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ON A GENERALIZED HYPERGEOMETRIC POLYNOMIAL

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ABSTRACT

This article aims to find out some formulae for a generalized hypergeometric polynomial by a simple method.

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1. Introduction

Making the use of a theorem due to Rainville [4, p.137 (Theorem 48)], the author has derived two formulae of very general nature, for a generalized hypergeometric polynomial defined as

$$f_n^{(\alpha,\beta)}(a\text{'}s;b\text{'}s;v) \equiv \frac{(1+\alpha)n}{n!}\ {}_{p+2}F_{q+1}\left[\begin{matrix} -n, 1+\alpha+\beta+n, a_1, \ldots a_p; \\ 1+\alpha, b_1, \ldots, b_q \end{matrix}\ v\right], \tag{1.1}$$

where n is a non–negative integer.

One more formula has been established by using the ordinary series manipulation technique.

In the last section of this article, numerous interesting known and unknown particular cases have also been discussed.

2. Results Obtained by the Theorem

By specialising the parameters, we have from [1, p.178 (2.3)]

$$(1-t)^{-1-\alpha-\beta}\ {}_{p+2}F_{q+1}\left[\begin{matrix} \frac{1+\alpha+\beta}{2}, \frac{2+\alpha+\beta}{2}, a_1, \ldots, a_p; \\ 1+\alpha, b_1, \ldots, b_q; \end{matrix}\ \frac{-4vt}{(1-t)^2}\right]$$
$$= \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)_n}{(1+\alpha)_n}\ f_n^{(\alpha,\beta)}(a\text{'}s;b\text{'}s;v)t^n\ .$$

Regarding the virtue of this generating relation, we mark that Theorem 48 [4, p.137], with

$$c = 1+\alpha+\beta, \gamma_n = \frac{(1+\alpha+\beta)_{2n}(a_1)_n \ldots (a_p)_n}{(1+\alpha)_n(b_1)_n \ldots (b_q)_n\ n!\ 4^n}\ ,$$

is applicable to the polynomial

$$f_n(v) = \frac{(1+\alpha+\beta)_n}{(1+\alpha)_n}\ f_n^{(\alpha,\beta)}(a\text{'}s;b\text{'}s;v)\ .$$

Therefore, the first property of the theorem yields the definition (1.1) of our polynomial $f_n^{(\alpha,\beta)}(a\text{'}s;b\text{'}s;v)$ immediately.

The second property of the theorem offers the expansion formula:

$$v^n = \frac{(1+\alpha)_n(b_1)_n \ldots (b_q)_n}{(a_1)_n \ldots (a_p)_n} \sum_{k=0}^{n} \frac{(-n)_k(1+\alpha+\beta+2k)(1+\alpha+\beta)_k}{(1+\alpha+\beta)_{n+k+1}(1+\alpha)_k} \times f_k^{(\alpha,\beta)}(a\text{'}s;b\text{'}s;v)\ . \tag{2.1}$$

The applications of the third and fifth property of the theorem provide the two formulae which can be had easily from [1, p.180 (3.1) and (3.6)].

Applying the fourth property of the theorem, we obtain the following formula:

$$v\ D\ f_n^{(\alpha,\beta)}(a\text{'s}; b\text{'s}; v) - n\ f_n^{(\alpha,\beta)}(a\text{'s}; b\text{'s}; v)$$
$$= \frac{-(1+\alpha)_n}{(1+\alpha+\beta)_n} \sum_{k=0}^{n-1} \frac{(1+\alpha+\beta)_k}{(1+\alpha)_k} \times$$
$$\times \left[(1+\alpha+\beta)\ f_k^{(\alpha,\beta)}(a\text{'s}; b\text{'s}; v) + 2v\ D\ f_k^{(\alpha,\beta)}(a\text{'s}; b\text{'s}; v)\right] ,$$

where

$$D \equiv \frac{d}{dv}, \qquad n \geq 1\ . \tag{2.2}$$

3. Another Formula

With the help of ordinary series manipulation, we prove the following formula:

$$\sum_{n=0}^{\infty} \frac{(h)_n}{n!} \frac{(1+\alpha)_m}{m!}\ {}_{p+3}F_{q+2}\left[\begin{matrix} -m, 1+\alpha+\beta+m, -n, a_1, \ldots, a_p; \\ 1+\alpha, h, b_1, \ldots, b_q; \end{matrix}\ v\right] t^n$$
$$= (1-t)^{-h}\ f_m^{(\alpha,\beta)}\left(a\text{'s}; b\text{'s}; \frac{-vt}{1-t}\right) \tag{3.1}$$

Proof: The L.H.S. of (3.1) =

$$= \frac{(1+\alpha)_m}{m!} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(h)_n(-m)_r(1+\alpha+\beta+m)_r(-n)_r(a_1)_r \ldots (a_p)_r\ t^n\ v^r}{n!r!(1+\alpha)_r(h)_r(b_1)_r \ldots (b_q)_r}$$
$$= \frac{(1+\alpha)_m}{m!} \sum_{r=0}^{m} \left[\frac{(-m)_r(h)_r(1+\alpha+\beta+m)_r(a_1)_r \ldots (a_p)_r(-vt)^r}{r!(1+\alpha)_r(h)_r(b_1)_r \ldots (b_q)_r} \sum_{n=0}^{\infty} \frac{(h+r)_n}{n!}\ t^n\right]$$
$$= \frac{(1+\alpha)_m}{m!} \sum_{r=0}^{m} \frac{(-m)_r(1+\alpha+\beta+m)_r(a_1)_r \ldots (a_p)_r(-vt)^r}{r!(1+\alpha)_r(b_1)_r \ldots (b_q)_r}\ (1-t)^{-h-r}$$

= R.H.S. of (3.1))

4. Particular Cases

The result (2.1) includes [2, p.161 (6.7)], [4, p.185 Ex.17, p.262(2), p.282(32), p.285(4), p.288(8), p.292(12)].

The result (2.2), for $p = q = 1$, yields a relation for generalized Rice's polynomial

$$v\ D\ H_n^{(\alpha,\beta)}(a_1, b_1, v) - n\ H_n^{(\alpha,\beta)}(a_1, b_1, v)$$
$$= \frac{-(1+\alpha)_n}{(1+\alpha+\beta)_n} \sum_{k=0}^{n-1} \frac{(1+\alpha+\beta)_k}{(1+\alpha)_k} \left[(1+\alpha+\beta)\ H_k^{(\alpha,\beta)}(a_1, b_1, v) + 2v\ D\ H_k^{(\alpha,\beta)}(a_1, b_1, v)\right] , \tag{4.1}$$

where $D \equiv \frac{d}{dv}$, $n \geq 1$.

The result (2.2) encompasses [4, p.185 Ex.18, p.262(4), p.282(34), p.286(6), p.288(9), p.292(14)].

The result (3.1) includes [3, p.431(6)] for $p = q = 0$. Also choosing $p = q = 1$ in (3.1), we get a relation for generalized Rice's polynomial

$$\sum_{n=0}^{\infty} \frac{(h)_n}{n!} \frac{(1+\alpha)_m}{m!} \, {}_4F_3 \left[\begin{matrix} -m, 1+\alpha+\beta+m, -n, a_1 \\ \\ 1+\alpha, h, b_1 \end{matrix} \; v \right] t^n$$

$$= (1 - t^{-h} \; H_m^{(\alpha,\beta)} \left(a_1, b_1, \frac{-vt}{1-t} \right) . \qquad (4.2)$$

In (4.2), on putting $\alpha = \beta = 0$, we get a similar type relation for Rice's polynomial.

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