led to the following equation for a,

$$\frac{2-a}{4(n+\frac{1}{2})}\Bigg/\left[\left(\frac{2-a}{4(n+\frac{1}{2})}\right)^2-1\right]^{\frac{1}{2}}=\nu_0=\frac{\mu_0}{\sqrt{N_0\omega}} \tag{15}$$

and the energy of the level with n radial quanta is obtained from one of the Eqs.(7)-(9), with a determined from Eq.(15).

Once obtained the energy (ν_0), we may estimate the level width in the quasiclassical (in $1/N$) approximation

$$\Gamma=m\exp-2n\int_{x_0}^{1}dx\sqrt{U(x)-\nu_0^4/4}\quad, \tag{16}$$

where $U(x)$ is the potential defined in Eq.(6). Let us note that for Eq.(16) to hold we shall obtain $\Gamma<<1$. The integral in Eq.(16) may be explicitly evaluated out to give

$$\Gamma=m\exp-\left\{\frac{\mu_0}{\omega}\sqrt{\mu_0^2-N_0\omega}-N_0\ln\frac{\mu_0+\sqrt{\mu_0^2-N_0\omega}}{\sqrt{N_0\omega}}\right\} \tag{17}$$

We shall discuss below the meaning of the widths of the levels computed from Eq.(4). An alternative estimation of the width of states with $J=0$ may be obtained by computing the transmission probability of the effective Coulomb barrier at $r=\frac{\mu}{\sqrt{2\omega}}$. This has been done in Ref.[13].

3. COMPARISON WITH REAL MESONS

Let us briefly describe the properties of mesons following from Eqs.(7)-(10), (15) and (17), and compare them with the properties of real mesons.

The first point to notice is that in our model there is no mixing between quarks. A flavour is defined by a value of the mass, m_q, and of the frequency, ω_q. This may be taken as an approximate description of $b\bar{b}$, $c\bar{c}$ and light isovector mesons in states with $J=L$.

So, we will consider the analogues of these mesons in our model. Let us start with the $b\bar{b}$. There is almost no uncertainty in the value of mass one may assign to the b quark. We take it from Ref.[4], $m_b\simeq 4.977$ GeV. The frequency may be chosen to fit the value of a physical

magnitude, for example, the energy gap to the first excited state in the subsector with $J=0$. This gap is expected to be 0.58 GeV [4]. So, from

$$\Delta E(n,J)=m_b\{\mu_{nJ}(\omega_b)-\mu_{00}(\omega_b)\}\quad, \tag{18}$$

and particularizing to $n=1, J=0$, one obtains $\omega_b=0.062$. It means that the $b\bar{b}$ system is contained in the frequency region $\omega<<1$ (perturbative). In this region, a may be looked for as a power series in ω. One obtains

$$a=4n+O(\omega)\quad, \tag{19}$$

and the mass spectrum coincides with the perturbative result found in Refs.[5,6]

$$\mu^{(\pm)}=2+\frac{\omega}{2}\left[3+2J+4n\pm\sqrt{J(J+1)}\right]+O(\omega^2) \tag{20}$$

Energy differences computed from Eq.(18) and the $\mu^{(+)}$ branch of Eq.(20) reproduce qualitatively the first two expected [4] (and partially observed [14]) Regge trajectories in bottonium in the sector with $J=L$, i.e. those corresponding to $n=0$ and $n=1$. The splitting of the two levels corresponding to a definite J is not, however, given by the difference $\mu^{(+)}-\mu^{(-)}$. In our model, this is a very strong splitting caused by a strong $L-S$ coupling, while in bottonium the splitting is supposed to be insignificant [4].

The experimental data available for $c\bar{c}$ mesons in states with $J=L$ are the following. Three lines with their corresponding widths have been reported [4]: $\eta_c(2980), \Gamma=10$ MeV; $\eta_c(3950), \Gamma=8$ MeV; $\chi_{c1}(3510), \Gamma=1.3$ MeV. On the other hand, one can get an idea of the expected radii of these mesons from the measured strong interaction radius of the J/ψ, $R^2=0.04$ fm^2 [15] (the J/ψ is not a state with $J=L$).

In Fig.2, excitation energies computed from Eqs.(7) and (15) are compared to the observed [14] (expected [4]) values for $c\bar{c}$ states with $J=L$. The mass $m_c=1.628$ GeV has been taken from Ref.[4], while the frequency ω_c has again been chosen to fit the energy gap in the subsector with $J=0$ (0.61 GeV). We obtained $\omega_c=0.235$.

The radius of the $\eta_c(2980)$ computed from Eq.(3) is shown to be $R_{\eta_c}=0.21$ fm, a very reasonable value. One shall note that for charmonium (and also for bottonium), x_0 defined in Eq.(10) is a very small magnitude as compared with the diameter of the region where the meson lives until it decays, i.e. $x_0<<1$, and thus this value cannot follow from the total cross section measured in a high energy scattering experiment [15].

The law (17) does not give the experimentally observed widths of $c\bar{c}$ states, but it leads to a qualitatively correct dependence on the quark flavour, n, and J. Indeed, according to Eq.(17) we obtain that $b\bar{b}$ levels have smaller widths than $c\bar{c}$, and that to excited $c\bar{c}$ states correspond smaller widths than to the ground state $\eta_c(2980)$. The latter is simply a consequence of the fact that for excited states of the $\mu^{(+)}$ branch it is harder to tunnel through the barrier at $x<1$. This statement, of course, must be understood as referring to states below the $D\bar{D}$ threshold.

The experimental excitation energies of $J = L$ isovector resonances are represented in Fig.3 [14]. The width of the level in GeV is given when available. Missing states (thin lines) are taken from Ref.[4]. Dashed lines are the results of our calculations.

In the present case, the uncertainty in the mass of the u quark is very high, and we took it as a free parameter, together with the frequency ω_u, to fit two experimentally observed magnitudes: the energy gap in the subsector with $J = 0$, 1.162 GeV, and the pion radius $R_\pi = 0.64$ fm. We estimated the radius as half the r_0 given in Eq.(11), as this value follows from an $x_0 \approx 1$. One obtains

$$m_u = 0.242 \text{ GeV} \qquad (21)$$

$$\omega_u = 7.48 \qquad (22)$$

As may be seen, light mesons are contained in the highly non-perturbative region $\omega >> 1$. In this frequency range some interesting phenomena take place. For example, the branches $\mu^{(+)}$ and $\mu^{(-)}$ change their relative disposition, i.e. now $\mu^{(-)}$ is higher in energy than $\mu^{(+)}$. Something analogous to this is experimentally observed at least in $J = 1$ states, in which $J^{PC} = 1^{+-}$ states are lower in energy than 1^{++} states [14], contrary to what is expected in charmonium: $1^{+-} > 1^{++}$ [4]. So, in Fig.3 we drew the $\mu^{(-)}$ branch except for the doublet in $J = 1, n = 0$ for which both branches are represented.

The level widths, calculated from Eq.(17), are also given in Fig.3. The agreement is not good, but some qualitative properties are reproduced. For example, the widths of light mesons are several orders of magnitude higher than for $c\bar{c}$ mesons, the widths of excited $u\bar{u}$ states do not decrease with increasing J, etc.

Concerning the radii of light mesons, we shall make an interesting remark. We choose m_u to fit R_π, but Eq.(11) contains more information than a simple number. From Eq.(11) we obtain for the radii is

$$R^2_{q\bar{q}} = \frac{0.01}{m_q^2}\left(\frac{3+2J}{2} + \frac{a}{2}\right) \text{ fm}^2 \qquad (23)$$

Eq.(23) reproduces qualitatively the known properties of strong interaction radii of light mesons [15]. The main contribution to $R^2_{q\bar{q}}$ (i.e. the term $\frac{3+2J}{2}$) is independent of the interaction potential (of ω). In non-relativistic potential models, an analogous term is usually interpreted as a relativistic smearing of quark coordinates. A soft dependence on ω comes from a. We shall stress that Eq.(23) follows from Eq.(7) in the $\omega >> 1$ limit. The latter is a natural definition of the meson radius.

4. SUMMARY AND DISCUSSION

The main result of the present paper is the qualitative picture of mesons following from Eq.(2): mesons are metastable states living in a region of squared radius $\frac{0.01}{m^2}\frac{\mu^2}{2\omega}\text{fm}^2$.

We compared the properties of our $b\bar{b}$, $c\bar{c}$ and $u\bar{u}$ model mesons with the properties of real mesons. To achieve this goal we determined the quark parameter ω_q as to fit the value of an observable magnitude, for example the energy gap in the subsector $J = 0$. This is, of course, a rough procedure because we are working in a *zeroth order approximation*, without realistic hyperfine and other forces. Nevertheless, we obtained reasonable values for the radii of the studied mesons. For light mesons we obtained an expression for the radius, which reproduces what is known about strong interaction radii of hadrons [15].

Excitation energies of states with $J = L$ were calculated by using mass formulae obtained by means of the $1/N$–expansion. The agreement with experiment is only qualitative due to the fact that the forces acting in the model are not realistic. In particular, the splitting of the doublets in subsectors with $J > 0$ is excessive. However, some qualitative properties are correctly reproduced. For example, the fact that the branches $\mu^{(+)}$ and $\mu^{(-)}$ change their relative disposition as we go from $\omega << 1$ (heavy quarkonia) to $\omega >> 1$ (light mesons) seems to have its analogue in the spectra of real mesons.

We estimated the level width by computing the quasiclassical (in $1/N$) transmission probability of the barrier at $x = 1$. This leads to qualitative correct predictions, for example that heavy mesons have smaller widths (according to the law $\Gamma \approx m \exp -\frac{4}{\omega}$, where ω decreases as m is increased). But the values we obtain for Γ are far from the real ones. This is due to the major role played by the vacuum in the decay of a meson, which is not taken into account in the two–particle equation (2). In particular, we obtain an instable π meson (against to its disintegration into u and $\bar{u}$), while in reality the π is stabilized because of the lack of final states for its decay to proceed.

We see that further developments of the present work may go along two directions. The first is to carry out the *zeroth–order analysis* for $J = L + 1$ states, and the second to include realistic forces. Both seem to be very promising.

Acknowledgments

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FIGURE CAPTIONS

Fig.1 Effective potential entering Eq.(2). The Coulomb-like barrier at $r = \frac{\mu}{\sqrt{2\omega}}$ is transparent. It means that this potential supports only metastable states (the mesons).

Fig.2 Excitation energies of $c\bar{c}$ mesons. The bold lines are experimentally observed levels, thin lines correspond to calculations by Godfrey and Isgur [4], while our model levels (the $\mu^{(+)}$ branch) are represented by dashed lines.

Fig.3 The same as Fig.2 for $u\bar{u}$ mesons, but in this case dashed lines represent levels calculated from the $\mu^{(-)}$ branch, except that the two terms of the $n = 0, J = 1$ doublet are drawn. Level widths are also indicated when available (in GeV).

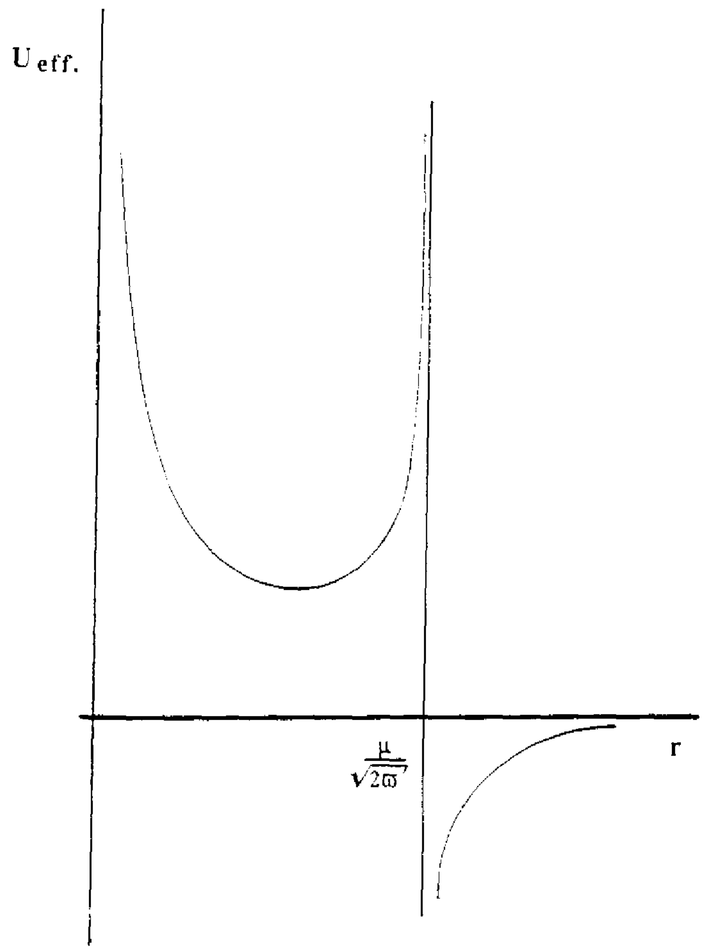


Fig.1

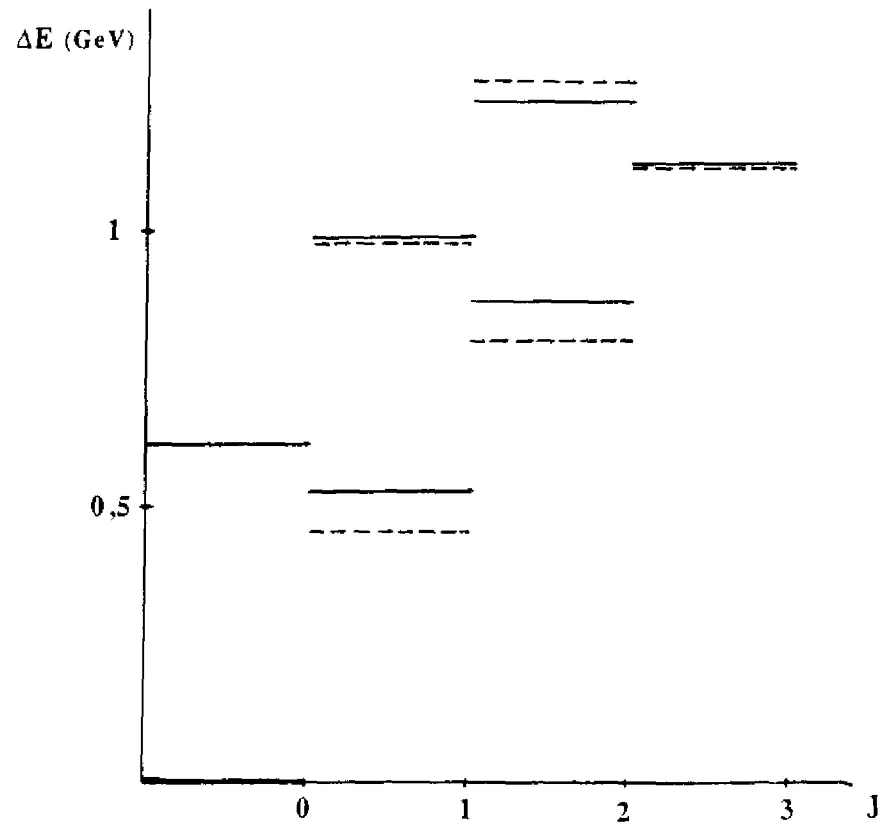
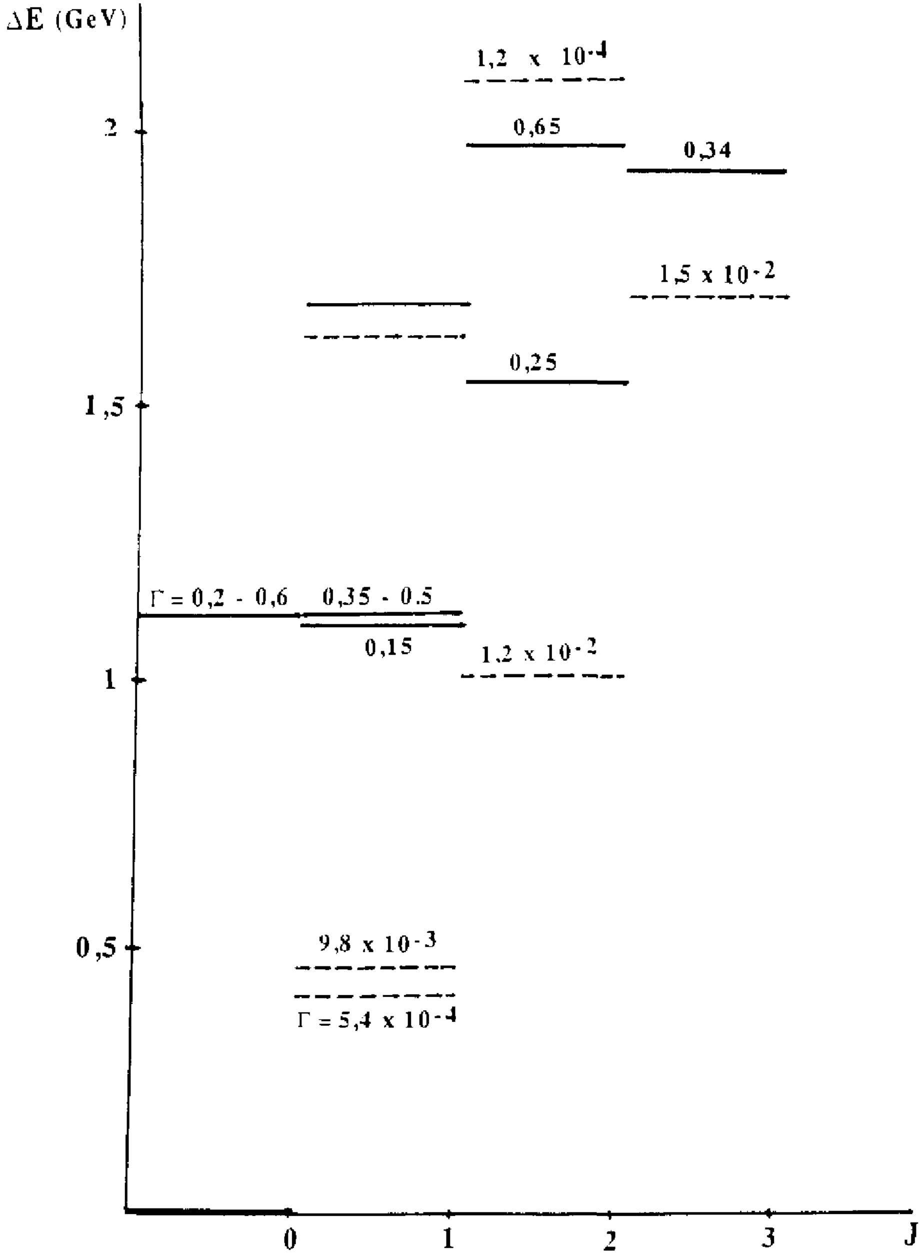


Fig.2



ΔE (GeV)
1,2 x 10-4
2
0,65
0,34
1,5 x 10-2
0,25
1,5
Γ = 0,2 - 0,6
0,35 - 0.5
0,15
1,2 x 10-2
1
0,5
9,8 x 10-3
Γ = 5,4 x 10-4
0
1
2
3
J

Fig.3