GYROINVARIANT SECOND ORDER DRIFT VELOCITIES AND CORRESPONDING GUIDING CENTRE POSITION; GENERAL TREATMENT

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I Introduction

In our previous papers gyroinvariant drift velocities have been obtained for non-uniform magnetic fields [1,2]. A general treatment is presented here for non-uniform and variable electro-magnetic fields. As in the mentioned papers we introduce complex variables and non-analytical functions, combined with sucessive integration by parts. There is no need to use non-canonical variables in Hamiltonian or Lagragian treatments[3,4,5]. Some preliminary ideas of this work were presented some time ago[6,7]. Here we define suitable quasiperiodic functions T(t) and a continuous averaging procedure in order to obtain the gyroinvariant drift velocities: Guiding center is defined in a general way, extending our previous definitions[1,2]. The guiding center position is found as a function of the instanteneous vector position to the second order. This allows us to perform the averaging integrations. Our results for guiding center position are coincident with those of previous authors. In relation to drift velocities they are coincident in the first order, but there is not way to compare them in the second order, since our results are gyroinvariant and the previous ones are not.

II. THEORETICAL TREATMENT

In this paper we are considering the non-relativistic motion of a charged particle under non-uniform and variable magnetic B (r,t) and physical electric field, E_{ph} (r,t)

$$\varepsilon \frac{dv}{dt} = \vec{E}_{ph} + \vec{v} \times \vec{B}(\vec{r},t) \; ; \; \varepsilon = m/q \qquad (2,1)$$

where q and m are respectively charge and mass particle.

As previous authors have done we are denoting by E_{ph} the physical electric field, which will be assumed to be of order ε. To simplify the ordering in the perturbation analysis we define $\vec{E}$, as in Ref. 5, by $\vec{E}_{ph}=\varepsilon\vec{E}$

The unitary magnetic field vector $\hat{b}$ $(\vec{r},t)$, perpendicular velocity $\vec{w}$, and parallel velocity u are defined as usual

$$\vec{v} = u\hat{b} + \vec{w} \; ; \; \vec{w} \cdot \hat{b} = 0 \qquad (2,2)$$

We choose a reference frame in such a way that the unitary vector $\hat{b}$ will be parallel to the z-axis at any time. That can be done if the reference frame rotates with the angular velocity $\vec{\omega}$, such that

$$\vec{\omega} = \hat{b} \times \frac{d\hat{b}}{dt} \; ; \; \frac{d\hat{e}_1}{dt} = \vec{\omega} \times \hat{e}_1 \qquad \frac{d\hat{e}_2}{dt} = \vec{\omega} \times \hat{e}_2 \qquad (2,3)$$

We introduce complex functions associated with every vector function in such a way that the real and imaginary parts of the complex function will be respectively equal to the components of the vector in the directions $\hat{e}_1$ and $\hat{e}_2$. In our notation we use the same letter for the complex function as for the vector but without the circumflex. This will be clearer looking at the equations that follow.

$$\vec{w} = w_x \hat{e}_1 + w_y \hat{e}_2 \qquad w = w_x + i w_y \qquad (2,4)$$

$$\vec{\nabla} = \partial_s \hat{b} + \partial_x \hat{e}_1 + \partial_y \hat{e}_2 \; ; \; \nabla = \partial x + \partial y \, i \qquad (2,5)$$

The Lorentz equation is written as

$$\epsilon(u\dot{b} + u\frac{db}{dt} + \frac{\delta w}{\delta t} + \vec{\omega}\times\vec{w}) = \vec{E} + B\,\vec{w}\times\hat{b} \qquad (2,6)$$

Where $\delta/\delta t$ is the relative derivative and d/dt is the absolute derivative,that is, $d/dt = \delta/\delta t + \omega\times$

The projection of Eq. (2.6) in the plane perpendicular and parallel to $\hat{b}$ can be written using complex variables as

$$\dot{w} + i\frac{B}{\epsilon}w = E - u\frac{db}{dt} \;;\; \dot{u} = E_{//} + \vec{w}\cdot\frac{d\hat{b}}{dt} = E_{//} + \frac{1}{2}(w\frac{d\bar{b}}{dt} + \bar{w}\frac{db}{dt}) \qquad (2,7)$$

where the point and the bar over a letter denote relative time derivative and the complex conjugate function respectively.

The absolute derivative can also be written in another form which is very convenient for our treatmen t.

$$\frac{d}{dt} = \frac{D}{Dt} + \frac{1}{2}(w\bar{\nabla} + \bar{w}\nabla) \;;\; \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial s} \qquad (2,8)$$

Returning now to Eq. (2.7) for w, that equation can be formally considered as a non-homogeneous first order differential equation, whose solution is

$$w = w_0 S + S\int_0^t S(E - u\frac{Db}{Dt})\,dt - S\int_0^t S\,\vec{w}\cdot\nabla\hat{b}\,dt \qquad (2.9)$$

as in previous papers we define

$$S(t) = \exp\left\{-\frac{i}{\epsilon}\int_0^t B[\vec{r}(t_1), t_1]\,dt_1\right\} \qquad (2.10)$$

We also denote by $Q(\vec{r},t)$ the reciprocal of the function $B(r,t)$, Ref. 1, $Q(\vec{r},t) = 1/B(\vec{r},t)$ The quasi-periodic functions $T^{(+)}(t)$, $T^{(-)}(t)$ and $T(t)$ will follow as a result of the periodicity of exponential function for arguments a multiple of 2π. Thus its definition is

$$\int_{t-T^{(-)}(t)}^{t} B[\vec{r}(t_1), t_1]\,dt1 = \int_t^{t+T^{(+)}} B[\vec{r}(t_1), t_1]\,dt_1 = 2\pi \qquad (2.11)$$

$$2\,T(t) = T^{(-)}(t) + T^{(+)}(t) \qquad (2.12)$$

For the particular time t = 0, the preceding functions are denoted by $T_0^{(+)}$, $T_0^{(-)}$ and T_0.

Our results for the first and second order drift velocities are

$$w_D^{(1)} = -i\;Q(E - u\frac{Db}{Dt}) - i\;\frac{|w_0|^2}{2}\nabla Q \qquad (2.13)$$

$$w_D^{(2)} = \epsilon^2 E_{//}Q^2\left(\frac{\partial b}{\partial t} + 2u\frac{\partial b}{\partial s}\right) - \frac{\epsilon^2}{2}|w_0|^2 Q^2\,\nabla\cdot\hat{b}\left(\frac{\partial b}{\partial t} + 2u\frac{\partial b}{\partial s}\right)$$

$$+\epsilon^2 Q\left[\frac{D(QE)}{Dt} - u\frac{D}{Dt}(Q\frac{\partial b}{\partial t}) - u^2\frac{D}{Dt}(Q\frac{\partial b}{\partial s})\right]$$

$$+\epsilon^2 u^2 Q^2\left[(E - u\frac{Db}{Dt})\cdot\nabla\right]b - \frac{\epsilon^2}{8}|w_0|^2 u\,Q\bar{\nabla}(Q\nabla b)$$

$$-\frac{\varepsilon^2}{8}|w_0|^2 \; Q^2 \frac{Db}{Dt}\bar{\nabla} b - \frac{\varepsilon^2}{2}|w_0|^2 < \bar{\nabla}(Q \frac{DQ}{Dt}) > \qquad (2.14)$$

$$-\varepsilon^2 \frac{|w_0|^2}{2} < Q \frac{\partial Q}{\partial s} \frac{Db}{Dt} > - \frac{\varepsilon^2}{8}|w_0|^2 < Q^2 \frac{Db}{Dt} \nabla b >$$

$$+\frac{\varepsilon^2}{4}|w_0|^2 < Q^2 \frac{Db}{Dt}\bar{\nabla} b > - \frac{\varepsilon^2}{8}|w_0|^2 < u \bar{\nabla}(Q^2 \nabla b)>$$

$$+\frac{\varepsilon^2}{4}|w_0|^2 < u \bar{\nabla}(Q^2 \nabla b) > + \frac{\varepsilon^2}{2}|w_0|^2 < \nabla [Q < \frac{DQ}{Dt} >] >$$

$$+\varepsilon^2 < Q (E - u \frac{Db}{Dt}) < \frac{DQ}{Dt} >>$$

Where the notation < > means

$$< F > = \frac{1}{2T_o} \int_{-T_o(-)}^{T_o(+)} F \, dt \qquad (2.15)$$

III . GUIDING CENTRE

As in our paper [1] we consider a fictitious particle whose velocity is just the drift velocity and to zeroth order of approximation is just the instantaneous vector portion $\vec{r}$. The vector position of that particle is what we called the guiding center position $\vec{R}$.

$$\frac{d\vec{R}}{dt} = w_D \qquad (3.1)$$

$$\vec{R} = \vec{r} + \varepsilon \vec{X}_1 + \varepsilon^2 \vec{X}_2 + \dots \qquad (3.2)$$

$$\vec{X}_1 = F_1 \hat{a} + f_1 \hat{c} + g_1 \hat{b} \qquad (3.3)$$

Where $\hat{c}$ is the unitary velocity vector $\hat{v}$ and $\hat{a}$ is $\hat{b} \times \hat{c}$.

The procedure is an extension of our previous papers [1,2].The results are:

$$\vec{R} = \vec{r} \quad w Q \hat{a} + \frac{\varepsilon^2}{2} [\frac{w^2}{2} \hat{a}\cdot\nabla Q^2 - w \frac{DQ^2}{Dt} + 3 u w Q^2 \hat{a}\hat{c} : \nabla \hat{b}$$

$$-6 u w Q^2 \hat{c}\hat{a} : \nabla \hat{b}] \hat{a} + \frac{\varepsilon^2}{2} [3 u w Q^2 \hat{a}\hat{a} : \nabla \hat{b} - u w Q^2 \hat{c}\hat{c} : \nabla \hat{b}] \hat{c}$$

$$+ \frac{\varepsilon^2}{2} [2 w Q^2 \hat{c} . \frac{D\hat{b}}{Dt} + w^2 Q^2 \hat{c}\hat{c} : \nabla \hat{b}] \hat{b} + O(\varepsilon^3) \qquad (3.4)$$

Using the guiding center position and the average procedure we obtain for second order drift velocities the following results.

$$\vec{V}_D = -\varepsilon Q \hat{b} \times (\vec{E}_\perp - u \frac{D\hat{b}}{Dt}) - \frac{w^2}{2} \hat{b} \times \nabla Q + \langle u \hat{b} \rangle$$

$$+ \frac{\varepsilon}{2} w^2 Q (\hat{b} \cdot \nabla \times \hat{b}) \hat{b} - \varepsilon^2 Q^2 (E_{\parallel} + \frac{w^2}{2} \nabla \cdot \hat{b})(\frac{\partial \hat{b}}{\partial t} + 2u \frac{\partial \hat{b}}{\partial s})$$

$$+ \varepsilon^2 Q [\frac{D(QE_\perp)}{Dt} - u \frac{D}{Dt}(Q \frac{\partial \hat{b}}{\partial t}) - u^2 \frac{D}{Dt}(Q \frac{\partial \hat{b}}{\partial s})]$$

$$+ \varepsilon^2 u^2 Q^2 (\vec{E} - \frac{D\hat{b}}{Dt} u) \cdot \nabla \hat{b} - \varepsilon^2 w^2 \frac{\partial Q^2}{\partial s} \frac{D\hat{b}}{Dt}$$

$$- \frac{\varepsilon}{2}^2 w^2 Q^2 \hat{b} \times [(\hat{b} \times \frac{D\hat{b}}{Dt}) \cdot \nabla \hat{b}] + \frac{\varepsilon}{4}^2 w^2 u (\nabla Q^2) \cdot \nabla \hat{b}$$

$$- \frac{7}{4} \varepsilon^2 w^2 u \hat{b} \times [(\hat{b} \times \nabla Q^2) \cdot \nabla \hat{b}] + \frac{\varepsilon^2}{2} \frac{DQ^2}{Dt} (\vec{E} - u \frac{D\hat{b}}{Dt})$$

$$+ \frac{\varepsilon^2}{6} w^2 (\nabla Q^3) \cdot \frac{D\hat{b}}{Dt} \hat{b} + \frac{\varepsilon^2}{2} w^2 \frac{DQ^2}{Dt} \nabla \cdot \hat{b} \hat{b}$$

$$+ \frac{\varepsilon^2}{16} w^2 u Q^2 (\nabla \hat{b}) : (\nabla \hat{b}) \hat{b} + O(\varepsilon^3) \qquad (3.5)$$

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