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Z(2N) PARAFERMIONS FROM U(1) LOOP GROUP

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ABSTRACT

We study $\mathbb{Z}_k$-charged sectors of the level k $U(1)$ loop group where $k \in 2\mathbb{Z}$ and show that the fields creating these charges obey $\mathbb{Z}_k$ parafermionic commutation relations.

KIVONAT

A k szintű $U(1)$ hurokcsoport $\mathbb{Z}_k$ töltésű szektorait tanulmányozzuk $k \in 2\mathbb{Z}$ esetén, és megmutatjuk, hogy az ezen töltéseket keltő terek parafermion csererelációknak tesznek eleget.

1. Introduction

Chiral conformal field theories living on the compactified lightcone $\mathbf{S}^1$ provide interesting examples of quantum field theories with strange statistics and unusual internal symmetries. For the analysis of particle statistics and internal symmetries the methods of algebraic quantum field theory offer a conceptually clear and mathematically sound basis [DHR]. However, the implementation of the algebraic method to conformal models requires the knowledge of an algebra of observables given - if possible - in terms of generators and relations. For example the $\mathbf{Z}(N)$ parafermion models of Fateev and Zamolodchikov [FZ] are conformal field theories defined by the help of operator product expansion of pointlike fields. To our knowledge it is not known how the C^*-algebra of observables of this theory can be formulated in terms of generators and relations.

The concept of the loop group [PS] lies very close to both CFT and algebraic QFT. It describes a conformal model in terms of bounded operators. In this paper we consider the simplest possibility, the central extended $U(1)$ loop group algebra spanned by operators $W(f)$, $f : \mathbf{S}^1 \longrightarrow \mathbf{R}$ satisfying Weyl algebra relations. Unlike [BMT] we investigate the possibility that the loop group element e^{if} represented by $W(f)$ does not necessarily lie in the identity component. This leads to a quantization of the level parameter k in the cocycle; k must be an even integer. Considering this "large" loop group algebra as our algebra of observables we study their $\mathbf{Z}_k$ type of superselection sectors and construct fields that create the $\mathbf{Z}_k$ charges. The commutation relations of these fields turn out to be of the parafermion type. The precise relation, however, of these parafermions to those of Fateev-Zamolodchikov requires further study.

2. The U(1) loop group

The $U(1)$ loop group is the abelian group of the smooth mappings from $\mathbf{S}^1$ into the group $U(1)$ with respect to the pointwise multiplication:

$$\mathcal{G} = \{\Phi : \mathbf{S}^1 \longrightarrow U(1) \quad |\Phi \in C^\infty\}$$

$$\Phi_1\Phi_2(\theta) = \Phi_1(\theta)\Phi_2(\theta) \qquad \theta \in \mathbf{S}^1$$

The part of $\mathcal{G}$, which is connected to the unit element, clearly forms a subgroup, $\mathcal{G}_o$. Since $\mathcal{G}$ is abelian $\mathcal{G}_o$ is a normal subgroup. The factor group $\mathcal{G}/\mathcal{G}_o$ is the fundamental group of S^1, the additive group of the integers. This means that $\mathcal{G} = \mathcal{G}_o \times \mathbf{Z}$. In this way the cosets can be labelled by the winding number: $n = \int_{S^1} \frac{d\theta}{2\pi} \Phi(\theta)^{-1} \frac{d}{d\theta} \Phi(\theta)$. $\mathcal{G}_o$ corresponds to $n = 0$.

2.1. Proposition: The elements of $\mathcal{G}_o$ can be parametrized with functions in $\mathcal{L}_o = C^\infty(S^1, \mathbf{R})$, as $\Phi(\theta) = e^{if(\theta)}$.

($\mathcal{L}_o$ can be treated as the Lie algebra of $\mathcal{G}$. Since it is a function space, the Lie bracket is trivial.)

Proof: Since $\Phi \in \mathcal{G}_o$ it is a smooth deformation of the unit element 1, there exists a smooth map Φ_t from the interval $[0,1]$ into $\mathcal{G}_o$ such that $\Phi_o = 1 \in \mathcal{G}_o$, $\Phi_1 = \Phi$.

Let us define $f_t(\theta) = -i \int_0^\theta \frac{d\tau}{2\pi} \Phi_t^{-1}(\tau) \frac{d}{d\tau} \Phi_t(\tau)$. Since $f_t(\theta = 0) = 0$ and $\frac{d}{d\theta}(e^{-if_t(\theta)} \Phi_t(\theta)) = 0$, $\Phi_t(\theta) = \Phi_t(0) e^{if_t(\theta)}$. Defining now $f(\theta)$ as $f_1(\theta)$, it is C^∞ type by construction: a smooth deformation of the function $f_o(\theta) \equiv 0$. ∎

The above parametrization of $\mathcal{G}_o$ is redundant. The elements f_1 and f_2 in $\mathcal{L}_o$ define the same element in $\mathcal{G}_o$ if and only if $f_1(\theta) = f_2(\theta) + 2\pi n$ for some $n \in \mathbf{Z}$. Considering $\mathcal{L}_o$ as an additive group, $\mathcal{G}_o$ is isomorphic to the factor group $\mathcal{L}_o / 2\pi\mathbf{Z}$.

The constant functions in $\mathcal{L}_o$ define a U(1) subgroup in $\mathcal{G}_o$. Therefore

$$\mathcal{G} = U(1) \times \Omega \times \mathbf{Z}. \tag{2.1}$$

where $\Omega = \{f \in \mathcal{L}_o | \quad f(0) = 0\}$ is a non-compact group. (Actually a linear space.)

3. Projective representations of $\mathcal{G}_o$

The map W from $\mathcal{G}_o$ into the group of unitary operators acting on some Hilbert space is called a projective unitary representation of $\mathcal{G}_o$ if

$$\begin{gathered} W(f)W(g) = W(f+g)\alpha(f,g), \\ W(f+2\pi) = W(f); \quad \alpha(f+2\pi, g) = \alpha(f, g+2\pi) = \alpha(f,g) \end{gathered} \tag{3.1}$$

for any f and g in $\mathcal{L}_o$. (The second line is necessary to obtain a representation of $\mathcal{G}_o$ not only of $\mathcal{L}_o$.)

Since the multiplication of the W operators must be associative, α must be a 2-cocycle, i.e. it must satisfy

$$\alpha(f_1, f_2)\alpha(f_1 + f_2, f_3) = \alpha(f_1, f_2 + f_3)\alpha(f_2, f_3) \tag{3.2}$$

for any $f_i \in \mathcal{L}_o \quad i = 1,2,3.$

3.1. Lemma: If B is a real bilinear form on the linear space $\mathcal{L}_o$, $\alpha(f,g) = e^{iB(f,g)}$ is a 2-cocycle on $\mathcal{L}_o$. Let us call the cocycles of the above form "quadratic" cocycles. If furthermore

$$B(f + 2\pi, g) = B(f,g) \ mod\ 2\pi \quad \text{and} \quad B(f, g + 2\pi) = B(f,g) \ mod\ 2\pi \tag{3.3}$$

then it defines a 2-cocycle on $\mathcal{G}_o$.

Proof: This follows immediately substituting α into (3.2). ∎

3.2. Proposition: The cohomology classes of the quadratic cocycles on $\mathcal{G}_o$ can be labelled with the antisymmetric real bilinear forms on $\mathcal{L}_o$ satisfying (3.3).

Proof: We have to prove first that all the quadratic coboundaries are defined by the help of symmetric bilinear forms, second that all the symmetric bilinear forms define cocycles that are coboundaries.

The first part follows immediately from the symmetric form of the coboundary operator on 1-cocycles:

$$\delta\beta(f,g) = \frac{\beta(f+g)}{\beta(f)\beta(g)},$$

so the bilinear form must be symmetric itself.

The opposite direction is also easily obtained: If B is symmetric and bilinear then

$$B(f,g) = \frac{1}{2}\Big(B(f,g) + B(g,f)\Big) = \frac{1}{2}\Big(B(f+g, f+g) - B(f,f) - B(g,g)\Big),$$

so the corresponding $\alpha(f,g)$ is the coboundary of the 1-cocycle $\beta(f) = e^{i\frac{1}{2}B(f,f)}$. ∎

In order for the W operators to generate the observable algebra of some conformal field theory we make further restrictions on the cocycles: we want them to be invariant under the reparametrization of $\mathbf{S}^1$:

$$\alpha(f,g) = \alpha(f \circ D, g \circ D), \quad \text{for any} \quad D \in Diff(\mathbf{S}^1) \tag{3.4}$$

3.3. Proposition: The 2-variable functional on $\mathcal{L}_o$

$$\alpha_k(f,g) = e^{i\frac{k}{2}\int_0^{2\pi}\frac{d\theta}{2\pi} f(\theta) g'(\theta)} \tag{3.5}$$

defines inequivalent reparametrization invariant quadratic 2-cocycles on $\mathcal{G}_o$ for any real parameter k.

Proof: $K(f,g) = k\int_0^{2\pi} \frac{d\theta}{2\pi} f(\theta)g'(\theta)$ is an antisymmetric real bilinear form on $\mathcal{L}_o$ satisfying (3.3), so according to the previous proposition, it defines inequivalent 2-cocycles on $\mathcal{G}_o$.

The reparametrization invariance can be checked by direct calculation. ■

As a result of this analysis we obtained a realization of the U(1) current algebra in terms of projective representations of the U(1) loop group. Writing $W(f)$ as $e^{i(J,f)}$ with $(J,f) = \int_0^{2\pi} \frac{d\theta}{2\pi} f(\theta)J(\theta)$, the relation $W(f)W(g) = W(f+g)\alpha_k(f,g)$ with the above choice of α_k is equivalent to the current algebra $[J(\theta_1), J(\theta_2)] = ik\delta'(\theta_1 - \theta_2)$, or using Fourier modes, to $[J_n, J_m] = -kn\delta_{n+m,0}J_{n+m}$.

4. Projective representations of $\mathcal{G}$

4.1. Lemma: The elements of $\mathcal{G}$ can be parametrized with functions in $\tilde{\mathcal{G}} = \{f \in C^\infty(\mathbf{R},\mathbf{R}) \ \ |f(\theta + 2\pi) = f(\theta) \ mod\ 2\pi\}$ as $\Phi(\theta) = e^{if(\theta)}$.

(The winding number $n = \frac{1}{2\pi}(f(\theta + 2\pi) - f(\theta))$ determines the decomposition of $\tilde{\mathcal{G}}$ as $\bigcup_{n\in\mathbf{Z}} \mathcal{L}_n$. $\mathcal{L}_o$ is the same as before. $\mathcal{L}_n$ is a linear space only for $n = 0$.)

Proof: Because of (2.1), any element of $\mathcal{G}$ can be written as $\Phi = \Phi_o e^{il_n}$, where Φ_o is an element of $\mathcal{G}_o$, and l_n is a representative of the $\mathcal{G}_o$ coset with the appropriate winding number. Since $l_n = n\theta$ is a correct choice and using Lemma 2.1 Φ_o can be parametrized as $\Phi_o = e^{if_o(\theta)}$ with $f_o \in \mathcal{L}_o$, $\Phi(\theta) = e^{i(f_o(\theta)+n\theta)}$ gives the required parametrization. ■

Similarly to the case of $\mathcal{G}_o$, $\mathcal{G}$ is isomorphic to the factor group $\tilde{\mathcal{G}}/2\pi\mathbf{Z}$.

In order to give the inequivalent projective representations of $\mathcal{G}$ we have to find the inequivalent 2-cocycles $\tilde{\alpha}$ on $\mathcal{G}$ satisfying (3.3).

As it can be easily seen Lemma 3.1 and Proposition 3.2 remain true with the modification that instead of bilinearity we require only additivity in each argument of B. (Since $\tilde{\mathcal{G}}$ is not invariant under scalar multiplication.) So we have to determine the real antisymmetric "biadditive" forms on $\mathcal{G}$. We still require reparametrization invariance, i.e.

$$\tilde{\alpha}(f,g) = \tilde{\alpha}(f \circ \tilde{D}, g \circ \tilde{D}) \tag{4.1}$$

for any lift $\tilde{D}$ of some $D \in Diff(S^1)$ in the sense that $\tilde{D} \in \mathcal{L}_1$ and for the canonical projection $p : \mathbf{R} \longrightarrow S^1 \quad p \circ \tilde{D} = D \circ p$.

We may require that the restriction of $\tilde{\alpha}$ to $\mathcal{G}_o$ is α_k for some $k \in \mathbf{R}$.

4.2. Proposition: **A class of mutually inequivalent reparametrization invariant quadratic 2-cocycles on $\mathcal{G}$ can be obtained in the following form:**

$$\alpha_k(f,g) = e^{i\frac{k}{2}\int_0^{2\pi}\frac{d\theta}{2\pi}\left(f(\theta)g'(\theta)-g(\theta)f'(\theta)\right)} \tag{4.2}$$

where k is an *even integer*.

Proof: To make the original α_k reparametrization invariant it is necessary to subtract the second term from the original bilinear form. In order to satisfy the analogue of (3.3) k must be an even integer. The cocycles of the above form are inequivalent for any allowed value of k, since they are antisymmetric as it can be checked by partial integration.

We have the further freedom to add any real antisymmetric reparametrization invariant bilinear form that vanishes on $\mathcal{G}_o$. This means that it is degenerate on the $\mathcal{G}_o$ cosets i.e. it may depend only on the winding numbers. Since all real bilinear forms on $\mathbf{Z}$ are symmetric, it corresponds to a coboundary. ∎

In the language of the current algebra the extension of $\mathcal{G}_o$ to $\mathcal{G}$ corresponds to the extension with a conjugate of J_o: $\mathcal{G}$ is generated by $\{J_n | n \in \mathbf{Z}\}$ and by $W(l_1)$ with the commutation relation

$$J_n W(l_1) = W(l_1)(J_n + k\delta_{n,0}).$$

J_o has to have integer spectrum in order for $e^{i(J,f)}$ to belong to $\mathcal{G}$. So $W(l_1)$ cannot change the *mod k* value of J_o.

5. The local net structure of the observable algebra

In the following we want to consider the above constructed *-algebra

$$\mathcal{A}_k = \langle W(f),\ f \in \tilde{\mathcal{G}}, | \quad \begin{array}{l} W(f)W(g) = W(f+g)\alpha_k(f,g) \\ W(f)^* = W(-f)\alpha_k(f,-f)\ f,g \in \tilde{\mathcal{G}} \rangle \end{array} \tag{5.1}$$

as the global observable algebra. The subalgebra of the observables localized in an interval $I \subset S^1$ is given by

$$\mathcal{A}_k(I) = \langle W(f) | f \in \tilde{\mathcal{G}}(I) \rangle \tag{5.2}$$

where $\tilde{\mathcal{G}}(I) = \{f \in \tilde{\mathcal{G}} | \quad f(\mathbf{R} \setminus p^{-1}(I)) \in 2\pi\mathbf{Z}\}$.

It can be easily checked that this structure really satisfies the locality requirement: the operators localized in disjoint intervals commute. Haag duality, however, is violated because there are global central elements in $\mathcal{A}_k$.

5.1. Proposition: The centre of $\mathcal{A}_k$ is generated by $Y := W(\frac{2\pi}{k})$.

Proof: The W operators are not only algebraic generators but they form a linear basis as well, so any element of $\mathcal{A}_k$ can be written as a finite linear combination: $X = \sum_f c_f W(f)$.

$$W(g)XW(g)^{-1} = \sum_f c_f W(g)W(f)W(g)^{-1} = \sum_f c_f \alpha_k(g,f)^2 W(f)$$

so X is central iff for every $f \in \tilde{\mathcal{G}}$ either $c_f = 0$ or $\alpha_k(g,f)^2 = 1$ for any $g \in \tilde{\mathcal{G}}$. The second case can occur only if $f' \equiv 0$ and $f(0) \in \frac{2\pi}{k}\mathbf{Z}$. ■

The elements of the subalgebra defined by the constant elements of $\tilde{\mathcal{G}}$ play a special role, they measure the winding number n_f of the function $f \in \tilde{\mathcal{G}}$:

$$W(c)W(f)W(c)^{-1} = W(f)e^{ikcn_f}.$$

6. Automorphisms on $\mathcal{A}_k$

Multiplying the W operators with the operators of a true unitary representation of $\mathcal{G}$ clearly defines a *-automorphism of $\mathcal{A}_k$. These automorphisms are characterized by linear maps K from $\mathcal{G}$ into $\mathbf{R}$:

$$\rho(W(g)) = W(g)e^{iK(g)} \tag{6.1}$$

Let us write these maps into the form

$$K(g) = \int \frac{d\theta}{2\pi} r(\theta)g(\theta) + sn_g \tag{6.2}$$

by the help of $r \in \tilde{\mathcal{G}}$, $s \in \mathbf{R}$. The integration is to be taken on some interval $(z, z+2\pi)$. In order for K to be well defined also on $\mathcal{G}$ not only on $\tilde{\mathcal{G}}$, $\int \frac{d\theta}{2\pi} r(\theta)$ must be integer. On composing morphisms both r and s are additive.

The inner automorphisms $Ad_{W(f)}$ have the above form (6.1) with $r = -kf'$, $s = kf(0)$. They have two important features: $r \in \mathcal{L}_0$ and $\int \frac{d\theta}{2\pi} r(\theta) \in k\mathbf{Z}$. (We do not have to deal with the value of s: since $f(0)$ can be chosen arbitrarily independently of f', two morphisms that differ only in the value of s are connected by an inner automorphism which is defined by the help of a constant function.) This means that we have two candidates for "quantum numbers" labelling the sectors:

(I): *The integral of r:* $\int_0^{2\pi} \frac{d\theta}{2\pi} r(\theta) \; mod \; k$. This is a $\mathbf{Z}_k$ charge.

(II): *The winding number of r.* Since this number is zero for the inner automorphisms, all automorphisms with different winding numbers of r are inequivalent. This is a $\mathbf{Z}$ valued charge.

To see whether these numbers really correspond to transportable local charges some further investigation is necessary. In the rest of the paper we carry out the detailed analysis for the (I) case.

As we have seen the equivalence classes of the morphisms of type (I) form a $\mathbf{Z}_k$ group with respect to the composition. The corresponding quantum number is the integral of the $r \in \mathcal{L}_0$ function appearing in (6.2). It is measured by the elements of the centre, i.e. Y is the global charge operator for it:

$$\rho(Y) = Y e^{i\frac{2\pi}{k}\int_0^{2\pi} \frac{d\theta}{2\pi} r(\theta)}.$$

Let us introduce a more compact notation by the help of functions in $\tilde{\tilde{\mathcal{G}}} := \left(\bigcup_{m=1}^{k} \mathcal{L}_{\frac{m}{k}}\right) \bigcup \tilde{\mathcal{G}}$. $\tilde{\tilde{\mathcal{G}}}$ is the generalization of $\tilde{\mathcal{G}}$ to $\tilde{\mathbf{S}}^1$, the k-fold covering space of $\mathbf{S}^1$:

$$\mathcal{L}_{\frac{m}{k}} := \{\tilde{f} : \mathbf{R} \longrightarrow \mathbf{R} | \quad \tilde{f} \in C^\infty, \; \tilde{f}(x+2\pi) = \tilde{f}(x) + m\frac{2\pi}{k}\}. \tag{6.3}$$

Then instead of (6.1) we can write

$$\rho_{\tilde{f}}(W(g)) = W(g) exp\{ik \int_{z_0}^{z_0+2\pi} \frac{d\theta}{2\pi} \tilde{f}'(\theta) g(\theta) - ik\tilde{f}(z_0) n_g\} \quad ; \quad \tilde{f} \in \mathcal{L}_{\frac{m}{k}} \tag{6.4}$$

In this notation $\tilde{f}(z_o)$ plays the role of the earlier s and $k\tilde{f}'$ the role of r. m is the quantum number labelling the sectors. $\rho_{\tilde{f}}$ is independent of the choice of z_o. Extending α_k to $\tilde{\tilde{\mathcal{G}}}$, we can use the shorthand notation

$$\rho_{\tilde{f}}(W(g)) = W(g)\alpha_k(g, \tilde{f})^2, \tag{6.5}$$

which shows again that $\rho_{\tilde{f}}$ is inner automorphism $Ad_{W(\tilde{f})}$ iff $\tilde{f} \in \mathcal{L}_{\frac{m}{k}}$ with $m = 0 \ mod\ k$. (Then α_k is just the original α_k on $\tilde{\mathcal{G}}$.)

The composition law in this notation is simply $\rho_{\tilde{f}_1} \circ \rho_{\tilde{f}_2} = \rho_{\tilde{f}_1+\tilde{f}_2}$

6.1. Proposition: $\rho_{\tilde{f}}$ is localized in an interval $I \subset \mathbf{S}^1$ (i.e. it acts trivially on the observables localized in $\mathbf{S}^1 \setminus I$) iff $\tilde{f} \in \mathcal{L}_{\frac{m}{k}}(I)$, where

$$\mathcal{L}_{\frac{m}{k}}(I) := \{\tilde{f} \in \mathcal{L}_{\frac{m}{k}} \mid \tilde{f}(\mathbf{R} \setminus p^{-1}(I)) \in \frac{2\pi}{k}\mathbf{Z}\}. \tag{6.6}$$

Proof: Let us first choose a $g \in \mathcal{L}_o(I)$. For all such g $\rho_{\tilde{f}}(W(g)) = W(g)exp\{ik\int_{z_o}^{z_o+2\pi} \frac{d\theta}{2\pi}\tilde{f}'(\theta)g(\theta)\} = W(g)$ must hold which is equivalent to $\tilde{f}'(\mathbf{R} \setminus p^{-1}(I)) = 0$ so $\tilde{f}(\mathbf{R} \setminus p^{-1}(I)) = const.$

Choosing now a $g \in \mathcal{L}_n(I)$ $n \neq 0$, $\rho_{\tilde{f}}(W(g)) = W(g)exp\{ik\int_{z_o}^{z_o+2\pi} \frac{d\theta}{2\pi}\tilde{f}'(\theta)g(\theta) - ik\tilde{f}(z_o)n_g\}$. Since this expression is independent of z_o, we may choose $p(z_o) \in \mathbf{S}^1 \setminus I$. Then $\rho_{\tilde{f}}(W(g)) = W(g)exp\{-ikn_g\tilde{f}(z_o)\} = W(g)$ iff $\tilde{f}(z_o) \in \frac{2\pi}{k}\mathbf{Z}$, so $\tilde{f}(\mathbf{R} \setminus p^{-1}(I)) \in \frac{2\pi}{k}\mathbf{Z}$. ∎

As a consequence of this result the composition of morphisms that are localized in the same interval is a morphism localized in this interval again.

6.2. Proposition: The localized morphisms are locally transportable. That is for any two morphisms $\rho_{\tilde{f}_1}$ and $\rho_{\tilde{f}_2}$ that correspond to quantum numbers $m_1 = m_2 \ mod\ k$ and are localized in intervals I_1 and I_2, respectively, there exists a sequence of localized operators $W(h_1)...W(h_n)$ in $\mathcal{A}_k$ such that

$$\rho_{\tilde{f}_2} = Ad_{W(h_n)} \circ ... \circ Ad_{W(h_1)} \circ \rho_{\tilde{f}_1}.$$

In fact n can be chosen to be $n \leq 2$ and $n = 2$ is necessary only if $I_1 \bigcup I_2$ covers $\mathbf{S}^1$.

Proof: Let us first consider the case when $S^1 \setminus (I_1 \bigcup I_2)$ is not empty. Then there exists $\theta_o \in S^1 \setminus (I_1 \bigcup I_2)$ for which both $\tilde{f}_1(\theta_o)$ and $\tilde{f}_2(\theta_o)$ are in $\frac{2\pi}{k}\mathbf{Z}$.

Computing $\rho_{\tilde{f}_2} \circ \rho^{-1}_{\tilde{f}_1}$ one finds that it is $Ad_{W(\tilde{f}_1 - \tilde{f}_2)}$. This is an inner automorphism since $\tilde{f}_1 - \tilde{f}_2 \in \tilde{\mathcal{G}}$.

Since $(\tilde{f}_1 - \tilde{f}_2)(\theta_o) = \frac{2\pi}{k}s$, where s is an integer we can define $h(\theta) := \tilde{f}_1(\theta) - \tilde{f}_1(\theta) - \frac{2\pi}{k}s \in \tilde{\mathcal{G}}(I)$. $I \subset S^1$ contains both I_1 and I_2 but not θ_o. Since $W(\tilde{f}_1 - \tilde{f}_2)$ and $W(h)$ differs only in a central element Y^*, we obtain $\rho_{\tilde{f}_2} = Ad_{W(h)} \circ \rho_{\tilde{f}_1}$.

In the case when $S^1 \setminus (I_1 \bigcup I_2)$ is empty, there exists a sequence of intervals $I_{(1)} ... I_{(n+1)}$ such that $I_{(1)} = I_1$, $I_{(n+1)} = I_2$ and $S^1 \setminus (I_j \bigcup I_{j+1})$ is not empty for any j. This sequence of intervals defines the required sequence $W(h_1)...W(h_n)$ of localized operators in $\mathcal{A}_k$ by the above construction. ■

7. The field algebra

As it is clear from the previous discussion the field algebra corresponding to the considered type of morphisms is generated by the observable algebra and only one new generator, the kth power of which is in $\mathcal{A}_k$.

The field algebra is then given by the relations

$$\mathcal{F}_k = \langle W(f), \Gamma \mid \; f \in \tilde{\mathcal{G}} \;\; \Gamma W(f) = \gamma(W(f))\Gamma, \\ \Gamma^k = W(-l_1)C \rangle \tag{7.1}$$

The operator C is a central element that can be chosen at will. For later convenience we choose it to be $C = -i^{(k-1)}\mathbf{1}$.

γ is a reference morphism a representative of the equivalence class $m = 1$. A convenient choice is $\gamma = \rho(\tilde{f} = \frac{1}{k}l_1)$.

7.1. Lemma: For $\tilde{f} \in \mathcal{L}_{\frac{m}{k}}$ the morphism $\rho_{\tilde{f}}$ is induced by the unitary element $\Psi(\tilde{f}) = W(\frac{m}{k}l_1 - \tilde{f})\Gamma^m$ of $\mathcal{F}_k$. That is $Ad_{\Psi(\tilde{f})} = \rho_{\tilde{f}}$.

Proof: Calculating directly $Ad_{\Psi(\tilde{f})}$ on some $W(h)$ immediately gives the above result. ■

Since the Ψ operators can be parametrized with $\tilde{\mathcal{G}}$ functions one already conjectures that their monodromy operator should be a central element, the kth power of which is trivial. So only the morphisms and not the field operators themselves are periodic.

7.2. Proposition: The product of local charge transporters (in the sense of Proposition 6.2) that carries m units of $\mathbf{Z}_k$ charge once around the circle (that is the monodromy of $\Psi(\tilde{f})$, $\tilde{f} \in \mathcal{L}_{\frac{m}{k}}$) is $Y^m = W(m\frac{2\pi}{k})$.

Proof: Let us take two field operators $\Psi(\tilde{f}_1)$ and $\Psi(\tilde{f}_2)$ creating the same charge m and localized in disjoint intervals I_1 and I_2, respectively. $\Psi(\tilde{f}_1)^{-1}\Psi(\tilde{f}_2) = W(\tilde{f}_1 - \tilde{f}_2)\alpha_k(-\tilde{f}_1, \tilde{f}_2)$. $W(\tilde{f}_1 - \tilde{f}_2)$ is in $\mathcal{A}_k$ but is not localized in general. There exist two different observables that differ from $W(\tilde{f}_1 - \tilde{f}_2)$ only in central elements and that are localized in the two possible intervals containing both I_1 and I_2. Let us choose elements z_o and $\hat{z}_o$ from the two components of $\mathbf{S}^1 \setminus (I_1 \bigcup I_2)$ in such a way that according to the orientation of the circle $z_o < I_1 < \hat{z}_o < I_2$. Then $W(\tilde{f}_1 - \tilde{f}_2 - \tilde{f}_1(z_o) + \tilde{f}_2(z_o))$ is localized in the interval that contains I_1, I_2 and $\hat{z}_o$, but not z_o. $W(\tilde{f}_1 - \tilde{f}_2 - \tilde{f}_1(\hat{z}_o) + \tilde{f}_2(\hat{z}_o))$ is localized in the interval that contains I_1, I_2 and z_o, but not $\hat{z}_o$. $W(\tilde{f}_1 - \tilde{f}_2 - \tilde{f}_1(z_o) + \tilde{f}_2(z_o))W(\tilde{f}_1 - \tilde{f}_2 - \tilde{f}_1(\hat{z}_o) + \tilde{f}_2(\hat{z}_o))^{-1} = W(-\tilde{f}_1(z_o) + \tilde{f}_1(\hat{z}_o) + \tilde{f}_2(z_o) - \tilde{f}_2(\hat{z}_o)) = W(m\frac{2\pi}{k}) = Y^m$. ∎

7.3. Proposition: The $\Psi(\tilde{f})$ operators satisfy the algebraic relations:

$$\Psi(\tilde{f})^k = W(-k\tilde{f})e^{-i\frac{k-1}{2}I(k\tilde{f})} \qquad (7.2)$$

$$\Psi(\tilde{f}_1)\Psi(\tilde{f}_2) = \Psi(\tilde{f}_2)\Psi(\tilde{f}_1)e^{-i\frac{m_1 m_2}{k}\pi}\alpha_k(\tilde{f}_1, \tilde{f}_2)^2. \qquad (7.3)$$

where $\tilde{f} \in \mathcal{L}_{\frac{m}{k}}$ $\tilde{f}_i \in \mathcal{L}_{\frac{m_i}{k}}$, i=1,2; $I(\tilde{f}) = \int_0^{2\pi} \frac{d\theta}{2\pi}\tilde{f}(\theta)$.

Proof: The statement follows by direct calculation using the concrete form of $\Psi(\tilde{f}) = W(\frac{m}{k}l_1 - \tilde{f})\Gamma^m$, the commutation relations of the $W(g)$ operators and the definition of the Γ generator. ∎

Computing the commutator in the case when the localization intervals I_1 and I_2 of $\Psi(\tilde{f}_1)$ and $\Psi(\tilde{f}_2)$ are disjoint, the result is

$$\Psi(\tilde{f}_1)\Psi(\tilde{f}_2) = \begin{cases} \Psi(\tilde{f}_2)\Psi(\tilde{f}_1)e^{+i\frac{m_1 m_2}{k}\pi + i\left(m_2\tilde{f}_1(z_o) - m_1\tilde{f}_2(z_o)\right)} & \text{if} \quad z_o < I_1 < I_2 < z_o \\ \Psi(\tilde{f}_2)\Psi(\tilde{f}_1)e^{-i\frac{m_1 m_2}{k}\pi + i\left(m_2\tilde{f}_1(z_o) - m_1\tilde{f}_2(z_o)\right)} & \text{if} \quad z_o < I_2 < I_1 < z_o \end{cases}$$

This formula apparently depends on the arbitrary point z_o in $\mathbf{S}^1 \setminus (I_1 \bigcup I_2)$. But moving z_o in one connected component of $\mathbf{S}^1 \setminus (I_1 \bigcup I_2)$ both $\tilde{f}_1(z_o)$ and $\tilde{f}_2(z_o)$ are unchanged.

Moving it through I_1 or I_2 the change of $\tilde{f}_1(z_\circ)$ or $\tilde{f}_2(z_\circ)$ is compensated by the change of the sign of $\frac{m_1 m_2}{k}\pi$.

In order to obtain Fröhlich type commutation relations for the field operators, at first we have to have an ordering of the intervals. This exists only if we cut the circle at a point $z_\circ$ and consider only those morphisms that are localized in $S^1 \setminus \{z_\circ\}$.

Let $\Psi(\tilde{f}_1)$ and $\Psi(\tilde{f}_2)$ be localized in disjoint intervals I_1 and I_2 both of them being in $S^1 \setminus \{z_\circ\}$. Both $\tilde{f}_1(z_\circ)$ and $\tilde{f}_2(z_\circ)$ are in $\frac{2\pi}{k}\mathbf{Z}$, so we can define $\Psi_{z_\circ}(\tilde{f}_i) := Y^{\frac{k}{2\pi}\tilde{f}_i(z_\circ)}\Psi(\tilde{f}_i)$, $i = 1, 2$. Since Y is a central element, these field operators induce the same morphisms as the original ones.

7.4. Proposition: The modified field operators obey $\mathbf{Z}_k$-parafermionic commutation relations:

$$\Psi_{z_\circ}(\tilde{f}_1)\Psi_{z_\circ}(\tilde{f}_2) = \Psi_{z_\circ}(\tilde{f}_2)\Psi_{z_\circ}(\tilde{f}_1)\, e^{\pm i \frac{m_1 m_2}{k}\pi}. \tag{7.4}$$

The + sign holds if I_1 is on the left of I_2 in the interval $(z_\circ, z_\circ + 2\pi)$ and the – in the opposite case. The statistical parameter of the sector of charge m is then $\lambda_m = e^{i\frac{m^2}{k}2\pi}$.

8. Dynamics on the field algebra

Since we are investigating a chiral model, the 'space' translation and the 'time' evolution is described by the same automorphism, $\tau_x(W(f)) = W(T_x f)$ where $T_x f(z) = f(z - x)$. We are looking for the extension of τ_x to the field algebra $\mathcal{F}_k$. The translation of the morphisms is uniquely determined to be $\rho^x_{\tilde{f}} = \tau_x \circ \rho_{\tilde{f}} \circ \tau_{-x}$. This defines the translation of the field operators up to a central element. Since the elements of the centre are not localized, this ambiguity is specified by the requirement that translating a local field operator the resulting operator must be local.

8.1. Proposition: $\tau_x(\Psi(\tilde{f})) := \Psi(T_x \tilde{f})$ defines the unique extension of τ_x to $\mathcal{F}_k$ that preserves locality of the field operators. (This corresponds to the definition $\tau_x(\Gamma) = W(\frac{x}{k})\Gamma$.)

Proof: Obviously, if $\Psi(\tilde{f})$ is localized in an interval I then $\Psi(T_x(\tilde{f}))$ is localized in the translated interval.

The fact that it implements the translated morphism can be checked by direct calculation. ∎

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