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Statistical Mechanical Properties of $Z_N^{\otimes n-1}$ Broken Model

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Abstract

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The formulation of the $Z_N^{\otimes n-1}$ broken model is given. The partition function is obtained under an assumption of analyticity. A string function for the local state probabilities has also been found.

Аннотация

С.М.Сергеев. Статистико - Механические Свойства $Z_N^{\otimes n-1}$ Нарушенной Модели: Препринт ИФВЭ 92-46. — Протвино, 1992. — 9 с., библиогр.: 8.

Дана формулировка $Z_N^{\otimes n-1}$ нарушенной модели. Статистическая сумма получена в предположении аналитичности. Также построено представление для вероятности локального спинового состояния в виде одномерной конфигурационной суммы.

1. Introduction

The $Z_N^{\otimes n-1}$ broken model was obtained in [1]. The Boltzmann weight S of this model, regarded as an R - matrix, acts in a $(N^{n-1})^2$ - state space and intertwines two L - operators for the Belavin $sl(n)$ R - matrix.

In this paper we investigate the statistical mechanical properties of the model. Note, that we didn't find any physical regime for the $Z_N^{\otimes n-1}$ model except for the $n=2$ case. Nevertheless the numerical calculations show that the partition function of the model for real spectral parameters inside the region (3.11) is positive and agrees with (3.10).

Throughout the paper we apply the Baxter's technique [2] for the calculation of the free energy and the local state probabilities (LSP).

The plan of the paper is following. In section 2 we give the formulation of the $Z_N^{\otimes n-1}$ model. In section 3 the free energy for the elliptic parametrization and for some triginometric limits are found. The string function and the matrix representation for the LSP are described in section 4.

2. Formulation of the model

First of all introduce the following notations. Let ε_μ, $\mu = 0, ..., n-1$ be an orthonormal basis of R^n, so that $\Pi = \{\alpha_\mu : \alpha_\mu = \varepsilon_{\mu-1} - \varepsilon_\mu, \mu = 1, ..., n-1\}$ be the basis of the A_{n-1} root system. Let

$$Q_N = \{a,\ a = \sum_{\mu=1}^{n-1} \tilde{a}_\mu \alpha_\mu = \sum_{\mu=0}^{n-1} a_\mu \alpha_\mu;\ 0 \leq \tilde{a}_\mu < N\} \tag{2.1}$$

be the Z_N - span of Π. Hereafter we shall use the Jacoby function θ_1 in the form

$$\theta_1(z,\tau) = \sum_{k \in Z} e^{i\pi\tau(k+1/2)^2 + 2\pi i(k+1/2)(z+1/2)} \tag{2.2}$$

Define a function $w_k(\sigma,u)$, $k,\sigma \in Z$, by

$$w_k(\sigma,u) = \begin{cases} \prod_{j=0}^{\sigma-1} \theta_1(\frac{k/n+j-u}{N},\tau) & , \ \sigma > 0 \\ \left(\prod_{j=0}^{-\sigma-1} \theta_1(\frac{k/n-j-1-u}{N},\tau)\right)^{-1} & , \ \sigma < 0 \end{cases} \tag{2.3}$$

The function $w_k(\sigma,u)$ is determined uniquelly by the relation

$$\frac{w_k(\sigma+1,u)}{w_k(\sigma,u)} = \theta_1(\frac{k/n+\sigma-u}{N},\tau). \tag{2.4}$$

Note that

$$w_k(\sigma,u) = \frac{(-)^\sigma}{w_{-k}(-\sigma,-1-u)}. \tag{2.5}$$

Define the next functions $W(m,n;u)$ and $\bar{W}(m,n;u)$ by

$$W(m,n;u) = \prod_{\mu,\nu=0}^{n-1} w_{\mu-\nu}(m_\mu - n_\nu;u), \tag{2.6a}$$

$$\bar{W}(m,n;u) = \frac{1}{W(m,n;u)}, \tag{2.6b}$$

$m,n \in Q_N$, $u = p - q$ being the spectral parameter (Fig. 1).

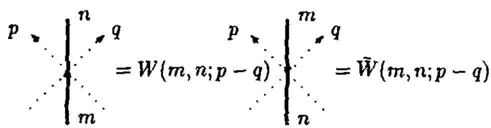

Fig. 1.

From (2.5), it follows that

$$\bar{W}(m,n;u) = W(n,m;-1-u) \tag{2.7}$$

The last function that we have to define is $D(m)$, $m \in Q_N$:

$$D(m) = \prod_{\mu<\nu} \frac{\theta_1(\frac{(\mu-\nu)/n+m_\mu-m_\nu}{N},\tau)}{\theta_1(\frac{\mu-\nu}{nN},\tau)} \tag{2.8}$$

We shall use a big full circle as the graphical notation for $D(m)$.

Now we can express the Boltzmann four - spin weight S in terms of the defined functions (Fig. 2)

$$S(m',n';m,n)_{pp'qq'} =$$

$$= D(m)D(n)\frac{W(n',m;q-p)W(m,n;p'-q)W(n,m';q'-p')}{W(n',m';q'-p)} \tag{2.9}$$

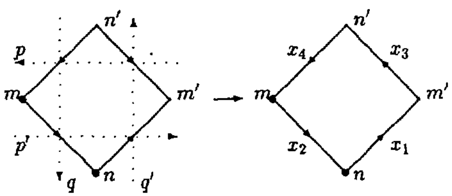

Fig. 2. $S(m',n';m,n)_{pp'qq'}$.

The statistical lattice is obtained by translating S.

The W - function obeys the unitarity relation

$$\sum_{l\in Q_N} W(m,l;u)W(l,n;-u)D(l)\sqrt{D(m)D(n)} = \Phi(u)\delta_{m,n} \tag{2.10}$$

where up to a constant factor

$$\Phi(u) = \frac{\theta_1(nu,N\tau)}{\theta_1(nu/N,\tau)}\left(\frac{\theta_1(u/N,\tau)}{\theta_1(nu,nN\tau)}\right)^n \times$$

$$\times \prod_{j=1}^{n-1}(\theta_1(j/nN-u/N,\tau)\theta_1(j/nN+u/N,\tau))^{n-j} \tag{2.11}$$

From (2.10) and (2.7), it follows that the unitarity relation for the S - matrix is given by

$$\sum_{n',m'\in Q_N} S(n'',m'';n',m')_{qq'pp'}S(m',n';m,n)_{pp'qq'} =$$

$$= \Phi(p-q)\Phi(q'-p')\delta_{n,n''}\delta_{m,m''} \tag{2.12}$$

It is useful to make the Jacoby transformation for the θ - functions. This transformation up to a gauge is equivalent to the replacement

$$\theta_1(z,\tau) \rightarrow \theta_1(\frac{z}{\tau}, -\frac{1}{\tau}) \tag{2.13}$$

in (2.3) and the function Φ in the unitarity condition becomes

$$\Phi(u) = \frac{\theta_1(nu/N\tau, -1/N\tau)}{\theta_1(nu/N\tau, -1/\tau)} \left(\frac{\theta_1(u/N\tau, -1/\tau)}{\theta_1(u/N\tau, -1/nN\tau)} \right)^n \times$$

$$\times \prod_{j=1}^{n-1} (\theta_1(j/nN\tau - u/N\tau, -1/\tau)\theta_1(j/nN\tau + u/N\tau, -1/\tau))^{n-j} \tag{2.14}$$

Throughout the paper the Jacoby transformation is implied and form (2.14) for the Φ is used.

3. Free energy

In the previous section we regarded the S matrix as the vertex one. It is convenient to consider the face formulation of the model. Let four - spin functions U and U' be

$$U(m_1, m_2, m_3, m_4)_{pp'qq'} = D(m_1) \frac{W(m_2, m_1; p' - p)}{W(m_3, m_4; p' - p)} \times$$

$$\times \sum_{m_0 \in Q_N} D(m_0) \frac{W(m_1, m_0; q' - p')W(m_0, m_4; p' - q)W(m_3, m_0, ; q - p)}{W(m_2, m_0; q' - p)} \tag{3.1a}$$

$$U'(m_1, m_2, m_3, m_4)_{pp'qq'} = D(m_1) \frac{W(m_1, m_4; q' - q)}{W(m_2, m_3; q' - q)} \times$$

$$\times \sum_{m_0 \in Q_N} D(m_0) \frac{W(m_0, m_3; q' - p')W(m_2, m_0; p' - q)W(m_0, m_1, ; q - p)}{W(m_0, m_4; q' - p)} \tag{3.1b}$$

The indentity (Fig. 3)

$$U(m_1, m_2, m_3, m_4)_{pp'qq'} = U'(m_1, m_2, m_3, m_4)_{pp'qq'} \tag{3.2}$$

is known as the Star - Star relation.

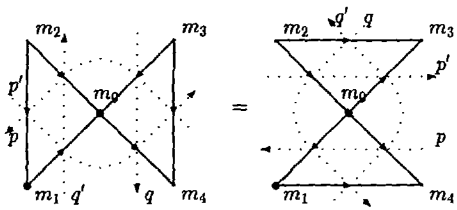

Fig. 3. The Star - Star Relation.

Clearly, translating U we obtain the same lattice as in the previous section.

Now consider transfer matrices T and T':

$$T_{\{\sigma'\}\{\sigma\}}(x_1, x_2) = \prod_i \frac{W(\sigma_i, \sigma'_i; x_1)W(\sigma'_i, \sigma_{i+1}; x_2)}{W(\sigma'_i, \sigma'_{i-1}; x_1 + x_2)} \tag{3.3a}$$

$$T'_{\{\sigma'\}\{\sigma\}}(x_1, x_2) = \prod_i \frac{W(\sigma_i, \sigma'_i; x_1)W(\sigma'_i, \sigma_{i-1}; x_2)}{W(\sigma_i, \sigma_{i+1}; x_1 + x_2)} \tag{3.3b}$$

Introduce four new variables x_1, x_2, x_3, x_4 instead of p, p', q, q':

$$\begin{aligned} x_1 &= q' - p1; & x_2 &= p' - q; \\ x_3 &= -1 - q' + p; & x_4 &= q - p; \end{aligned} \tag{3.4}$$

Using relation (2.7) and translating the star - star relation in the horizontal direction, we obtain

$$T'(x_1, x_2)T(x_3, x_4) = T(x_3, x_4)T(x_1, x_2), \tag{3.5}$$

hence the free energy is a sum of two terms, one depends on x_1 and x_2 and the other - on x_3 and x_4. Applying the same procedure to the vertical direction we can write

$$Q(x_1, x_3)Q'(x_4, x_2) = Q'(x_4, x_2)Q(x_1, x_3) \tag{3.6}$$

for appropriately defined Q and Q'. Note, that in terms of x_i the partition function is symmetric with respect to the cyclic permutation of x_i (see Fig. 2), so using (3.5), (3.6), (2.12) and (2.14) we derive the following equations for the free energy $k(x_1, x_2, x_3, x_4)$:

$$\begin{cases} k(x_1, x_2, x_3, x_4) &= k(x_1) + k(x_2) + k(x_3) + k(x_4) \\ k(x) + k(-x) &= A \\ k(x) + k(-1 - x) &= \log \Phi(x) + B \end{cases} \tag{3.7}$$

where A and B are some constants. Supposing the function $k(x)$ to be the analytical one, we can define it together with A and B uniquely.

Let

$$w = e^{\frac{2\pi i u}{N\tau}}, \; q = e^{\frac{-2\pi i}{N\tau}}. \tag{3.8}$$

Using w and q, system (3.7) can be written as

$$\begin{cases} k(w) + k(w^{-1}) & = \log\Phi(w) + B \\ k(w) + k(xw^{-1}) & = A \end{cases} \tag{3.9}$$

Expanding $\log\Phi(w)$ into a power series of w, converging absolutely inside the ring

$$|q|^{1/n} < |w| < |q|^{-1/n}$$

and assuming the existence of a $k(w)$ power series of w, converging in the ring

$$|q|^{1+1/n} < |w| < |q|^{-1/n},$$

we obtain

$$k(w) = \sum_{k=1}^{\infty} -\frac{[q^{k(N-1)/2}][w^{kn}q^{kn/2}]}{k[q^{k/2}][q^{kN/2}][q^{kn/2}]} + \frac{[q^{k(N-1)/2}][w^k q^{k/2}]}{k[q^{kN/2}][q^{k/2n}]^2}, \tag{3.10}$$

where $[z]$ stands for $z - z^{-1}$. This series converges inside the strip

$$-1 - \frac{1}{n} < \mathrm{Re}\, u < \frac{1}{n} \tag{3.11}$$

for a pure imaginary τ.

In the trigonometric limit $\tau = 0$ of (2.11) system (3.7) can be solved by the Fourier transformation inside the strip of analyticity (3.11). The result is similar to (3.10):

$$k(x) = \int_0^\infty \frac{dk}{2k}\left(-\frac{sh(kn/2 + knx)sh(k(N-1)/2)}{sh(kn/2)sh(k/2)sh(kN/2)} + \right.$$

$$\left. + \frac{sh(k/2 + kx)sh(k(N-1)/2)}{sh(kN/2)sh^2(k/2n)}\right) \tag{3.12}$$

An other trigonometric limit of (2.11), obtained with the help of the $\Gamma(N)$ modular transformation, is the trigonometric limit of the $Z_N^{\otimes(n-1)}$ generalization of the chiral Potts model [3,4,1]:

$$\Phi(x) = \frac{\sin nNx}{\sin nx}\left(\frac{\sin x}{\sin Nx}\right)^n \tag{3.13}$$

Using the analyticity inside (3.11), we find

$$k(x) = \int_0^\infty \frac{dk}{2k} \left(n \frac{sh(k(2Nx+\pi))sh((N-1)\pi k)}{sh(\pi k)sh(N\pi k)} - \right.$$

$$\left. - \frac{sh(k(2Nx+\pi))sh((N-1)\pi k/n)}{sh(\pi k/n)sh(N\pi k/n)} \right) \quad (3.14)$$

4. Local State Probabilities

Let $a \in Q_N$ be a spin of a fixed site s_0 on the lattice. Define the local state probability conventionally:

$$< a >= \sum_{\{\sigma_s\}:\sigma_{s_0}=a} \prod_s S(\{\sigma\}) \quad (4.1)$$

Let $V_T(\sigma)$ be the eigenvector of $T(x_1, x_2)$, corresponding to the maximal eigenvalue. Clearly, $V_T(\sigma)$ depends on x_1 and x_2 via $x_1 + x_2 = -1 - (x_3 + x_4)$. Similarly, considering $Q(x_1, x_4)$, $V_Q(\sigma)$ depends on x_1 and x_4 via $x_1 + x_4$. Since

$$< a >= \sum_{\{\sigma\}:\sigma_0=a} \tilde{V}_T(\sigma)V_T(\sigma) = \sum_{\{\sigma\}:\sigma_0=a} \tilde{V}_Q(\sigma)V_Q(\sigma), \quad (4.2)$$

we conclude that $< a >$ is independent of the spectral parameters. For the sake of simplicity we set

$$x_1 = x_2 = u;\ x_2 = x_3 = -\frac{1}{2} - u; \quad (4.3)$$

Such choice allows us to use directly the Corner Transfer Matrix method for the face weights U and U'.

The SE corner matrix and the NW one, expressed in the terms of U, are the same. Denote them as $A(u)$. Making the Star - Star transformation (3.2) for the SW and NE matrices, we find them equal to $A(-1/2-u)$. So, the vector of local state probabilities is

$$< a >= trace_{\sigma_0=a}(A(u)A(-1/2-u))^2 \quad (4.4)$$

Using the algebraic limit of U, one can find the diagonal form of $A(u)$ and substitute it to (4.4).

To write out the answer for $< a >$ we need some new notations. Let t be a real number. Denote

$$S_N(t) = -(\delta_t + \frac{1}{2}), \quad (4.5)$$

where $t = \delta_t N + \bar{t}$ so that $0 \leq \bar{t} < N$, δ_a be an integer. Futher denote

$$f_k(\sigma) = \begin{cases} \sum_{j=0}^{\sigma-1} S_N(k/n + j), & \sigma > 0 \\ -\sum_{j=0}^{-\sigma-1} S_N(k/n - j - 1), & \sigma < 0 \end{cases} \tag{4.6}$$

$$g_k(\sigma) = \begin{cases} \sum_{j=0}^{\sigma-1} S_N(-k/n - j - 1/2), & \sigma > 0 \\ -\sum_{j=0}^{-\sigma-1} S_N(-k/n + j + 1/2), & \sigma < 0 \end{cases} \tag{4.7}$$

Let $m, n \in Q_N$. Define

$$F(m) = \sum_{\alpha,\beta} f_{\alpha-\beta}(m_a - m_b) \tag{4.8}$$

$$G(m,n) = \sum_{\alpha,\beta} g_{\alpha-\beta}(m_\alpha - n_\beta) \tag{4.9}$$

Using the introduced notations, the local state probability can be written as

$$< a >= \sum_{\{\sigma\}:\sigma_0=a} q^{\sum_{k=1}^{\infty} k(G(\sigma_k,\sigma_{k-1})+G(\sigma_k,\sigma_{k+1})+2F(\sigma_k))} \tag{4.10}$$

$$= \sum_{\{\sigma\}} A^{(1)}_{a,\sigma_1} A^{(2)}_{\sigma_1,\sigma_2} A^{(3)}_{\sigma_2,\sigma_3} \cdots \ , \tag{4.11}$$

where

$$A^{(k)}_{\sigma,\sigma'} = q^{k(G(\sigma',\sigma)+F(s'))+(k-1)(G(\sigma,\sigma')+F(\sigma))} \tag{4.12}$$

More simply expressions we can write for the $n = 2$ case, which is closely connected to the Kashiwara - Miwa model [5]. In this case σ and σ' are nonnegative integers less than N.

$$A^{(k)}_{\sigma,\sigma'} = q^{(2k-1)(f(\sigma+\sigma')+|\sigma-\sigma'|)-kf(2\sigma')-(k-1)f(2\sigma)}, \tag{4.13}$$

where

$$f(a) = \begin{cases} 0, & a = 0 \\ a - 1, & a \leq N \\ 3a - 2N - 2, & a > N \end{cases} \tag{4.14}$$

Contrary to the RSOS [6,7] models these local state probabilities do not correspond to any GKO - construction directly. Assuming the following assymptotic for LSP [8] in the limit when $q = e^{2\pi i r} \to 1^-$

$$< a > \to e^{\frac{\pi i z}{12 r}}, \tag{4.15}$$

z being a central charge, we estimate for $n = 2$

$$z(N) = \frac{3(N-1)}{2(N+1)}. \tag{4.16}$$

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