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Quantum Hamiltonian Differential Geometry

- How does quantization affect phase space ? -

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QUANTUM HAMILTONIAN DIFFERENTIAL GEOMETRY

- How does quantization affect phase space ? -

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Abstract

Quantum Phase Space is given a description which entirely parallels the usual presentation of Classical Phase Space. A particular Schwinger unitary operator basis, in which the expansion of each operator is its own Weyl expression, is specially convenient for the purpose. The quantum Hamiltonian structure obtains from the classical structure by the conversion of the classical pointwise product of dynamical quantities into the noncommutative star product of Wigner functions. The main qualitative difference in the general structure is that, in the quantum case, the inverse symplectic matrix is not simply antisymmetric. This difference leads to the presence of braiding in the backstage of Quantum Mechanics.

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1. INTRODUCTION

Consider the classical phase space $\mathbf{E}^{2n}$ of some mechanical system with generalized coordinates $q = (q^1, q^2, \ldots, q^n)$ and conjugate momenta $p = (p_1, p_2, \ldots, p_n)$. Dynamical quantities $F(q, p)$, $G(q, p)$, etc, are functions on $\mathbf{E}^{2n}$ (the euclidean 2n-dimensional space) and constitute an associative algebra with the usual pointwise product like $(F \cdot G)(x) = F(x)G(x)$ as operation. Given any associative algebra one may get a Lie algebra with the commutator as operation. Of course, due to the commutativity, the classical Lie algebra of dynamical functions coming from the pointwise product is trivial. In Classical Mechanics, it is the peculiar noncommutative Lie algebra defined by the Poisson bracket which is physically significant. This is a rather strange situation from the mathematical point of view, as natural brackets are those coming as commutators in associative algebras and the Poisson bracket does not come from any evident associative algebra of functions. We know however that a powerful geometric background, the Hamiltonian (or symplectic) structure, lies behind the Poisson bracket, giving to its algebra a meaningful and deep content.

In Quantum Mechanics, the product in the algebra of dynamical functions (that is, the operators) is noncommutative and the consequent commutator has a fundamental role. Nevertheless, despite the foresight of Dirac who, in his basic paper [1], calls commutators "quantum derivations", the well known noncommutativity in Quantum Mechanics has more of algebra than of geometry. The difference rests, of course, in the absence of specifically geometric structures in the algebra of operators, such as differentiable structure, differential forms, connections, metrics and the like - in a word, in the absence of a *differential* geometry. It is true that in this respect much knowledge has come up from those quantization procedures related to geometrical quantization. For instance, there does exist a certain connection underlying prequantization, but the whole procedure does not lead to full quantization [2] and its study is still in progress. On the other hand, the real mechanics of Nature is Quantum Mechanics, and classical structures must come out as survivals of those quantal characteristics which are not completely "erased" in the process of taking the semiclassical limit. It is consequently amazing that precisely those quantal structures backing the basic Hamiltonian formalism of Classical Mechanics, in particular the symplectic structure, be poorly known.

The recent developments in noncommutative geometry [3] have led to renewed attempts to make explicit the quantum symplectic structure. Manin's "quantum space" [4], for example, is a step towards it. Our objective here is to study the question of quantum differential geometry from this point of view and to unravel the problem of the symplectic structure. We shall restrict ourselves to a specially simple case in which everything seems to work quite well: the lattice 2n-torus, whose continuum classical limit case is precisely the above E^{2n}. Notice that E^{2n}, the simplest example of phase space, is enough to model any case in which the configuration space is a vector space. It is always good to have such a fair, straightly working case in sight before one proceeds to more involved problems.

In crude language, the usual lore of noncommutative geometry [5] runs as follows [6]. Functions on a manifold M constitute an associative algebra C(M) with the pointwise product. This algebra is full of content because it encodes the manifold topology and differentiable structure. It contains actually all the information about M. The differentiable structure of smooth manifolds, for instance, has its counterpart in terms of the derivatives acting on C(M), the vector fields. On usual manifolds (point manifolds, as the phase space above), this algebra is commutative. The procedure consists then in going into C(M) and working out everything in it, but "forgetting" about commutativity, proceeding as if the product were noncommutative, while retaining associativity. In the E^{2n} above, this would mean that F·G is transformed into some non-abelian product F o G, with "o" being a new operation. The resulting geometry of the underlying manifold M will thereby "become" noncommutative. Recall that a manifold is essentially a space on which coordinates (an ordered set of real, commutative point functions) may be defined. When we go to noncommutative manifolds the coordinates, like the other functions, become noncommutative. Differentials come up in a very simple way through the (then nontrivial) commutator. Associativity of the product F o G implies the Jacobi identity (i.e, the character of Lie algebra) and this makes of the commutator $[F, G] = F \circ G - G \circ F$, with fixed F, a derivative "with respect to F": $[F, [G, H]] = [[F, G], H] + [G, [F, H]]$ is just the Leibniz rule for the "product" defined by [,]. It is known since long [7] that the product "o" related to quantization is the so called "star-product" and that classical-quantal relationship is better seen in the Weyl-Wigner picture of Quantum Mechanics. In that picture, quantum operators are obtained from classical dynamical functions via the Weyl prescription and the quantum formalism is expressed in terms of Wigner functions, which are "c-number" functions indeed but multiply each other by the star product. It will be seen that this procedure is in one-to-one correspondence with the operator point of view in which, instead of the function algebra, a matrix algebra is at work.

We intend first to establish a general notion of the Weyl prescription, by recognizing as such the expressions of quantum operators in a certain well-chosen basis of unitary operators. Then, with that notion in mind, we examine the differential geometry of the space of quantum operators, with central interest in its symplectic structure. As already said, we adopt the lattice torus as a privileged example: it exhibits the basic ideas in a simple way and the continuum classical limit gives the usual euclidean phase space $\mathbf{E}^{2n}$ of very simple systems. While from time to time making contact with the continuum limit, we use actually the discrete Weyl-Wigner procedure inherent to the lattice, which avoids most of the difficulties involving integration measures and reduces quantum operators to finite matrices. The Hamiltonian quantum structure comes up in a direct way and it is an easy task to compare their characteristics with those of its classical correspondent.

The "well-chosen" unitary basis alluded to is a symmetrized Schwinger basis for the Weyl realization of the Heisenberg group. In that basis, operators appear in a luminous way as the quantum versions of classical dynamical quantities, whenever the latter exist. Relationship between classical and quantum objects are of permanent interest, but here Weyl-Wigner transformations will help bridging the gap between classical differential geometric concepts, of which the symplectic form is an example, and their quantum counterparts. The operators belonging to Schwinger's basis are labelled by a double-integer index forming a lattice torus, which we call quantum phase space (QPS). The coefficients in the operator expansions are functions defined on QPS. An exposition [8] (to which we shall refer as "I") has been given previously of the role and basic characteristics of this space.

Two ingredients will be essential to establish a differential geometry on the space of quantum operators and/or on the space of Wigner functions. The first will be the Weyl-Wigner transformation. Section 2 is devoted to introducing the necessary notation, state properties of Schwinger's symmetrized basis, recall some general facts on QPS given in I and add some new necessary material. Operator expansions in that basis are related to Weyl-Wigner transformations in section 3. Also a short incursion is made into twisted convolutions and products with the aim of making clear the interpretaton of the Weyl prescription as a Fourier operator expansion. Although we avoid as a rule the rather involved mathematics which stays behind the whole subject, a few comments of a more formal character are made in section 4, whose main intension is to explain why our case is so simple that full quantization can be achieved. Quantum groups are sketchly discussed, as well as their relationship to braiding and Yang-Baxter equations. The quantum analogue to classical phase space is fundamentally a matrix space, and here comes the second basic ingredient: it will be the "matrix noncommutative differential geometry". We shall here simply apply to the space of quantum operators the matrix geometry [9] introduced by Dubois-Violette, Kerner and Madore (DKM hereafter), which is summarized in section 5. Besides other important characteristics, these authors have found a natural symplectic structure on differential matrix space, leading to a naturally defined "Poisson bracket". The differential geometry of operator space is examined in section 6. It is known that the structure corresponding to the quantum commutator in a c-number formulation is not the Poisson bracket, but the Moyal bracket [10]. We find this bracket as a natural feature of matrix differential geometry. It is also shown the perfect isomorphism between this operator formalism and the formalism using Wigner functions with the star product. In section 7, we look for braiding behind the formalism and exhibit nontrivial solutions for the braid (or Yang-Baxter) equations. The existence of these solutions is related to a specific property of the quantum case: the inverse quantum symplectic matrix is not an antisymmetric matrix. Unless explicitly indicated, we shall dispense with Einstein convention on repeated indices and indicate explicitly summations wherever they appear.

2. THE WEYL-SCHWINGER BASIS

Weyl realizations [11] of the Heisenberg group are built up in terms of two "conjugate" unitary operators U and V satisfying the basic relation $VU = \omega\, UV$, where ω is a complex number. Finite dimensional representations are obtained by taking for U and V matrices NxN such that $U^N = 1$, $V^N = 1$. There is one realization for each integer number N. It follows by taking determinants in the basic relation that $\omega = \exp[i\frac{2\pi}{N}]$. The group elements are then power products of the type $U^m V^n \omega^p$, which satisfy automatically the Heisenberg group defining relations in terms of triples, $(m, n, p) * (r, s, q) = (m + r, n + s, p + q + \frac{1}{2}[ms - nr])$.

These monomials in U and V constitute a complete basis for all quantum operators related to the physical system. In a basis for the space of states given by orthonormalized kets $|v_k\rangle$, with k integer, , U is defined by

$$U|v_k\rangle = |v_{k+1}\rangle\ ,\ \text{with}\ |v_{k+N}\rangle \equiv |v_k\rangle. \qquad (2.1)$$

Of course,

$$U^m|v_k\rangle = |v_{k+m}\rangle \qquad (2.2)$$

and the cyclic condition imposed on the kets ensures $U^N = 1$. The eigenvalues $u_k = e^{i(2\pi/N)k}$ of U correspond to eigenkets fixed by

$$U|u_k\rangle = u_k|u_k\rangle . \tag{2.3}$$

The other operator V is given by

$$V|u_k\rangle = |u_{k-1}\rangle \tag{2.4}$$

and

$$V^n|u_k\rangle = |u_{k-n}\rangle, \text{ with } |u_{k-N}\rangle \equiv |u_k\rangle. \tag{2.5}$$

Then also $V^N = 1$ and the V eigenvalues are $v_k = e^{i(2\pi/N)k}$. The eigenkets $|v_k\rangle$ such that

$$V|v_k\rangle = e^{i(2\pi/N)k}|v_k\rangle \tag{2.6}$$

are just those we have started with. Of course, $V^n|v_k\rangle = e^{i(2\pi/N)kn}|v_k\rangle$. A direct calculation shows that the basic relation

$$V U = e^{i(2\pi/N)} U V , \tag{2.7}$$

or

$$V^n U^m = e^{i(2\pi/N) m n} U^m V^n, \tag{2.8}$$

is implied by the ket cyclic conditions. It follows also that all numbers k, m, n, etc, are defined mod(N).

The above expressions are invariant under the simultaneous changes $U \to V$, $V \to U^{-1}$, $m \to n$, $n \to -m$. This symmetry leaves invariant the operators

$$S_{(m,n)} = e^{i(\pi/N) m n} U^m V^n,$$

which are such that $S_{(0,0)} \equiv 1$, $S^{-1}_{(m,n)} = S^{\dagger}_{(m,n)} = S_{(-m,-n)}$ and constitute a complete orthonormal (in a natural metric given below) basis to whose special virtues we shall repeatedly refer. Further properties are: associativity, $(S_{(m,n)}S_{(r,s)})S_{(k,l)} = S_{(m,n)}(S_{(r,s)}S_{(k,l)})$; quasi-periodicity, $S_{(N,p)} = (-)^p S_{(0,p)}$ and $S_{(p,N)} = (-)^p S_{(p,0)}$; and two-foldedness: $S_{(N,N)} = (-)^N S_{(0,0)}$ (more on this property may be seen in [12]). Powers are given by $S^p{}_{(m,n)} = S_{(pm,pn)}$. The explicit form of $S_{(m,n)} = \sum_k e^{i(\pi/N)(2k+m)n} |v_{k+m}\rangle\langle v_k|$ in the V-diagonal ket basis is

$$S_{(m,n)} = \omega^{mn/2} \begin{pmatrix} 0 & \begin{matrix} \omega^{n(N-m)} & 0 & 0 & \dots & 0 \\ 0 & \omega^{n(N-m+1)} & 0 & \dots & 0 \\ 0 & 0 & \omega^{n(N-m+2)} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \omega^{n(N-1)} \end{matrix} \\ \begin{matrix} \omega^0 & 0 & \dots & \dots & 0 \\ 0 & \omega^n & 0 & \dots & 0 \\ 0 & 0 & \omega^{2n} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \omega^{n(N-m-1)} \end{matrix} & 0 \end{pmatrix} \tag{2.9}$$

The upper-left zero bloc is an m x (N-m) matrix. Each dynamical quantity will be an operator **A** belonging to the group algebra, and will be written in this basis as $\mathbf{A} = \frac{1}{N}\sum_{m,n} A^{(m,n)} S_{(m,n)}$, with complex coefficients $A^{(m,n)} = \mathrm{tr}\,[S^\dagger{}_{(m,n)}\mathbf{A}]$. It is easily checked that $\mathrm{tr}[S_{(m,n)}] = \delta_{m,0}\delta_{n,0}N$, $\mathrm{tr}\mathbf{A} = A^{(0,0)}$, $A^{(m,n)} = [S^\dagger{}_{(m,n)}\mathbf{A}]^{(0,0)}$ and $\mathrm{tr}\ \mathbf{A}^\dagger\mathbf{A} = \sum_{m,n}|A^{(m,n)}|^2$. The trace actually defines an internal product on the algebra by $\langle\mathbf{A},\mathbf{B}\rangle = \mathrm{tr}\,(\mathbf{B}^\dagger\mathbf{A})$. A trivial metric is thereby defined,

$$g_{(m,n)(r,s)} = \frac{1}{N}\mathrm{tr}[S^\dagger{}_{(m,n)}S_{(r,s)}] = \delta^{(2)}{}_{(m,n)(r,s)} = \delta_{m,r}\delta_{n,s}.$$

The basis $\{S_{(m,n)}\}$ is orthonormal in this metric. The set of operators constitutes thus a metric algebra with unity. The double indices are a token of phase space, as will become clear below when we discuss the classical limit. We shall most of the time use a compact notation in terms of two-vectors $\mathbf{m} = (m_1, m_2)$, $\mathbf{n} = (n_1, n_2)$, $\mathbf{r} = (r_1, r_2)$, $\mathbf{0} = (0, 0)$, $-\mathbf{m} =$

$(-m_1, -m_2)$, $\mathbf{m} + \mathbf{r} = (m_1 + r_1, m_2 + r_2)$, $\mathbf{m} \times \mathbf{r} = (m_1r_2 - m_2r_1)$, $\mathbf{m} \cdot \mathbf{r} = (m_1r_1 + m_2r_2)$, etc. The basis members are then

$$S_{\mathbf{m}} = e^{i(\pi/N)m_1m_2} U^{m_1} V^{m_2} \tag{2.10}$$

and the above properties become $S_0 = 1$, $S^{-1}_{\mathbf{m}} = S^{\dagger}_{\mathbf{m}} = S_{-\mathbf{m}}$, etc. We introduce a double delta $\delta_{\mathbf{m},\mathbf{r}}$ in terms of which the metric is simply

$$g_{\mathbf{mr}} = \frac{1}{N}\mathrm{tr}[S^{\dagger}_{\mathbf{m}}S_{\mathbf{r}}] = \delta^{(2)}_{\mathbf{m},\mathbf{r}} = \delta_{\mathbf{m},\mathbf{r}}. \tag{2.11}$$

Operators are then written

$$\mathbf{A} = \frac{1}{N}\Sigma_{\mathbf{m}}A^{\mathbf{m}}S_{\mathbf{m}}\ , \tag{2.12}$$

with

$$A^{\mathbf{m}} = \mathrm{tr}\left[S^{\dagger}_{\mathbf{m}}\mathbf{A}\right]\ . \tag{2.13}$$

$\Sigma_{\mathbf{m}}$ means summation over all the distinct pairs $\mathbf{m} = (m_1, m_2)$.

It is immediate to find that

$$S_{\mathbf{m}}\,|v_k\rangle = e^{i\alpha_1(k;\,\mathbf{m})}\,|v_{k+m_1}\rangle,$$

where

$$\alpha_1(k;\mathbf{m}) = \frac{\pi}{N}(2k+m_1)m_2. \tag{2.14}$$

For each value of N, the $S_{\mathbf{m}}$'s realize a representation of the Heisenberg group as a projective representation (or abelian extension) of the double cyclic group $Z_N \otimes Z_N$. They indeed satisfy

$$S_{\mathbf{r}}\, S_{\mathbf{m}} = e^{2i\alpha_2(\mathbf{m},\mathbf{r})}S_{\mathbf{m}}\, S_{\mathbf{r}} = e^{i\alpha_2(\mathbf{m},\mathbf{r})}\, S_{\mathbf{m}+\mathbf{r}}, \tag{2.15}$$

with

$$\alpha_2(\mathbf{m},\mathbf{r}) = \frac{\pi}{N}(m_1r_2-m_2r_1) = \frac{\pi}{N}(\mathbf{m}\times\mathbf{r}). \tag{2.16}$$

Unlike α_1, the phase α_2 is actually independent of the state label. There are many helpful identities and properties, such as

$$\frac{1}{N}\sum_k e^{i(2\pi/N)k(r-p)} = \delta^{rp} \tag{2.17}$$

for simple index summation; or, for double indices,

$$\frac{1}{N^2}\sum_m e^{2i\alpha_2(m,\, r-s)} = \delta^{r,s} \tag{2.18}$$

and

$$\frac{1}{N}\sum_m S_m A S^\dagger_m = (\mathrm{tr}\, A)\, \mathbf{I} . \tag{2.19}$$

They are of calculational interest and shall be stated when needed.

The operators U and V, introduced by Weyl, were brought to the forefront by Schwinger, who in a series of papers [13] stressed both their role as purveyors of a complete basis and the optimal uncertainty provided by their conjugacy (2.7). The basic facts of Quantum Mechanics were expressed in this formalism. A fact of particular interest is the following: when N is a prime number, the pair (U, V) describes one degree of freedom taking on N possible values. When N is not prime, it is a product of prime numbers and the basis $\{S_m\}$ factorizes correspondingly into a product of independent sub-bases, one for each prime factor, that is, one for each degree of freedom. This fact leads to a classification of the quantum degrees of freedom in terms of prime decompositions of integer numbers. If we want to work simultaneously with two or more degrees of freedom of a physical system, we must use for N a well-chosen non-prime value, but it is, in general, more convenient to analyse the system into its independent degrees and examine each one at a time. In any case we must qualify our previous statement concerning expression (2.12): it holds for any operator which is dependent on the degrees of freedom under consideration (and of course of the corresponding conjugate momenta).

Although the set of products $\{U^m V^n\}$ constitute by itself a complete basis for the operator algebra, the complete symmetrized basis $\{S_{\mathbf{m}}\}$ is of particular interest because of its many remarkable properties. It is immediate from (2.9) that the S_{mn}'s reduce to the Pauli matrices for $N = 2$. For $N \geq 2$, they are those generalizations of Pauli matrices providing the finest grading of the linear complex Lie algebra gl(N,C) [14]. Gradings of Lie algebras are physically important because they establish the existence of preferred basis admitting additive quantum numbers. $\{S_{\mathbf{m}}\}$ is a preferred basis in this sense. Furthermore, as seen below, this basis is directly related to the semiclassical limit and to Weyl-Wigner transformations.

The lattice torus spanned by the labels (m, n) has been called quantum phase space (QPS) in I. Because of their property of two-foldedness, $S_{(N,\,N)} = (-)^N S_{(0,\,0)}$, the operators $S_{\mathbf{m}}$ constitute a double covering modulo N of the torus. It was shown in I how closed paths on QPS lead to open paths in operator space and how this fact is related to noncommutativity. The numbers m and n may be seen as coordinates mod N. Or else, we may think of U^m and V^n as global "coordinates", with values in the group $Z_N \otimes Z_N$. In this case, due to the projective character of the representation, such coordinates will appear as noncommutative "point functions". The usual continuum limit corresponds to taking to infinity both the torus radii, while $N \to \infty$, each number of form $\sqrt{2\pi/N}\, m \to$ some constant a, $\sqrt{2\pi/N}\, \Sigma_k \to \int da$, and the area $(2\pi/N)$ of each elementary lattice plaquette on QPS tends to zero.

The resulting expressions are dependent on Planck's constant and the classical limit is, of course, $h \to 0$.

The phases in (2.14) and (2.16) have been shown in I to be the result of the action of algebraic cochains on the group elements, $\alpha_1(k; \mathbf{m}) = \alpha_1(k; S_{\mathbf{m}})$ and $\alpha_2(\mathbf{m}, \mathbf{r}) = \alpha_2(k; S_{\mathbf{m}}, S_{\mathbf{r}})$. On such cochains a derivative δ satisfying $\delta^2 = 0$ is defined, and the resulting cohomology gives information on the projective representation involved. Thus, if α_1 is exact, i.e, $\alpha_1 = \delta\alpha_0$ for some 0-cochain α_0, α_1 may be eliminated by adding a phase α_0 to the wavefunctions. When α_2 is exact, $\alpha_2 = \delta\beta$, it can be eliminated by absorbing its "integral" phase β in the operators, which thereby appear as "gauged", or state-dependent. It turns out actually that $\alpha_2 = \delta\alpha_1$ in the present case, so that $\delta\alpha_2 = 0$. This means that the two-cochain α_2 is a cocycle, a condition which is equivalent to associativity $(S_m S_r)S_k = S_m(S_r S_k)$. The two-cochain α_2 being a cocycle means also that it might be gauged out by modifying the $S_{\mathbf{m}}$'s. We shall not do it because some important aspects are more easily seen when its presence is explicit, but the whole formalism keeps in consequence some analogy to gauge theories. We may call α_2 the "fundamental cocycle" because, beside being the analogue on QPS of the symplectic form in Classical Mechanics, it actually tends to it in the classical limit. It defines a "pre-symplectic" structure on QPS, which is at the origin of the classical Hamiltonian structure. The cochain α_1, on the other hand, has a role analogous to Liouville's canonical form. When there exists a limiting classical system, such cochain properties will have classical counterparts.

The usual case of the position and momentum operators $\mathbf{q}$, $\mathbf{p}$ turns up when we choose $S_{(m,0)} = U^m = e^{i\sqrt{2\pi/N}\, m\mathbf{q}}$ and $S_{(0,n)} = V^n = e^{i\sqrt{2\pi/N}\, n\mathbf{p}}$. In the continuum limit, expressions like $\sqrt{2\pi/N}\, m$ and $\sqrt{2\pi/N}\, n$ tend to constants a_1, a_2, etc, so that the eingenvalues of q and p, which are of the form $\sqrt{2\pi/N}\,k$, tend to numbers q, p. Thus, $S_{(m,0)} \to S_{(a_1,0)} = e^{ia_1q}$, $S_{(0,n)} \to S_{(0,a_2)} = e^{ia_2p}$ and

$$S_m \to S_a = e^{i(a_1q+a_2p)} \tag{2.20}$$

$$= e^{-ia_1a_2/2}e^{ia_1q}e^{ia_2p} = e^{-ia_1a_2/2}S_{(a_1,0)}S_{(0,a_2)}$$

On the other hand, $S_{(a_1,0)}\,S_{(0,a_2)} = e^{i\alpha_2[(0,a_2),(a_1,0)]}\,S_{(a_1,a_2)}$ by the very definition of α_2, so that $\alpha_2[(a_1,0),(0,a_2)] = -\frac{1}{2}a_1a_2$. We have been using $\hbar = 1$, which can be corrected when necessary by simple dimensional analysis: while a commutator has no dimension by itself, the Poisson bracket has dimension $[\text{action}^{-1}]$. The case above has an especially simple classical limit, as higher order tems vanish in the Weyl-Wigner transformation. Only the Poisson bracket comes out, and we find that $\alpha_2[(a_1,0),(0,a_2)] = \alpha_2(e^{ia_1q}, e^{ia_2p}) = -(\hbar/2)\{a_1q, a_2p\}$. Notice that the operators appear in the exponents. The relationship will become more clear in section 6, where we shall see that the symplectic structure on the operator algebra is in reality related to the sine of α_2. But one learns already from the example above that, as soon as one leaves the group to consider general operators, one's interest becomes focused on the operator space itself: one goes from the group elements to members of the algebra they generate. In classical Hamiltonian formalism, we may start with phase space as defined (say) by the (q, p) pair, but canonical invariance implies that any other pair of conjugate dynamical functions F(q,p), G(q,p) may be used as well. And, indeed, only through the use of general dynamical functions do we arrive at the complete picture. Here, although starting from QPS, we shall be led to work on the operator algebra. It will be on this algebra that the quantum symplectic structure will find its general expression.

The phase chosen in (2.10) is at the origin of the nice properties of basis $\{S_m\}$. The special form (2.20) of the continuum limit exhibits the main reason for that choice: it gives to $\{S_m\}$ the role of a Fourier basis and renders it fundamental to Weyl-Wigner transformations.

3. WEYL-WIGNER TRANSFORMATIONS

The operator expansions (2.12) are discrete versions of the Weyl prescription giving the quantum operator in correspondence with a classical dynamical quantity. Let us only recall in general lines how the prescription [15] works for the $\mathbf{E}^{2n}$ coordinate-momentum case. The Wigner functions $A_W(q, p)$ are written as Fourier transforms [16] of the Wigner densities $A(a, b)$,

$$A_W(q, p) = F[A] = \iint da\,db\; e^{i(aq+bp)} A(a, b). \tag{3.1}$$

Then the Weyl operator $\mathbf{A}(\mathbf{q}, \mathbf{p})$, function of operators $\mathbf{q}$ and $\mathbf{p}$ which corresponds to A_W, will be

$$\mathbf{A}(\mathbf{q}, \mathbf{p}) = \iint da\,db\; e^{i(a\mathbf{q}+b\mathbf{p})} A(a, b). \tag{3.2}$$

We may denote by $\hat{F}$ this operator Fourier transform, so that

$$\mathbf{A} = \hat{F}\,[F^{-1}[A_W]] \tag{3.3}$$

and

$$A_W = F[\hat{F}^{-1}[\mathbf{A}]]\ . \tag{3.4}$$

As is well known, the Wigner functions are c-number representatives of quantum quantities (they will include powers of $\hbar$, for example) which tend to the classical quantities in correspondence when $\hbar \to 0$. The densities $A = F^{-1}[A_W] = \hat{F}^{-1}[A]$ include commonly Dirac deltas and their derivatives. From (2.20),

$$A(q, p) = \iint da\,db\, S_{(a,b)}\, A(a, b), \qquad (3.5)$$

of which (2.12) is a discrete version, with the coefficients $A^{\mathbf{m}}$ as Wigner densities. This also shows how quantization casts its roots in Fourier analysis: an operator is given as a Fourier expansion. We may, if we wish, use also here a double-index notation $\mathbf{x} = (q, p)$, $\mathbf{a} = (a_1, a_2)$ so as to be able to write $e^{i(a_1 q + a_2 p)} = e^{i\,\mathbf{a}.\mathbf{x}}$, $A(\mathbf{x}) = \int d\mathbf{a}\, S_{\mathbf{a}}\, A(\mathbf{a})$, etc.

The product of two operators is

$$AB = \frac{1}{N^2} \Sigma_{\mathbf{m}} \Sigma_{\mathbf{r}} A^{\mathbf{m}} B^{\mathbf{r}} e^{i\,\alpha_2(\mathbf{r},\mathbf{m})} S_{\mathbf{m}+\mathbf{r}} =$$

$$= \frac{1}{N} \Sigma_{\mathbf{p}} \Big[\frac{1}{N} \Sigma_{\mathbf{m}} A^{\mathbf{m}} B^{\mathbf{p}-\mathbf{m}} e^{i\,\alpha_2(\mathbf{p},\mathbf{m})} \Big] S_{\mathbf{p}}, \qquad (3.6)$$

from which the densities

$$(AB)^{\mathbf{p}} = \mathrm{tr}\,[S^{\dagger}_{\mathbf{p}} AB] = \frac{1}{N} \Sigma_{\mathbf{m}} A^{\mathbf{m}} B^{\mathbf{p}-\mathbf{m}} e^{i\,\alpha_2(\mathbf{p},\mathbf{m})} \qquad (3.7)$$

are directly read. This expression is a discrete twisted convolution, as seen below.

The commutators

$$[S_{\mathbf{r}}, S_{\mathbf{m}}] = 2i\sin[\alpha_2(\mathbf{m}, \mathbf{r})] S_{\mathbf{m}+\mathbf{r}} \qquad (3.8)$$

lead, for two general operators, to

$$[A, B] = \frac{1}{N^2} \Sigma_{\mathbf{m}} \Sigma_{\mathbf{r}} A^{\mathbf{m}} B^{\mathbf{r}}\, 2i\sin[\alpha_2(\mathbf{r}, \mathbf{m})] S_{\mathbf{m}+\mathbf{r}} =$$

$$= \frac{1}{N^2} \Sigma_p \Sigma_m A^m B^{p-m}\, 2i \sin[\alpha_2(p, m)] S_p . \qquad (3.9)$$

This is the Weyl prescription for the commutator and the expressions

$$[A, B]^p = tr(S^\dagger_p [A, B]) =$$

$$= \frac{1}{N} \Sigma_m A^m B^{p-m}\, 2i \sin[\alpha_2(p, m)] \qquad (3.10)$$

will be the densities of the discrete version of the Moyal bracket. α_2 alone will appear in the first order of the sine expansion. It is obvious that α_2, which had in (2.15) marked the projective character of the representation, appears as the source of noncommutativity in (3.9). For sake of completeness and to allow immediate comparison with the classical case, we write down also the continuum versions, for once reintroducing Planck's constant wherever due:

$$[S_a, S_b] = \frac{2i}{\hbar} \sin [\hbar \frac{b \times a}{2}] S_{a+b};$$

$$[A, B] = \frac{2i}{\hbar} \int da \int db\, A(b) B(a-b) \sin [\hbar \frac{a \times b}{2}] S_a .$$

Summing up, the Schwinger symmetrized basis appears as a (discrete finite at first, or continuum infinite in the limit) operator Fourier basis [17]. In the expansion (2.12), for example, the operator **A** is just the operator Fourier transform of the Wigner density A^m. An example of Wigner density is related to the power of the Weyl operator U, $(U^j)^m = N \delta^{m_1}{}_j \delta^{m_2}{}_0$, leading to $U_W(r, s) = e^{i (2\pi/N) jr}$, independent of s and coherent with the previously indicated continuum limit:

$$U_W(r,s) = e^{i \sqrt{2\pi/N}\, j \sqrt{2\pi/N}\, r} \Rightarrow e^{i \sqrt{2\pi/N}\, jq} \Rightarrow e^{iaq}.$$

The star product (or twisted product) is introduced in a simple way through the notion of twisted convolution [18]. Let us again consider the phase space $\mathbf{E}^{2n}$, using now the notation $\mathbf{x} = (x_1, x_2, \ldots, x_n, x'_1, x'_2, \ldots, x'_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n, y'_1, y'_2, \ldots, y'_n)$ for its points. The complex functions f, g defined on $\mathbf{E}^{2n}$ constitute a commutative algebra with the usual pointwise product. Such products of functions are the Fourier transforms of convolutions. More precisely, if $F[f]$ is the Fourier transform of f, the pointwise product is $f \cdot g = F^{-1}[F[f] * F[g]]$. By the way, to the usual convolution of two functions,

$$(f * g)(\mathbf{x}) = \int f(\mathbf{y}) g(\mathbf{x}-\mathbf{y}) d\mathbf{y},$$

will clearly correspond the expression $\frac{1}{N} \Sigma_m A^m B^{p-m}$ in the discrete case.

One may realize the passage from the commutative to the noncommutative, alluded to in the Introduction, by going from the pointwise-product algebra $C^{\infty}(\mathbf{E}^{2n})$ of complex differentiable functions on $\mathbf{E}^{2n}$ to a noncommutative-product algebra in the following way. The twisted convolution of index c is defined by

$$(f *_c g)(\mathbf{x}) = \int e^{i(c/2)(\mathbf{x} \wedge \mathbf{y})} f(\mathbf{y}) g(\mathbf{x}-\mathbf{y}) d\mathbf{y}, \qquad (3.11)$$

where $(\mathbf{x} \wedge \mathbf{y}) = x_i y'_i - x'_i y_i$. The corresponding expression in our case,

$$\frac{1}{N} \Sigma_m A^m B^{p-m} e^{i(c/2)\, m \wedge p}, \qquad (3.12)$$

shows that $(\mathbf{AB})^p$ given by (3.7) is just a twisted convolution whose index is the area $\frac{2\pi}{N}$ of an elementary lattice plaquette. The twisted product "o" stands to twisted convolution as the pointwise product stands to usual convolution: it is defined as an inverse Fourier transform of the twisted convolution of Fourier transforms:

$$f \circ g = F^{-1}[F[f] *_c F[g]]. \qquad (3.13)$$

It defines thus on $C^{\infty}(E^{2n})$ a new, noncommutative algebra, a deformation of the algebra defined by the pointwise product.

We have up to now avoided giving the explicit form of the Wigner functions, or the discrete version of (3.1). The reason is that Fourier transforms are expansions in the irreducible unitary representations and in our case we should use actually not unitary representations, but projective representations. Recall that, in order to have a Weyl realization of the Heisenberg group, one needs to perform an extension of the group $Z_N \otimes Z_N$. The truly unitary representations would in reality be related to $Z_N \otimes Z_N$ and not to the Heisenberg group. We are now in position to circumvent this difficulty. We start by learning how to read the coefficients, or Wigner densities, and establishing as a rule that *densities never simply multiply, but always convolute in the twisted way given by* (3.7). Once this is kept in mind, we may use a unitary representation and write

$$A_W(\mathbf{r}) = \frac{1}{N} \Sigma_{\mathbf{m}} A^{\mathbf{m}} e^{i(2\pi/N)\, \mathbf{m}.\mathbf{r}} . \qquad (3.14)$$

This expression, we repeat, only makes sense if the rule (3.7) for twist-convolving the coefficients is used every time some multiplication is performed. $A_W(\mathbf{r})$ *with* $\mathbf{r} = (r_1, r_2) \in Z_N \otimes Z_N$ *may be seen as a function* on QPS (in reality, on its Fourier dual). Either we use operators (2.12) with the coefficients $A^{\mathbf{m}}$ taken as functions belonging to the usual commutative algebra of pointwise products and usual convolutions or we use Wigner representation (3.14) with $A^{\mathbf{m}}$ belonging to the twisted algebra. This point is of extreme importance, for it allows us to keep using usual Fourier transformations, while taking noncommutativity into account exclusively in the coefficients. We shall below (after equation 3.18)) give an alternative which is formally simpler. By using $\delta_{rm} = \frac{1}{N} \sum_k e^{i(2\pi/N)k(r-m)}$, (3.14) gives immediately

$$A^m = \frac{1}{N} \sum_r e^{-i(2\pi/N)(m.r)} A_W(r). \qquad (3.15)$$

It becomes also immediate to verify that the Wigner function corresponding to an hermitian operator $A = \frac{1}{N} \sum_m A^m S_m = \frac{1}{N} \sum_{m,n} A^{(-m)*} S_m$ is a real function.

Looking at (3.6) we recognize that $AB = \hat{F}[F^{-1}[A_W] *_c F^{-1}[B_W]]$. It follows $(AB) = \hat{F}^{-1}[AB] = [F^{-1}[A_W] *_c F^{-1}[B_W]]$ and consequently, from (3.4),

$$(AB)_W = F[[F^{-1}[A_W] *_c F^{-1}[B_W]]] = A_W \circ B_W, \qquad (3.16)$$

a twisted product in which the phase α_2 provides the twisting, i.e, the deformation of the function algebra. As $\alpha_2 \neq 0$, the "classical" dynamical quantities A_W, B_W in correspondence with quantum dynamical quantities multiply each other no more by the pointwise product, but by the noncommutative twisted product. If α_2 vanished, twisted convolution would turn into simple convolution and twisted product would reduce to pointwise product. But this is never the case and twisting is an inevitable mark of quantum behavior. With the twisted-convolution prescription we are able to circumvent a further mathematical complication, whose discussion we shall leave to the next section.

Let us consider an example in the case N = 2. It is immediate to obtain from (2.9) the Pauli matrices $\sigma_1 = S_{10}$, $\sigma_2 = S_{11}$ and $\sigma_3 = S_{01}$. We obtain the Wigner densities from the general operator expressions like $\sigma_1 = \frac{1}{2} \Sigma_m (2\delta^{m_1}{}_1 \delta^{m_2}{}_0) S_m$ and Wigner functions as their Fourier transforms, $(\sigma_1)_W(\mathbf{r}) = \frac{1}{2} \Sigma_m (2\delta^{m_1}{}_1 \delta^{m_2}{}_0)\, e^{i(2\pi/N)\mathbf{m}.\mathbf{r}}$, etc. The results are $(\sigma_1)_W(r_1, r_2) = e^{i\pi r_1}$, $(\sigma_2)_W(r_1, r_2) = e^{i\pi(r_1+r_2)}$ and $(\sigma_3)_W(r_1, r_2) = e^{i\pi r_2}$. The twisted convolutions are of the type $(\sigma_3 *_c \sigma_1)^{(i_1, i_2)} = \frac{1}{2} \Sigma_{m,n} (2\delta^{m_1}{}_0 \delta^{m_2}{}_1)(2\delta^{i_1 - m_1}{}_1 \delta^{i_2 - m_2}{}_0) e^{i(\pi/2)(i_1 m_2 - i_2 m_1)} = 2\delta^{i_1}{}_1 \delta^{i_2}{}_1 e^{-i(\pi/2)}$. The twisted product comes then as $((\sigma_3)_W \circ (\sigma_1)_W)(r_1, r_2) = i\, e^{i\pi(r_1+r_2)} = (\sigma_2)_W(r_1, r_2) = ((\sigma_3\, \sigma_1)_W)(r_1, r_2)$. This example illustrates the extreme simplicity which comes from discreteness. It also exhibits clearly the role of $Z_N \otimes Z_N$. But above all it shows how the noncommutativity is taken into account exclusively through the twisting.

It will not be too bad to repeat the main points. It is well known that the geometry of a manifold M is "encoded" in the properties of the algebra C(M) of functions defined on it. The very definition of a differentiable manifold, for example, is based on the differentiability of some special functions, the coordinate transformations. The subset $C^0(M)$ of continuous functions will contain information on general topological properties, but no information on the differentiable structure. All such information is, nevertheless, contained in the set $C^\infty(M)$ of indefinitely differentiable functions. Such function spaces are usually taken as algebras with the operation of pointwise product. Mathematically, they are simple cases of $*$-algebras. We have seen the emergence of noncommutative geometry related to QPS coming precisely from the fact that, in order to keep the quantum information, the functions representing quantum quantities - the Wigner functions - do not belong to an algebra built up with this simple commutative product, but require a twisted product. We have passed into a noncommutative $*$-algebra.

Mathematicians use such commutativity-breakings in *-algebras of functions on a manifold as the starting point to uncover noncommutativity in the manifold itself. We have said in the Introduction that in a noncommutative geometry coordinates also become noncommutative. After all we have seen, the pair of coordinates q, p in Quantum Mechanics is a paradigmatic example: they must multiply each other via the star product and their commutator coincides with that of the operators **q**, **p**.

Let us go back to (3.14). It represents a Fourier expansion in the basis $\{\varphi_{\mathbf{m}}(\mathbf{r}) = e^{i(2\pi/N)\mathbf{m}.\mathbf{r}}\}$ of functions on the lattice torus. This is actually a basis of $Z_N \otimes Z_N$ characters. The Wigner functions $(S_{\mathbf{m}})_W$ corresponding to the basis operators will be simply

$$(S_{\mathbf{m}})_W(\mathbf{r}) = \varphi_m(\mathbf{r}). \qquad (3.17)$$

Thus, in the Weyl prescription, the Wigner functions corresponding to the basic Schwinger operators are just the Fourier basic functions. This is one more quality of the symmetrized Schwinger basis. The correspondence between functions and operators is then complete. We may for instance find the "translations" in terms of Wigner functions of the operator properties given in section 2. With the help of $\frac{1}{N^2}\Sigma_r\varphi^*_{\mathbf{m}}(\mathbf{r})\varphi_{\mathbf{n}}(\mathbf{r}) = \delta_{\mathbf{m},\mathbf{n}}$, the properties involving the trace come up directly, like (3.15), $A^{\mathbf{m}} = \frac{1}{N}\Sigma_{\mathbf{r}}\varphi^*_{\mathbf{m}}(\mathbf{r})A_W(\mathbf{r})$. We shall only make explicit the main results. One finds the twisted product $(S_{\mathbf{r}})_W \circ (S_{\mathbf{u}})_W(\mathbf{q}) = e^{i\,\alpha_2(\mathbf{u},\mathbf{r})}(S_{\mathbf{r}+\mathbf{u}})_W(\mathbf{q})$, leading to the basic result

$$\varphi_{\mathbf{r}}(\mathbf{q}) \circ \varphi_{\mathbf{u}}(\mathbf{q}) = (S_r)_W(\mathbf{q}) \circ (S_u)_W(\mathbf{q}) = e^{i\,\alpha_2(u,r)}\varphi_{\mathbf{r}+\mathbf{u}}(\mathbf{q}). \qquad (3.18)$$

Notice that the characters, initially providing a unitary representation. give now a projective one. And this leads to another point of view concerning the use of the star product. Instead of convoluting the coefficients, we may write $A_W(\mathbf{r}) = \frac{1}{N}\Sigma_m A^m \varphi_m(\mathbf{r})$ and use directly (3.18). With the twisted product as the function algebra operation, the measure of its noncommutativity gives automatically the Moyal bracket for the Wigner functions

$$[\varphi_r , \varphi_m]_o (q) =$$

$$= [\varphi_r \circ \varphi_m - \varphi_m \circ \varphi_r] (q) = 2i\sin[\alpha_2(m, r)]\, \varphi_{r+m}(q). \qquad (3.19)$$

which is the Weyl-Wigner version of (3.8). The counterpart of (3.6) will be

$$(A_W \circ B_W)(q) = \Sigma_p (AB)^p \varphi_p(q) \qquad (3.20)$$

with the coefficients given by (3.7). Finally, (3 9) is translated into

$$[A_W , B_W]_o(q) = \frac{1}{N^2}\Sigma_m \Sigma_r A^m B^r\, 2i\sin[\alpha_2(r, m)]\varphi_{m+r} (q) =$$

$$= \frac{1}{N^2}\Sigma_p \Sigma_m A^m B^{p-m}\, 2i\sin[\alpha_2(p, m)]\varphi_p(q) \qquad (3.21)$$

We see that the parallelism is complete and that quantization may be seen (i) as a passage from classical dynamical quantities to operators or (ii) a deformation in the algebra of the coefficients or still (iii) as a deformation in the algebra of the basic functions, from pointwise product to twisted product. It will be seen in the following that the parallelism goes over to the Hamiltonian structure.

4. SOME INTERMEDIATE COMMENTS

It is convenient to make here a parenthesis and digress briefly on some mathematical topics before proceeding to the final assault to our main problem. The discussion, though not quite indispensable, will help us to get a better understanding both of what we have being doing and of the forthcoming results. In special, it will bring to light the main reasons for the particular simplicity of our case.

One of the most fascinating developments of the last decade has been the discovery of the relation of Hopf algebras (or bialgebras, or quantum groups) to Yang-Baxter equation and, consequently, to braid groups [19]. The simplest example of bialgebra in Physics appears in the sum of two angular momenta, usually written $J = J_1 + J_2$. This operator applies however on direct-sum kets $|\Psi\rangle = |\Psi_1\rangle \oplus |\Psi_2\rangle$, which shows that J should in reality be written $J = J_1 \otimes 1 + 1 \otimes J_2$ and is consequently an object belonging to a Hopf algebra. Further investigation has uncovered the relationship of Hopf algebras to an impressive number of seemingly disparate topics. It has established the existence of common points in as distinct subjects as statistical lattice models, integrability of non-linear equations, infinite algebras, the inverse scattering method, topology of low-dimensional manifolds, low-temperature superconductivity, non-abelian harmonic analysis, conformal models in field theory, noncommutativity geometry, quantum Hall effect, etc, etc. We shall of course ignore almost everything of all that and concentrate into a few topics of immediate interest to our theme.

There are many ways to approach Hopf algebras [20] besides the purely algebraic one. Among physicists, the most usual method involves group deformations [21], but we may also generalize the method used to introduce the classical groups as those transformation groups which preserve given sesquilinear forms [22]. Or still find a gate through free calculus, as in knot theory [23], where they make a brief act of presence in terms of Alexander matrices, but are immediately abelianized in the calculations leading to invariant polynomials. The main point is that the entries of the usual matrices appearing in "classical" subjects become noncommutative, although subject to certain rules. In very very rough lines, it happens the following. The entries t^i_j of usual matrices consist of real or complex numbers and consequently $t^i_j t^m_n = t^m_n t^i_j$. Matrices constitute Lie groups, that is, smooth manifolds. To each group element corresponds a matrix. Each point on a manifold, each matrix in the case, is entitled to have coordinates, a set of real numbers describing it. Thus, each matrix will have its coordinates, which are just the entries t^m_n if they are real and, if they are complex, their real and imaginary parts. If now we take the very entries of the matrices as noncommutative, the new "coordinates" will be of a new kind. Noncommutative geometry comes to the fore. The noncommutativy is written in the form $R^{rs}_{im} t^i_j t^m_n = R^{pq}_{jn} t^r_p t^s_q$, where the R^{ij}_{kl}'s are complex coefficients. Some restrictions, in the form of constraints on these coefficients, are imposed to ensure a minimum of "respectability" to the new algebraic structure which comes out. In particular, the imposition of associativity leads to the Yang-Baxter equation, whose form and relationship to braid groups are given in section VII. The resulting new structure is a Hopf algebra. Thus, "quantum groups" is a general name for some sets of matrices whose entries are themselves non-commutative. They are not groups at all but structures trying to generalize them. In our case, we shall find "hypermatrices", matrices whose entries are themselves matrices which can be expanded in Schwinger's basis.

We may say a few words in the same naïve style on another approach [24], which is the nearest to our purposes. It emphasizes the role of Fourier transformations and is concerned with harmonic analysis on groups. It takes as starting algebras the spaces of functions on groups. Recall classical Fourier transformations on the line or on Euclidean 3-space: they establish a duality between the space of functions on the original space and the space of the Fourier transforms. The original space is actually a translation group T and its dual $\hat{T}$ is the space of (equivalence classes of) unitary irreducible representations of T. In this classical, abelian case, the dual $\hat{T}$ is another group. It happens that, when the original group G is locally compact commutative and compact, or discrete, the dual set $\hat{G}$ is a locally compact commutative group, which is furthermore respectively discrete or compact. This is the "Pontriagin duality". Actually, the transformations take place between the respective group rings, to which the representations can be extended. Important examples of Fourier-dual spaces are: the group R of the real numbers, dual to itself; the circle S^1 and the group Z of integers are dual to each other; and, specially important for our purposes, the cyclic group Z_N is dual to itself. This is a first reason for the simplicity of our case: we remain in the same Z_N while going to and fro by Fourier transformations.

A great change comes up when the original group is noncommutative: the corresponding dual set is no more a group. Much is known for the case in which G is compact. The dual $\hat{G}$ is then a category, that of the finite dimensional representations of G, that is, a category of vector spaces (or an algebra of blocs). This new duality, between a group and a category, is called Tanaka-Krein duality. The reason for the special simplicity of abelian groups is that their unitary irreducible representations have dimension one and the tensor product of two such representations is another one-dimensional representation. Each such representation may be considered simply as a complex function f, f: G → C, g → f(g), with $f(g_1g_2) = f(g_1)f(g_2)$. Tensor product is then reduced to the simple pointwise product of functions and the set of inequivalent unitary irreducible representations is then itself a group. This is the property which does not generalize to the noncommutative case. The good question then is: is it possible to enlarge the notion of group to another object, so that its dual come to be an object of the same kind ? The complete answer has been found in the case of finite groups: the more general objects required are precisely Hopf algebras. So, quantum groups are that generalization of groups allowing for a "good" notion of Fourier duality.

We have in the previous section prepared ourselves, through the resource of absorbing the noncommutativity in the coefficients, to perform the Fourier transformation starting from the commutative group $Z_N \otimes Z_N$. The Heisenberg group is non-abelian, but with the twisted-convolution prescription we are able to circumvent this problem. The algebra of functions on $Z_N \otimes Z_N$ with multiplication given by "$*_c$" is Fourier-transformed into the algebra of functions on $Z_N \otimes Z_N$ with multiplication "o". One works as if the domain space were in both cases commutative groups. The prescription is a trick allowing one to use the simpler formalism of Pontrjagin duality, instead of working with Tanaka-Krein duality. Or, if one prefers, it allows one to avoid the explicit use of Hopf algebras. This is another reason for the simplicity of our case.

In the case $N = \infty$, we touch the subject of infinite algebras [25]. The $\{S_m\}$ basis has been a source of ideas in model building. It was extensively used by Floratos, first in the study of the (discretized) torus membrane [26], then to obtain special representations of the $GL_q(2)$ quantum group [27] and finally to relate Manin quantum space to Quantum Mechanics [28]. The double-graded algebra (3.8), with its "fermionic" anticommutator counterpart

$$[S_r , S_m]_+ = 2 \cos [\alpha_2(k; S_m, S_r)] S_{m+r},$$

have been used [29] in the $N = \infty$ case as a supersymmetric example of Kac-Moody type algebras.

But the main analogues to our case are the so called algebras of area-preserving transformation generators [30]. Let us recall the main points for the special case of a phase space. One starts by supposing that a set $\{\varphi_m\}$ of basic functions exists, in terms of which every function on the space may be expanded. Given the Poisson bracket $\{,\}_P$, the basis functions will establish an algebra $\{\varphi_m, \varphi_n\}_P = c^r{}_{mn}\, \varphi_r$. In Hamiltonian theory, each function $F = \frac{1}{N}\Sigma_m F_m \varphi_m$ on phase space will define a Hamiltonian field L_F which is such that $L_F\, G = \{F, G\}_P$ for all G. With the notation $\mathbf{x} = (x_1, x_2, \ldots , x_n, x'_1, x'_2, \ldots , x'_n)$ for the coordinates on phase space, $L_F = \Sigma_k \partial_k F \partial_{k'} - \partial_{k'} F \partial_k = (\partial^{\rightarrow} F) \times \vec{\partial}$ is the field generating the canonical transformations whose generating function is F. We may thus attach to the basis $\{\varphi_m(\mathbf{r})\}$ a basis $\{L_m\}$ of Hamiltonian fields defined by $L_m G = \{\varphi_m, G\}_P$ for all G. An algebra of differential operators results, with

$$[L_m, L_n] = c^r{}_{mn}\, L_r.$$

Notice that $L_m\varphi_n = \{\varphi_m, \varphi_n\} = c^r{}_{mn}\varphi_r$. The L_m's will generate canonical transformations, preserving the area (or volume) by Liouville theorem. The formalism is more general, as such area-preserving transformations may be defined also on manifolds which are not phase spaces. According to Fletcher [25], every infinite double-indexed Lie algebra like the above one may be put into the Moyal form (or its particular case, the Poisson form) *in some conveniently chosen function basis* . Actually, he shows that any (infinite double-indexed) bracket algebra may be transformed into Moyal's, provided only that it satisfies Jacobi identities. The systems of basic functions $\{\varphi_m(x)\}$ depend, of course, on which phase space is considered. Proposed basis are, for instance: $\{\varphi_m(x) = x_1^{m_1+1}x_2^{m_2+1}\}$ for the plane and $\{\varphi_m(x) = x_1^{m_1-1}e^{im_2x_2}\}$ for the cylinder. In reality, such basis are chosen so as to provide infinite algebras of proper interest. But the question which remains is: when we expand general functions in terms of such basis, how do we calculate the coefficients ? In order to have well-defined situations, functions should constitute an inner-product space, with an integration measure provided. Notice further that, in the regular representation needed, also the coefficients represent the group. The basis proposed above are local basis and do not allow expansions of functions on the plane and the cylinder, but only on limited domains of them. However, different domains mean actually different spaces. It is enough to recall one true basis for the plane, the quickly converging Hermite functions, to realize that the behaviour at infinity must be accounted for in some way. Furthermore, for quantization we need unitary characters and polynomials are not enough. Abraham and Marsden [31], for instance, have considered the Lie algebra of real-valued polynomials on E^{2n}, but showed soon after that no full quantization is possible. Non-unitary characters [32] may be very helful for other ends, such as analytic continuations to make contact with

Laplace transform, but quantization is related to true Fourier expansions, summations over (non equivalent) unitary irreducible representations. Thus, in what concerns quantization, the whole issue is not clear and deserves further study. The third reason for the simplicity of our case is that our basis $\{\varphi_{\mathbf{m}}(\mathbf{r}) = e^{i(2\pi/N)\mathbf{m}\cdot\mathbf{r}}\}$ for the torus and its limit do respect the unitarity requirement, and that is why we shall be able to obtain a full quantization. In what follows we shall do something similar to the infinite algebra procedure, but finding directly the Moyal bracket $\{f, g\}_{Moyal} = \lim_{x\to x'} [\frac{1}{s}\sin(s\vec{\partial}\times\vec{\partial}')f(x)g(x')]$, and with full justification coming from operator space via Weyl-Wigner transformations. Let us say once more that ours is the simplest case and that quantization on non-trivial spaces is a vast and very difficult problem [33]. Finally, we are here comitted to Weyl prescription. There are others [34], leading to other infinite algebras, and to other kinds of quantization. They are ignored in this paper.

5. MATRIX DIFFERENTIAL GEOMETRY

We have previously established that functions describing quantum phenomena are noncommutative. Noncommutativity of functions being a signal of geometrical noncommutativity, let us address ourselves to the differential geometrical aspects. We look for the Hamiltonian structure in the space of quantum operators, which on the lattice torus are finite matrices. What follows in this section is a simplified resumé of the DKM results. Let us start by reccalling some basic facts of algebra. Given the set M_N of NxN matrices of complex elements, the set $End(M_N)$ of all the endomorphisms (linear operators) $M_N \to M_N$ is an associative algebra and the set of its commutators, indicated by $[End(M_N)]$, is a Lie algebra. Any endomorphism D: $M_N \to$

M_N such that $D(ab) = (Da)b + a(Db)$ is a derivation. The Lie algebra $[End(M_N)]$ contains $D(M_N)$, the subspace of all the derivations of M_N, as a Lie subalgebra: the commutator of two derivations is a derivation. Each element $a \in M_N$ defines a derivation $ad(a) = ad_a$ by the adjoint action, $ad_a(b) = [a, b]$. Conversely, every derivative on M_N is of the form ad_a for some $a \in M_N$. We state these well known facts to emphasize the distinction between an algebra and its derived algebra. Perhaps the example which best illustrates this point comes from Classical Mechanics, where the derivative corresponding to a member of the algebra of the dynamical quantities, say $F(q, p)$, is the corresponding Hamiltonian field X_F, whose effect on another quantity G is given by the "commutator" involved, the Poisson bracket: $X_F(G) = \{F, G\}$. The Hamiltonian field X_F is the infinitesimal generator of the canonical transformation whose generating function is F.

On a differentiable manifold V, differential forms map fields into the space $C^\infty(V)$ of differentiable functions on V. $C^\infty(V)$ is, with the operation of pointwise multiplication, a commutative algebra. We shall find out a non-comutative geometry by replacing functions by matrices, the space $C^\infty(V)$ by M_N. Let us then introducce differentials in matrix space. Differential forms will become M_N-valued and some new properties will come forth.

A set of SU(N) matrices E_k, with $[E_j, E_k] = C^i{}_{jk}E_i$, provides a basis for the algebra M_N, and the derivative algebra will have a basis $\{e_k\}$, with $e_k = ad_{E_k}$. A general derivative X will be written as $X = X^k e_k$. In a way similar to differential forms, a whole graded algebra $\Omega = \oplus_p \Omega^p(M_n)$ of matrix-valued forms is given in the following way [35]. $\Omega^0(M_n)$ is identified to M_N itself. Just as the differential of a function f acts on a field X according to $df(X) = X(f)$, the space $\Omega^1(M_n)$ of 1-forms is introduced once a differential operator d is defined by

$$dM(X) = X(M) = ad_X M = [X, M]. \qquad (5.1)$$

For higher orders, $d : \Omega^p \rightarrow \Omega^{p+1}$ is given by

$$(p+1)(d\,\Xi)(e_1,e_2, \ldots, e_{p+1}) =$$
$$= e_1\Xi(e_2, e_3, \ldots, e_{p+1}) - e_2\Xi(e_1, e_3, \ldots, e_{p+1}) + \ldots + (-)^p e_{p+1}\Xi(e_1, e_2, e_4, \ldots, e_p) - \Xi([e_1, e_2], e_3, \ldots, e_{p+1}) + \Xi([e_1, e_3], e_2, \ldots, e_{p+1}) - \Xi([e_1, e_4], e_2, \ldots, e_{p+1}) + \ldots + (-)^p\, \Xi([e_1, e_{p+1}], e_2, \ldots, e_p) + \Xi(e_1, [e_2, e_3], \ldots, e_{p+1}) + \ldots + (-)^p\,\Xi(e_1, e_2, e_3, \ldots, [e_p,e_{p+1}]). \qquad (5.2)$$

This "d" is the well known derivative [36] used in the study of Lie algebra cohomology [37]. Each expression like $e_1\Xi(e_2, e_3, \ldots, e_{p+1})$ means here that e_1 is to operate as in (5.1) on the result of applying Ξ to $(e_2, e_3, \ldots, e_{p+1})$, that is, for instance, $e_1\Xi(e_2, e_3, \ldots, e_{p+1}) = [e_1, \Xi(e_2, e_3, \ldots, e_{p+1})]$. It comes out from this definition that $d^2 \equiv 0$. For example, $(d^2M)(e_1,e_2) = e_1(dM(e_2)) - e_2(dM(e_1)) - dM([e_1, e_2]) = e_1(e_2(M)) - e_2(e_1(M)) - [e_1, e_2](M) = 0$. Now that the functions of usual differential calculus have been replaced by matrices, notice that $dM\,N\,(X) = dM(X)\,N = [X, M]\,N \neq N\,dM(X) = N\,[X, M]$, that is, $dM\,N \neq N\,dM$.

An interior product is provided by an antiderivation i_X corresponding to each derivation X of the algebra. It is given by the requirement that

$$(i_X\,\Xi)(X_1, X_2, \ldots, X_{p-1}) = \Xi(X, X_1, \ldots, X_{p-1}) \qquad (5.3)$$

hold for all sets $\{X_1, X_2, \ldots, X_{p-1}\}$ of derivatives. Then $L_X = d \circ i_X + i_X \circ d$ is a derivation, the matrix analog to the Lie derivative. Properties alike to those valid in differential calculus, such as $d \circ L_X = L_X \circ d$, $\{i_{X_1}, i_{X_2}\} = 0$, $[L_{X_1}, L_{X_2}] = L_{[X_1, X_2]}$, etc, may be shown to keep holding.

It is immediate that $dE_k(e_j) = e_j(E_k) = [E_j, E_k] = C^i{}_{jk}E_i$. The forms dE_k constitute a basis for $\Omega^1(M_n)$. Higher order multilinear mappings would constitute the analogous to higher order forms, but there is an important difference: due to matrix noncommutativity, $dM\, dN \neq - dN\, dM$ in general. The basis $\{dE_k\}$ has this defect. A far more convenient basis is formed by those Θ^j which are dual to the e_j's,

$$\Theta^j(e_k) = \delta^j{}_k. \qquad (5.4)$$

In this basis,

$$dE_k = C^i{}_{jk}E_i\Theta^j. \qquad (5.5)$$

Now the Θ^j's, unlike the dE_k's, do satisfy

$$\Theta^i\, \Theta^j = - \Theta^j\, \Theta^i, \qquad (5.6)$$

and furthermore

$$d\, \Theta^i = - \tfrac{1}{2} C^i{}_{jk}\Theta^j\, \Theta^k, \qquad (5.7)$$

a formula of Maurer-Cartan type.

Introducing the "canonical form"

$$\Theta = E_j\Theta^j, \qquad (5.8)$$

DKM have shown that the two-form

$$\Omega = d\Theta \qquad (5.9)$$

provides a natural symplectic structure on the matrix space, defining consequently a Poisson bracket. In Classical Mechanics, the Poisson bracket of two dynamical functions F and G equals the symplectic form applied to the corresponding hamiltonian fields X_F and X_G: $\{F, G\} = \Omega(X_F, X_G) = - X_F(G) = X_G(F) = dF(X_G) = - dG(X_F)$. On the matrix space, to each $A \in M_N$ will correspond a hamiltonian field $X_A \in D(M_N)$ which is such that $\Omega(Y, X_A) = Y(A) = dA(Y)$ for all $Y \in D(M_N)$. The matrix Poisson bracket is defined by

$$\{A, B\}_M = \Omega(X_A, X_B) . \qquad (5.10)$$

Due to linearity, it is enough to examine the relations for the basis members. Thus, $\{E_i, E_j\}_M = \Omega(e_i, e_j) = dE_j(e_i) = e_i(E_j) = i[E_i, E_j]$ and in consequence $\{A, B\}_M = [A, B]$. We shal find it convenient, by reasons of hermiticity, to define the derivatives as $X(M) = ad_{iX}M = i[X, M]$ in the following. In this convention,

$$\{A, B\}_M = i\,[A, B]. \qquad (5.11)$$

Thus, the Poisson bracket coming out from the natural Hamiltonian structure in a matrix manifold is just the commutator. We shall in the following show that, despite some peculiarities, once the formalism is applied to the quantum operator algebra, this symplectic structure is exactly that provided by the cocycle α_2 and the Poisson bracket is directly related to the Moyal bracket. Many more things may be introduced on matrix space, such as integration, coderivative and the Laplacian, connections and their curvatures. Also general theorems like the Hodge decomposition have been proved by DKM, but we shall not need them here. We only remark that on Manin space a similar differential calculus has been defined [38].

6. QUANTUM SYMPLECTIC GEOMETRY

Excluding the unit $S_0 = S_{(0,0)}$, the S_m's are n^2-1 unitary traceless matrices and may be used as a basis for the algebra of the special unitary group SU(N). The dynamical quantities, as expressed in (2.12), belong to the algebra M_N. In reality, the S_m's generate a subgroup of the complex linear group GL(N,C) with the center as commutator subgroup. Taking them as a basis for the algebra, we consider the related derivative algebra generated by the operators $e_m = ad(iS_m) = ad_{iS_m}$ (for all $S_m \neq S_0$). Using the Jacobi identity, we find that, for all S_k, $[e_r, e_m](S_k) = 2\sin[\alpha_2(r, m)]\, e_{m+r}(S_k)$, so that the general commutators are

$$[e_r, e_m] = 2\sin[\alpha_2(r, m)]\, e_{m+r}. \tag{6.1}$$

The structure coefficients

$$C^p{}_{rm} = 2\sin[\alpha_2(r, m)]\, \delta^p{}_{m+r} \tag{6.2}$$

(where $\delta^p_{m+r} = \delta^{p_1}_{m_1+r_1}\delta^{p_2}_{m_2+r_2}$) have the symmetry $C^p{}_{rm} = C^r{}_{-m,p}$ and define a Cartan-Killing metric,

$$k_{rs} = -\frac{1}{2N^2}\Sigma_m\Sigma_p\, C^p{}_{rm}C^m{}_{sp} = \delta_{r,-s}\,. \tag{6.3}$$

This has an obvious relation to the metric (2.11): $k_{r,-s} = \delta_{r,s} = \frac{1}{N}\mathrm{tr}(S_r S^\dagger_s) = g_{rs}$. It is important to keep in mind that all numbers, also in the Kronecker deltas, are mod N and that, for all that concerns the derivative fields, the vanishing index $\mathbf{0} = (0, 0)$ is excluded. For this reason we must use an adaptation of (2.18) in the above calculations: if we indicate by Σ', with primed indices, the summation excluding the (0, 0) contributions, it becomes

$$\sum'_{m} e^{2i\,\alpha_2(m,\, r\text{-}s)} = N^2\delta^{r,\,s} - 1. \qquad (6.4)$$

The metric k_{rs} establishes a relationship with the hermitian conjugates, which furthermore allows an improvement in notation. In fact, we verify that, defining $S^m = k^{mn}S_n$, such "covariant" versions of the "contravariant" S_m's are just their hermitian conjugates:

$$S^m = k^{mn}S_n = S_{-m} = S^{\dagger}_{m}.$$

Thus, (2.13) is also $A^m = \mathrm{tr}\,[S^m A]$.

We introduce a basis of 1-forms θ^r dual to the e_r,

$$\theta^m(e_r) = \delta^m{}_r\, I$$

which satisfy

$$\theta^m\, \theta^r = -\,\theta^r\, \theta^m. \qquad (6.5)$$

It is immediately seen that

$$dS_m\,(e_r) = e_r(S_m) = [iS_r, S_m] = C^p{}_{rm}S_p = 2\sin[\alpha_2(r, m)]S_{m+r},$$

so that

$$dS_m = S_p\, C^p{}_{rm}\, \theta^r = \Sigma_r\, 2\sin[\alpha_2(r, m)]S_{m+r}\theta^r. \qquad (6.7)$$

Calculating $d\theta^p\,(e_m, e_r)$ by (5.2), we find

$$d\theta^p = -\tfrac{1}{2}\, C^p{}_{rm}\, \theta^r\, \theta^m = \Sigma_m 2\sin[\alpha_2(p, m)]\; \theta^m\; \theta^{p-m}. \qquad (6.8)$$

We define then the matrix-valued 1-form

$$\Theta = \Sigma_r S_r \theta^r \qquad (6.9)$$

and the symplectic two-form

$$\Omega = d\Theta . \qquad (6.10)$$

It follows that

$$\Omega = S_p C^p{}_{mr}\, \theta^m \theta^r = \Sigma_p \Sigma_m 2\sin[\alpha_2(m, p)]\; S_p \theta^m\; \theta^{p-m} ,$$

or equivalently

$$\Omega = \Sigma_r \Sigma_m \sin[\alpha_2(r, m)]\; S_{m+r}\, \theta^r\; \theta^m , \qquad (6.11)$$

or still

$$\Omega = \tfrac{1}{2} \Sigma_r \Sigma_m \Omega_{rm}\, \theta^r \theta^m . \qquad (6.12)$$

Here we have isolated the components

$$\Omega_{rm} = C^p{}_{rm} S_p = i\, [S_r, S_m] \qquad (6.13)$$

in terms of which

$$e_m(A) = \tfrac{1}{N} \Sigma_r \Omega_{mr} A^r ; \qquad (6.14)$$

$$[A, B] = - \frac{i}{N^2} \Sigma_{r,m} A^r \Omega_{rm} B^m ; \qquad (6.15)$$

$$dS_m = \Sigma_r \Omega_{rm} \theta^r ; \qquad (6.16)$$

$$dA = \tfrac{1}{N} \Sigma_{r,m} A^r \Omega_{mr} \theta^m ; \text{ etc.}$$

We may form a matrix with entries Ω_{rm}, $\hat{\Omega} = (\Omega_{rm})$, which will be the analogue of the classical symplectic matrix. $\hat{\Omega}$ will be a hypermatrix, a matrix each entry of which is itself a matrix, and this is an intimation of a Hopf algebra.

The hypermatrix $\hat{\Omega}$ has actually an inverse, defined as a hypermatrix $\hat{r}$ with the properties $\hat{r}\,\hat{\Omega} = \hat{\Omega}\,\hat{r} = \hat{1}$. One verifies using (6.4) that

$$r^{ij} = \frac{i}{N^2} S^j S^i = \frac{i}{N^2} e^{i\alpha_2(i,j)} S^{i+j} = \frac{i}{N^2} e^{i(\pi/N) i \times j} S^{i+j} . \qquad (6.17)$$

We find also

$$\theta^j = -\frac{1}{N} \Sigma_m r^{jm} dS_m = \frac{1}{N} \Sigma_m dS_m r^{mj} . \qquad (6.18)$$

We are used to the fact that the inverse to an antisymmetric matrix is another antisymmetric matrix. This is not necessarily true for hypermatrices, and indeed r^{ij} is not antisymmetric. Using (6.17), we find

$$r^{ij} = e^{2i\alpha_2(i,j)} r^{ij}, \qquad (6.19)$$

and (6.13) tell us that

$$r^{ji} - r^{ij} = \Omega^{ij} = k^{im} k^{jn} \Omega_{mn} . \qquad (6.20)$$

Given any field $X = \frac{1}{N} \Sigma_m X^m e_m$, the interior product $i_X\, \theta = \theta(X) = \frac{1}{N} \Sigma_m S_m \theta^m(X) = \frac{1}{N} \Sigma_{m,n} S_m X^m$. We verify easily that $d\, i_X\, \theta = -\, i_X \Omega$. Thus, the Lie derivative of θ vanishes for all X. Then $L_X\, \Omega = L_X \circ d\theta = d \circ L_X\, \theta = 0$, and consequently, in classical terminology, all fields X are strictly hamiltonian [39]. The operators of the form $\theta(X) = \frac{1}{N} \Sigma_m S_m X^m$ play the role of generating functions: $d[\theta(X)]\,(Y) = \Omega(X, Y) = -\, d[\theta(Y)]\,(X)$ for all X, Y. This corresponds to a result of I, where the fundamental cocycle α_2 has been shown to vanish under the action of the Lie derivative defined by any transformation.

Conversely, to operators $\mathbf{A} = \frac{1}{N}\Sigma_m A^m S_m$ and $\mathbf{B} = \frac{1}{N}\Sigma_r B^r S_r$, will correspond hamiltonian fields $X_A = -\frac{i}{N}\Sigma_m A^m e_m$ and $X_B = -\frac{i}{N}\Sigma_r B^r e_r$ [the factors (-i) being necessary if the coefficients are to remain the same of (3.10)]. The corresponding "Poisson bracket" is defined by (5.10) and it follows that

$$\{\mathbf{A}, \mathbf{B}\}_M = \frac{2}{N^2}\Sigma_p\Sigma_m A^m B^{p-m} \sin[\alpha_2(\mathbf{m}, \mathbf{p})]\, S_p. \qquad (6.21)$$

A glimpse at (3.9) confirms (5.11),

$$\{\mathbf{A}, \mathbf{B}\}_M = i\,[\mathbf{A}, \mathbf{B}], \qquad (6.22)$$

and we can read the (discrete version of the) Moyal bracket components in (6.21). We see thus that the DKM symplectic form gives directly the Moyal bracket. Concerning the relationship between α_2 and Ω, we might say that α_2 stands to Ω as the Poisson bracket stands to Moyal's. As a last point, it is easy to verify that $[e_m, e_r]\,(S_p) = d\{S_m, S_r\}\,(e_p) = e_p(\{S_m, S_r\})$, from which follows that

$$[X_A, X_B]\,(C) = d\{\mathbf{A}, \mathbf{B}\}\,(X_C) \qquad (6.23)$$

for all operators **A**, **B**, **C**. This means that the matrix Poisson bracket of two operators works as the generating function of the commutator of the respective derivative fields. In reality, the classical formula

$$\{\mathbf{A}, \mathbf{B}\} = \Omega\,(X_A, X_B) = e_i(B)\, r^{ij}\, e_j(A) \qquad (6.24)$$

keeps holding here. We must only be attentive to the fact that, as each factor is now a matrix, this expression holds only in that precise given order.

Equation (6.22) does not come as a real surprise. It has a clear analogue in Classical Mechanics: suppose the Liouville canonical form s, (which is such that the symplectic form is $\Omega = -d\sigma$) has vanishing Lie derivatives with respect to all members of a field basis $\{e_k\}$, such that $[e_i, e_j] = c^k{}_{ij}e_k$. Then the fields have global generating functions $\sigma(e_k)$ and their Poisson algebra just mimic the field algebra, $\{\sigma(e_i), \sigma(e_j)\} = c^k{}_{ij}\sigma(e_k)$. However, this is not always the case in Classical Mechanics. For classical systems in general Hamiltonian fields are only *locally* related to a generating function, while we have found above that all fields are *strictly* Hamiltonian in the quantum case. In classical systems, the commutator of two fields is not necessarily generated by the Poisson bracket of the respective generating functions [40]. The difficulty in the classical case comes from the necessity of global fields. Basically, it supposes the phase space to be a parallelizable manifold, such as a Lie group. Here, by taking a case whose semiclassical continuum limit is purely euclidean, we have avoided such difficulties of the general classical case.

Let us now quickly examine the Weyl-Wigner counterparts of all that. With the star product, the functions constitute a unit algebra, as $\varphi_0(p) = \varphi_{(0,0)}(p) = 1$; $\varphi_r(p) \circ \varphi_0(p) = \varphi_r(p)$, etc. We could obtain the whole structure without further resource to the operator version, through the use of the derivatives $L_r\varphi_m(q) = \{\varphi_r(p), \varphi_m(p)\}_{Moyal}$ and their dual forms α^s such that $\alpha^s(L_r) = \delta^s{}_r$. We shall however prefer to use the parallelism of both formalisms. The Weyl-Wigner versions of the symplectic form entries are

$$\omega_{ij}(q) = (\Omega_{ij})_W(q) = \Sigma_p C^p{}_{ij}\varphi_p(q) =$$

$$= L_r\varphi_m(q) = \{\varphi_r(q), \varphi_m(q)\}_{Moyal} \; ; \qquad (6.25)$$

and of the inverse,

$$\rho^{ij}(q) = (\Gamma^{ij})_W(q) = \frac{i}{N^2}\, e^{i\alpha_2(i,j)}\varphi^*{}_{i+j}(q)\,. \qquad (6.26)$$

Good use of (6.4) leads indeed to

$$\Sigma_{j'}\rho^{ij}(q) \circ \omega_{jk}(q) = \delta^i{}_k. \qquad (6.27)$$

As expected, we find also that

$$d\varphi_m(p)(L_r) = L_r\varphi_m(p) = \{\varphi_r(p), \varphi_m(p)\}_{Moyal} = \omega_{rm}(p), \qquad (6.28)$$

so that

$$d\varphi_m(p) = \Sigma_r\{\varphi_r(p), \varphi_m(p)\}_{Moyal}\, \alpha^r = \Sigma_r\omega_{rm}(p)\, \alpha^r\,. \qquad (6.29)$$

On the other hand, we have $\alpha^r(p) = -\Sigma_m\rho^{rm}(p)d\varphi_m(p)$ and

$$\Sigma_r\Sigma_m\omega_{rm}(p)\alpha^r(p)\alpha^m(p) = \Sigma_m d\varphi_m(p)\, \alpha^m(p) =$$

$$= -\Sigma_m\Sigma_k d\varphi_m(p)\rho^{mk}(p)\, d\varphi_k(p)\,.$$

It follows then from $0 = d^2\varphi_m(p) = L_iL_r\varphi_m(p)\alpha^i(p)\alpha^r(p) + [L_r\varphi_m(p)]$ $d\alpha^r(p)$ that

$$d\alpha^r(p) = -\frac{1}{2}\, \Sigma_i\Sigma_j C^r{}_{ij}\alpha^i(p)\alpha^j(p)\,, \qquad (6.29)$$

and

$$\Sigma_m d(\varphi_m(p)\alpha^m(p)) = \tfrac{1}{2}\,\Sigma_i\Sigma_j\omega_{ij}\alpha^i(p)\alpha^j(p). \qquad (6.30)$$

These are the Weyl-Wigner versions of (6.12) and (6.10).

There is a complete isomorphism of differential algebras, a direct correspondence between the operator algebra and the noncommutative algebra of Wigner functions. We may use either the hypermatrices with matrix entries or usual matrices with entries in the "star-product algebra". The statement that the dynamical quantities are the same in Classical and in Quantum Mechanics, only (the product defining) their algebra being different, is thereby vindicated.

7. THE QUANTUM WEB: BRAIDING BEHIND PHASE SPACE

We shall examine now the presence, in the above formalism, of solutions of Yang-Baxter equations, or, equivalently, of solutions of the relations defining braid groups.

We have been finding matrices the entries of which are themselves NxN matrices. The natural abodes of such hypermatrices are direct product spaces. Recall that in direct product index notation the product A ⊗ B of two matrices has entries

$$\langle ij|A \otimes B|mn\rangle = \langle i|A|m\rangle \langle j|B|n\rangle ,$$

the direct product of three matrices will have elements

$$\langle ijk|A \otimes B \otimes C|mnr\rangle = \langle i|A|m\rangle \langle j|B|n\rangle\langle k|C|r\rangle ,$$

and so on. If $R = A \otimes B$, and E is the N x N identity matrix, we use the compact writing

$$R_{12} = A \otimes B \otimes E,$$

$$R_{13} = A \otimes E \otimes B,$$

$$R_{23} = E \otimes A \otimes B, \text{etc.}$$

Matrix elements are $\langle ij|R|mn\rangle = R^{ij}{}_{mn}$ and, more important for what follows, $\langle ijr|R \otimes E|mns\rangle = \delta^r{}_s R^{ij}{}_{mn}$.

Representations of the braid groups [41] come out very simply in this notation. Recall that, for the n-strand group, we may use as basis a set $\{\sigma_j\}$ obeying the relations

$$\sigma_i \, \sigma_{i+1} \, \sigma_i = \sigma_{i+1} \, \sigma_i \, \sigma_{i+1}, \text{ for } \; i = 1, 2, \ldots, n\text{-}1;$$

$$\sigma_i \, \sigma_j = \sigma_j \, \sigma_i \;\text{ for } |i\text{-}j| \geq 2.$$

The symmetric group S_n is the special case coming up when all these σ_i's satisfy the additional conditions $(\sigma_i)^2 = 1$. To see how the direct product representation works, look for the braid group generators σ_1 and σ_2 as $\sigma_1 = B_{12}$ and $\sigma_2 = B_{23}$, B being some direct product as above. Then,

$$\langle kji| \, \sigma_1 \, \sigma_2 \, \sigma_1 |mnr\rangle = \langle kji| \, B_{12}|abs\rangle\langle abs|B_{23} \, |ucx\rangle\langle ucx|B_{12}|mnr\rangle =$$

$$= B^{kj}{}_{ab} B^{bi}{}_{cr} B^{ac}{}_{mn};$$

on the other hand,

$$\langle kji| \, \sigma_2 \, \sigma_1 \, \sigma_2 |mnr\rangle = \langle kji| \, B_{23}|abs\rangle\langle abs|B_{12} \, |ucd\rangle\langle ucd|B_{23}|mnr\rangle =$$

$$= B^{ji}{}_{bd} B^{kb}{}_{mc} B^{cd}{}_{nr}.$$

The braid equation $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$, or

$$B_{12}B_{23}B_{12} = B_{23}B_{12}B_{23}, \qquad (7.1)$$

becomes

$$B^{kj}{}_{ab}B^{bi}{}_{cr}B^{ac}{}_{mn} = B^{ji}{}_{ca}B^{kc}{}_{mb}B^{ba}{}_{nr}\,. \qquad (7.2)$$

Given B satisfying this equation, the matrix $R = PB$, defined by the index permutation P fixed by $R^{ij}{}_{mn} = B^{ji}{}_{mn}$, will satisfy the Yang-Baxter equation,

$$R^{jk}{}_{ab}R^{ib}{}_{cr}R^{ca}{}_{mn} = R^{ij}{}_{ca}R^{ck}{}_{mb}R^{ab}{}_{nr}\,, \qquad (7.3)$$

which in compact notation is

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}\,. \qquad (7.4)$$

There is a direct realization of these relations in terms of the metric k_{ri}: $R^{ij}{}_{mn} = (\delta^i{}_n\delta^j{}_m + a\, k^{ji}\, k_{mn})$ is a solution provided the complementary condition $a + a^{-1} + N^2 = 0$ is satisfied. A quantum group may be introduced in a way analogous to a classical group, as that Hopf algebra of hypermatrix transformations preserving a given bilinear form. The above R matrix is just the one related to the preservation of the metric [42]. The complementary condition is better understood in terms of braids: it corresponds to the equality of two braids differing only by a Reidemeister move (of type II), by which one changes the relative position of two strands without any "passing through" between them.

We shall however show the presence of another kind of solution, which is non-trivial due to the multiplicity hidden in (2.15). When we write $S_r\, S_m = e^{2i\alpha_2(m,\, r)} S_m\, S_r$, we are not actually telling the whole story: this gives simply the commutation condition for two fixed matrices

S_r and S_m. But $S_r S_m$ is actually related by phases to any other product of pairs of basis matrices such that the sum of the indices equals $\mathbf{r} + \mathbf{m}$.

Let us define the set $B = \{B^{ij}{}_{kl}\}$ of commutation coefficients between the members of the algebra basis by:

$$S^i S^j = \Sigma_{m,n} B^{ij}{}_{mn}\, S^m S^n . \qquad (7.5)$$

Then the general solution is

$$B^{ij}{}_{mn} = \delta^{i+j}{}_{m+n}\, e^{i[\alpha_2(m,\,n)\, -\, \alpha_2(i,\,j)]} . \qquad (7.6)$$

We may check directly that this does respect equation (7.2), but we may also use another way which, though lengthier, shows clearly how representations of the braid groups come up very simply from any associative algebra. We shall here follow a simple method leading directly to the braid equations for the Schwinger basis, but it will be clear that the procedure is valid for more general cases. The only specificity will be that here all indices are double. Let us impose coherence of the commutation coefficients with associativity and compare two different series of ways of bracketing the S^i's:

$$S^k S^j\; S^i = (S^k S^j) S^i = B^{kj}{}_{ab} S^a (S^b S^i) = B^{kj}{}_{ab} B^{bi}{}_{cr} (S^a S^c) S^r =$$

$$= B^{kj}{}_{ab} B^{bi}{}_{cr} B^{ac}{}_{mn} S^m\, S^n S^r ,$$

on the other hand,

$$S^k S^j\; S^i = S^k (S^j\; S^i) = B^{ji}{}_{ca} (S^k S^c)\, S^a =$$

$$= B^{ji}{}_{ca} B^{kc}{}_{mb}\, S^m\, (S^b\, S^a) = B^{ji}{}_{ca}\, B^{kc}{}_{mb}\, B^{ba}{}_{nr}\, S^m\, S^n S^r .$$

Consequently,

$$B^{kj}{}_{ab}B^{bi}{}_{cr}B^{ac}{}_{mn} = B^{ji}{}_{ca}\, B^{kc}{}_{mb}\, B^{ba}{}_{nr}$$

appears as an acceptable condition. This is just the braid equation (7.1), though here with double indices.

Notice to begin with the role of B as a kind of "grandmatrix", representing the exchange of matrices. More interesting is the fact that the subindices reflect just the order used in the bracketing: we have, to obtain the left-hand side of $B_{12}B_{23}B_{12} = B_{23}\, B_{12}\, B_{23}$, taken first the 1rst and the 2nd of the S's (which finally yield B_{12}), then the 2nd and the 3rd (to obtain B_{23}), finally again the 1rst and the 2nd of the S's (to get the final B_{12}). And quite analogously for the right-hand side. Each S is related to a strand of the braid.

Equation (7.1) is a presentation of the 3rd braid group B_3, corresponding to only three strands. For higher number of strands, we may consider higher order direct products and obtain elements of higher order braid groups. Adjacent bracketings lead to expressions of the type $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ above. Non contiguous bracketings will then lead to exchanges in which the S^j's ignore each other, and thus to expressions of the type $\sigma_k\sigma_j = \sigma_j\sigma_k$ for $|i-k| \geq 2$. Untouched S^j's correspond to non-exchanging strands.

Let us proceed to a inspection of the matrix elements of (7.6): (i) in the diagonal, m = i and n = j, we have just the identity; (ii) the case m = j; n = i, with

$$B^{ij}{}_{kl} = \delta^j{}_k\delta^i{}_l e^{2i\alpha 2(j,\, i)} = \delta^j{}_k\delta^i{}_l\, e^{2i\alpha 2(k,\, l)}\, , \qquad (7.7)$$

corresponds to (2.15). This case is trivial in the sense that $B^2 = I$, so that it is not a representation of the braid group, but one of the symmetric group which is meant. The other cases are non-trivial and come up from the mentioned fact that the product $S^i S^j$ is, up to the compensating phases of (7.6), equal to any other product $S^m S^n$, with the only proviso that $i + j = m + n$. The existence of non-trivial matrices B is directly related to the fact that r^{ij} is not simply antisymmetric. Indeed, (7.5) is the same as $r^{ji} = \Sigma_{m,n} B^{ij}{}_{mn}\, r^{nm}$. If the matrix B were a negative-diagonal hypermatrix, then $B^2 = I$ and only the symmetric group would be present. Braiding as found above is consequently a specifically quantum effect.

A negative point from the practical point of view is that the matrices are of very high orders. Because indices are double, the matrices are $N^4 x N^4$, too large for direct manipulations. For N = 2, B is a 16x16 matrix; for N = 3, it is a 81x81 matrix, etc. It remains anyhow that we have found braids at work in the backstage of Quantum Mechanics, although their meaning and importance (if any) are yet to be examined.

8. FINAL REMARKS

As said at the end of section 2, we have started with QPS as we would have started with the simple pair (q, p) in Classical Mechanics, and finished by finding a symplectic structure in the very algebras of general operators and/or Wigner functions, which appears directly as the quantum version of the algebra of general dynamical functions. Quantization appears directly as a deformation of the algebra of classical observables, as a manifestation of the non-commutative character the geometry of physical observables. Perhaps the most striking aspect of the formalism above is precisely this directness, an obvious consequence of working at the same time with Weyl-Wigner transformations and with matrices. The expression of each operator is automatically a discrete version of the Weyl prescription relating classical dynamical quantities to their quantum representatives through their Fourier components. The advantages of the discrete formalism come from its finiteness and formal simplicity.

The differential geometric treatment provides a clear distinction between the operators and the corresponding derivative fields (the true "quantum derivations") and clarifies some aspects of I. It was noticed there that the relation between the fundamental cocycle α_2 and the classical symplectic form was of exponential type. We see now that it is the sine in Moyal's bracket which establishes the exact correspondence. The DKM natural symplectic structure on the differential operator space is related to the Moyal bracket, thereby vindicating those authors' assertion that "quantum mechanics is noncommutative symplectic geometry".

Structures of Classical Mechanics come in principle from quantum mechanical structures, although something may be lost or distorted in the limiting process. From all that was said here, the classical symplectic structure stems in the long run from the cocycle α_2 and is, as a consequence, a relic signal of the basic noncommutative character of quantum mechanical geometry. We have in the introduction started with the set of dynamical functions F(q, p) defined on classical phase space. With the "classical" pointwise product that set constitutes a trivial algebra. Quantization changes neither the functions nor their set, but changes their product. And even the mathematically "strange" Poisson bracket, "strange" because it did not come from an associative product, comes out now as a limiting case of the quantum bracket, thus as part of the quantum heritage of Classical Mechanics.

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