D.I.Kazakov*, S.N.Solodukhin**

ON QUANTUM DEFORMATION
OF THE SCHWARZSCHILD SOLUTION

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*e-mail: kazakovD@thsunl.jinr.dubna.su
**e-mail: solodukhin@mainl.jinr.dubna.su

1   Introduction

One of the most important unresolved problems in general relativity is the problem of singularities. According to the results of Penrose and Hawking [1], the space-time singularities are typical for a classical theory of gravitation. Under rather general assumptions about the properties of the matter they occur in the Universe and inside black holes. The curvature of space-time increases without limit near a singularity. In such circumstances the classical theory is not applicable and, in particular, we cannot believe in its predictions concerning the complete global structure of space-time.

On the other hand, it is commonly believed that a successful quantization of gravity will provide us with modifications to the theory which are necessary to avoid the prediction of geodesically incomplete space-time manifolds [2,3]. Quantum corrections may completely change the gravitational equations and the corresponding space-time geometry at the Planck scale. The main problem on this way is the non-renormalizability of the Einstein gravity since the straightforward exploiting of the standard perturbation methods leads to inconsistent quantum theory. However, quantum gravity can be treated semi-classically [4] and the obtained results are sensible in some regimes when part of gravitational degrees of freedom in the leading order can be considered as classical [5], while the other part is described by exactly solvable quantum theory.

In this paper, we are trying to take into account the influence of quantum corrections on the behaviour of the Schwarzschild solution. This solution is probably the most important one in general relativity. It describes the space-time outside the gravitating body of mass M and allows maximal Kruskal extension which has a singularity at the radial parameter $r = 0$.

Our strategy is the following. We are interested in spherically symmetric solution of gravitational field equations and its deformation due to quantum excitations of the metric and the matter fields. Therefore, it is naturaly to assume that the general element $g_{\mu\nu}$ of the space of all metrics (over which we have to integrate in functional integral) in the neighbourhood of the classical configuration is represented as a sum of a spherically-symmetric part $g_{\mu\nu}^{sph}$ and a non-spherically symmetric perturbation $h_{\mu\nu}$:

$$g_{\mu\nu} = g_{\mu\nu}^{sph} + h_{\mu\nu}. \tag{1.1}$$

We do not assume the spherically symmetric part $g_{\mu\nu}^{sph}$ to be small and will quantize it exactly. Instead a non-spherically symmetric deviation $h_{\mu\nu}$ is assumed to be small and we will take it into account perturbatively under quantization. Spherically symmetric excitations of the metric do not contain propagating modes while the modes of $h_{\mu\nu}$ do propagate. So in the first order, eq.(1.1) is a separation on propagating and non-propagating modes. Bearing in mind the non-renormalizability of quantum

gravity with the Einstein-Hilbert action

$$S_{gr}[g_{\mu\nu}] = -\frac{1}{16\pi\kappa} \int d^4x \sqrt{-g} R \qquad (1.2)$$

one can expect that it is related to the contribution of the propagating h-modes (gravitons) to the Feynman diagrams. Thus, quantum theory blows up already in the first non-trivial order in h.

Inserting (1.1) into the Einstein-Hilbert action (1.2) we get in the second order in h :

$$S_{gr}[g_{\mu\nu}] = S_{gr}[g_{\mu\nu}^{sph}] + \int h\hat{D}h, \qquad (1.3)$$

where $\hat{D}$ is the second order differential operator determined with respect to the metric $g_{\mu\nu}^{sph}$. (Actually, the h-modes can be expanded with respect to the basis of spherical harmonics and be represented as an infinite set of two-dimensional fields (functions of only time and radial coordinates (t, r)). The operator $\hat{D}$ is then the corresponding two-dimensional second order operator.)

To the leading order (which we only will consider here) the non-spherical excitations $h_{\mu\nu}$ can be considered as classical and consequently we can assume that $h_{\mu\nu} = 0$ in (1.3). So far, as non-spherical excitations h are concerned, we are in the classical regime. In this case, the 4D theory of gravity with the Einstein-Hilbert action (1.2) reduces to quantum theory of only spherical excitations of the metric with the action $S_{gr}[g_{\mu\nu}^{sph}]$ describing effective exactly solvable two-dimensional theory of the 2D dilaton gravity. In the leading order, the effective theory describes the non-propagating spherically symmetric modes and is correctly tractable under the quantization (in the sense of generalized renormalizability). It is possible to compute the higher order corrections due to the presence of propagating gravitons, although at some point one is bound to encounter the problem of non-renormalizability of quantum gravity. The still unknown correct theory of quantum gravity is likely to avoid this problem but we expect that the modifications (which are not correctly calculable at the present moment) will not drastically alter the leading order result.

The 2D dilaton gravity has widely been investigated recently [6-14]. The review with detailed references can be found in [7]. The 4D dimensionally reduced models are discussed in [8,9].

2   Effective two-dimensional theory

The classical dynamics of a gravitational field interacting with the matter is determined by the standard Einstein-Hilbert action

$$S = \int d^4x \sqrt{-g^{(4)}} \left(-\frac{1}{16\pi\kappa} R^{(4)} + \mathcal{L}_{mat}\right), \qquad (2.1)$$

where $R^{(4)}$ is the scalar curvature determined by a four-dimensional metric $g_{\mu\nu}^{(4)}$ and L_{mat} is the Lagrangian of matter fields. The gravitational constant κ has the dimensionality of length squared $[l^2]$.

The metric $g_{\mu\nu}^{(4)}$ and the matter fields are assumed to be consistent with the condition of spherical symmetry. Let us consider in detail the gravitational part of the action (2.1). An arbitrary spherically symmetric metric can be written in the form

$$ds^2 = g_{\alpha\beta}(z)dz^\alpha dz^\beta - r^2(z)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{2.2}$$

where we assume that four-dimensional space-time is covered by the coordinates $(z^0, z^1, \theta, \varphi)$. Note that $g_{\alpha\beta}$ in (2.2) plays the role of a metric on the 2D space-time covered by the coordinates (z^0, z^1) and $r^2(z)$ is a function on this two-dimensional space. (We will use letters $\mu, \nu = 0, 1, 2, 3, 4$ for curved indices in four dimensions, while the corresponding indices in two dimensions will be denoted by $\alpha, \beta = 0, 1$).

In a usual way we can choose the coordinates (z^+, z^-) in which the first term in (2.2) takes the conformally flat form

$$ds^2s = e^{\sigma(z^+, z^-)}dz^+ dz^- - r^2(z^+, z^-)(d\theta^2 + \sin^2\theta d\varphi^2). \tag{2.3}$$

In the case of the Schwarzschild metric, (z^+, z^-) are the Kruskal coordinates defining the maximal extension of the black hole space-time and "r" is indeed the radius measured from the singularity located at the point $r = 0$.

The non-zero Ricci tensor components for the metric (2.3) are the following:

$$\begin{aligned}
R_{+-} &= \partial_+\partial_-\sigma + \partial_+\partial_-\ln r^2 + \frac{1}{2}\partial_+\ln r^2 \partial_-\ln r^2, \\
R_{\theta\theta} &= -1 - 2e^{-\sigma}\partial_+\partial_- r^2, \\
R_{\phi\phi} &= \sin^2\theta R_{\theta\theta}, \\
R_{\pm\pm} &= \partial_\pm^2 \ln r^2 + \partial_\pm\ln r^2 \partial_\pm\ln r^2 - \partial_\pm\sigma\partial_\pm\ln r^2.
\end{aligned} \tag{2.4}$$

and the scalar curvature $R^{(4)} = 4R_{+-}e^{-\sigma} - \frac{2}{r^2}R_{\theta\theta}$ is:

$$R^{(4)} = 4e^{-\sigma}\partial_+\partial_-\sigma - 2e^{-\sigma}\partial_+\ln r^2 \partial_-\ln r^2 + \frac{2}{r^2} + \frac{8}{r^2}e^{-\sigma}\partial_+\partial_- r^2 \tag{2.5}$$

Note that the first term in (2.5) coincides with the scalar curvature $R^{(2)}$ of the two-dimensional metric $g_{+-} = e^{\sigma(z^+, z^-)}$: $R^{(2)} = 4e^{-\sigma}\partial_+\partial_-\sigma$. Analogously, the whole expression (2.5) can be written in the covariant form with respect to the two-dimensional metric $ds^2 = g_{\alpha\beta}dz^\alpha dz^\beta$

$$R^{(4)} = R^{(2)} - \frac{2}{r^2}(\nabla r)^2 + \frac{2}{r^2} + \frac{2}{r^2}\Box r^2. \tag{2.6}$$

where $\Box = \nabla^2 = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$.

Since the determinant of the metric (2.3) is $g^{(4)} = -\frac{1}{4}e^{2\sigma}r^4\sin^2\theta$, the gravitational action

$$S_{gr} = -\frac{1}{16\pi\kappa} \int d^2z \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sqrt{-g^{(4)}} R^{(4)} \qquad (2.7)$$

takes the form:

$$S_{gr} = -\frac{1}{8\kappa} \int d^2z [r^2 R^{(2)} - 2(\nabla r)^2 + 2]. \qquad (2.8)$$

where we have omitted the integral of $\Box r^2$ which is the total derivative and does not affect the equations of motion. The action (2.8) determines the dynamics of spherically symmetric excitations of the 4D gravitational field. On the other hand, (2.8) describes the effective two-dimensional scalar-tensor theory of gravity. It is worth noting that this theory is indeed of the 2D dilaton gravity type. It is easy to see this introducing the "dilaton" field $\phi = \ln(\frac{r^2}{\kappa})$. Then eq.(2.8) takes form of the dilaton gravity [6-7]:

$$S_{gr} = -\frac{1}{8} \int d^2z [e^\phi(R - \frac{1}{2}(\nabla\phi)^2) + U(\phi)], \qquad (2.9)$$

where the "dilaton" potential is $U(\phi) = \frac{2}{\kappa}$. This observation is important for us since it allows one to use all the methods previously developed for the 2D dilaton gravity [6-7,12-14]. Note that usually one considers the following dilaton-gravity action

$$S_{str} = -\frac{1}{4} \int d^2z \sqrt{-g} e^\phi [R - (\nabla\phi)^2 + \lambda], \qquad (2.10)$$

which is inspired by string models. The essential difference between (2.9) and (2.10) lies in the quantum region. The string-inspired action (2.10) is shown to be finite, while the action (2.9) is renormalizable in the generalized sense: quantum corrections change the form of the potential $U(\phi)$. Having this in mind, let us consider instead of (2.8) the generalized action with an arbitrary "dilaton" potential $U(r)$:

$$S_{gr} = -\frac{1}{8} \int d^2z \sqrt{-g} [r^2 R^{(2)} - 2(\nabla r)^2 + \frac{2}{\kappa}U(r)], \qquad (2.11)$$

where we have introduced the dimensionless variable $r \to \frac{r}{\sqrt{\kappa}}$. Varying (2.11) with respect to the 2D metric $g_{\alpha\beta}$ and r leads to the equations of motion

$$2r\nabla_\alpha\nabla_\beta r = g_{\alpha\beta}[\frac{1}{\kappa}U(r) + 2r\Box r + (\nabla r)^2],$$

$$2\Box r + rR + \frac{1}{\kappa}U'(r) = 0. \qquad (2.12)$$

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The first equation in (2.12) means the existence of the two-dimensional Killing vector [10-12] $\xi_\alpha = \varepsilon_\alpha{}^\beta \partial_\beta r$ $(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0)$. Thus, we can choose the field r as one of the coordinates (which is space-like) and use the Schwarzschild gauge where the metric takes the form:

$$ds^2 = g dt^2 - g^{-1}\kappa dr^2, \tag{2.13}$$

where $g = g(r)$. For the metric (2.13) we get $\Box r = -\frac{g'}{\kappa}$, $(\nabla r)^2 = -\frac{g}{\kappa}$. Consequently, one has the following solution of equations (2.12):

$$g(r) = -\frac{2M}{r} + \frac{1}{r}\int^r U(\rho)d\rho, \tag{2.14}$$

where $M = const$. If $U(r) = 1$, then $g(r) = 1 - \frac{2M}{r}$ and we obtain the Schwarzschild metric. It is not surprising since the Schwarzschild metric is the unique spherically symmetric solution of Einstein equations in empty space. The constant M coincides with the ADM mass calculable at space-like infinity. It is worth noting that for $U = 1$ eqs.(2.12) are the Einstein equations in empty space

$$G_{\mu\nu} \equiv R_{\mu\nu}^{(4)} - \frac{1}{2}g_{\mu\nu}^{(4)}R^{(4)} = 0 \tag{2.15}$$

considered on the spherical metric (2.2-3). Hence, the reduction to the effective 2D theory (2.8) is self-consistent and we obtain again the Einstein equations.

To complete our consideration we present here the expressions for the effective two-dimensional and four-dimensional scalar curvature valid for the metric (2.2), (2.13-14):

$$R^{(2)} = -g'' = -\frac{2M}{r^3} + \frac{2}{\kappa r^3}\int^r U(\rho)d\rho - \frac{2}{\kappa r^2}U + \frac{U'}{\kappa r} \tag{2.16}$$

$$R^{(4)} = \frac{2}{\kappa r^2}(1 - U(r)) - \frac{U'}{\kappa r} \tag{2.17}$$

Note that for $U = 1$ we get $R^{(4)} = 0$ everywhere in the external region of gravitating body, as it follows from eqs.(2.15). In the next sections, we will show that taking into account quantum spherically symmetric excitations leads to the deformation of the form of the potential $U(r)$. According to (2.17), it manifests itself in non-zero value of $R^{(4)}$ outside the gravitating body. The Einstein tensor $G_{\mu\nu}$(2.15) for the metric (2.2) can also be written covariantly with respect to the 2D metric $g_{\alpha\beta}$:

$$\begin{aligned}
G_{\alpha\beta} &= \frac{2}{r}\nabla_\alpha\nabla_\beta r - g_{\alpha\beta}(\frac{1}{r^2} + \frac{2}{r}\Box r + \frac{(\nabla r)^2}{r^2}), \ \alpha,\beta = 0, 1, \\
G_{\theta\theta} &= \frac{r^2}{2}(R^{(2)} + \frac{2}{r}\Box r), \\
G_{\varphi\varphi} &= \sin^2\theta G_{\theta\theta}.
\end{aligned} \tag{2.18}$$

Eqs.(2.12) look like the quantum-corrected Einstein equations

$$G_{\mu\nu} = T_{\mu\nu}^{eff},$$ (2.19)

where

$$T_{\alpha\beta}^{eff} = \frac{g_{\alpha\beta}}{r^2}[U(\frac{r}{\sqrt{\kappa}}) - 1],$$

$$T_{\theta\theta}^{eff} = -\frac{1}{2\kappa}r\partial_r U(\frac{r}{\sqrt{\kappa}}),$$

$$T_{\varphi\varphi}^{eff} = \sin^2\theta T_{\theta\theta}$$ (2.20)

is the effective energy-momentum tensor which is induced due to quantum spherically symmetric excitations.

3 Quantization of spherically-symmetric excitations

Consider now quantum theory of spherically symmetric excitations of the metric described by the two-dimensional effective theory with the action (2.8):

$$S_{gr} = -\frac{1}{8}\int d^2z\sqrt{-g}[r^2R^{(2)} - 2(\nabla r)^2 + \frac{2}{\kappa}],$$ (3.1)

where r is dimensionless. Note that the dimensional gravitational constant κ from the combination $\frac{1}{\kappa}R^{(4)}$ in (2.1) has moved to the λ-term in the 2D action (3.1). It reflects the fact that the 2D effective theory (3.1) has a better renormalizable property than the initial 4D-action (2.1). As it has been noted in Sect.2, the action (3.1) takes the form of the 2D dilaton gravity which is widely investigated in recent years in connection with the interest in two-dimensional black holes [6-7]. In particular, the theory (3.1) was shown to be generally renormalizable in the sense that the renormalized action takes the same form as the original action (3.1) with some new potential $U(r)$:

$$S_{gr} = -\frac{1}{8}\int d^2z\sqrt{-g}[r^2R^{(2)} - 2(\nabla r)^2 + \frac{2}{\kappa}U(r)].$$ (3.2)

In the conformal gauge $g_{\alpha\beta} = e^{2\sigma}\bar{g}_{\alpha\beta}$ the action (3.2) takes the form

$$S_{gr} = \int d^2z\sqrt{-\bar{g}}[\frac{1}{2\psi}\bar{g}^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi + 2\bar{g}^{\alpha\beta}\partial_\alpha\psi\partial_\beta\sigma - \psi\bar{R} - \frac{1}{4\kappa}U(\psi)e^{2\sigma}],$$ (3.3)

where we denoted $\psi = \frac{r^2}{8}$ and $\bar{R} = R^{(2)}[\bar{g}]$.

Let us start with neglecting a possible anomalous term in the quantum version of the action (3.3). Of course, it is just an approximation, but its advantage is that the results obtained have a very simple analytic form and, moreover, possess the same interesting properties as in a more general case.

The divergencies now can be calculated by the background-field method [12]. It is useful to interpret (3.3) as the $D = 2$ σ-model (Note that our definition of curvature differs in sign from that of the paper [12].):

$$S = \int d^2 z \sqrt{-g} \left[\frac{1}{2} G_{ij}(X) g^{ab} \partial_a X^i \partial_b X^j - \frac{1}{2} R \Phi(X) + T(X) \right],$$

$$X^i = (\psi, \sigma); \Phi = 2v; \ T = -\frac{1}{4\kappa} U(\psi) \epsilon^{2\sigma};$$

$$G_{ij} = \begin{pmatrix} \frac{1}{\psi} & 2 \\ 2 & 0 \end{pmatrix}. \tag{3.4}$$

Since the metric G_{ij} is flat and the dilaton Φ is a linear function, the unique nontrivial divergency corresponds to the renormalization of the tachyon T. The coefficients of Weyl anomaly corresponding to the tachyon coupling take the form [12]:

$$\beta^T = -G^{ij} \nabla_i \nabla_j T + (G^{ij} \partial_i \Phi \partial_j T - 2T) \equiv \hat{\beta}^T + \Delta \beta^T \tag{3.5}$$

For (3.4) one gets

$$\Delta \beta^T = 0, \ \hat{\beta}^T = -\frac{1}{4\kappa} \epsilon^{2\sigma} \left(\frac{1}{\psi} U - 2\partial_\psi U \right) \tag{3.6}$$

Due to the factorization of the tachyon T (3.4) one obtains from (3.6) the β- function for the dilaton potential $U(v)$:

$$\beta^U = \left(\frac{1}{v} U - 2\partial_v U \right). \tag{3.7}$$

The fixed point of (zero of the β^U-function) corresponds to the potential

$$U(\psi) = c \sqrt{\psi} = \tilde{c} r. \tag{3.8}$$

In this case, the theory is finite [12] with the potential corresponding to the string-inspired dilaton gravity (2.10). A weaker renormalization condition is satisfied if $\left(\frac{1}{\psi} U - 2\partial_\psi U \right)$ is proportional to the potential itself, i.e., in the case of a Yukawa-like potential

$$U(r) = c \epsilon^{-\lambda r^2} r. \tag{3.9}$$

Then, the divergency can be absorbed into a renormalization of c. Inserting potentials (3.8) or (3.9) into (2.14) we obtain a metric corresponding to the UV fixed point. However, there exists a problem of reaching the fixed point since the classical ("bare") potential $U(r) = 1$ can be out of the attraction region of the UV fixed

point. Therefore we have to consider the renormalization group equation for the potential U:

$$\partial_t U = \beta^U = \frac{1}{\psi} U - 2\partial_\psi U, \tag{3.10}$$

where $t = \ln \frac{\mu}{\mu_0}$, μ being a scale parameter.

One should add the "initial" condition to eq.(3.10). We will assume that at $\mu = \mu_0$ ($t = 0$) the potential $U(\psi, t)$ coincides with the bare potential

$$U(\psi, t = 0) = 1. \tag{3.11}$$

It is easy to find a general solution of eq.(3.10):

$$U(\psi, t) = \sqrt{\psi} f(\psi - 2t), \tag{3.12}$$

where $f(...)$ is still an arbitrary function to be chosen from the initial condition.

Note that in our case eq.(3.10) is considered in the region $\{t \geq 0, \psi \geq 0\}$ being the transport equation with the characteristic line $\psi - 2t = 0$. Therefore, the solution U at the point (ψ, t) below this line ($\psi > 2t$) is obtained by transporting the initial condition, in our case it is $U(\psi, t = 0) = 1$, along the characteristic. Hence, for $\psi > 2t$ we get from (3.11-12) that $f(\psi) = \frac{1}{\psi^{1/2}}$ and consequently

$$U(\psi, t) = \frac{\psi^{1/2}}{(\psi - 2t)^{1/2}}, \quad \psi > 2t. \tag{3.13}$$

On the other hand, above the line $\psi = 2t$ (i.e. for $\psi < 2t$) the boundary condition $U(\psi, t)|_{\psi=0} = u(t)$ is "transported". We do not have this kind of a boundary condition from our problem. However, as one can see from the form of the general solution (3.12), the value of $U(\psi, t)$ for $\psi = 0$ cannot be different from zero. In other words, there is the unique possible boundary condition

$$U(\psi, t)|_{\psi=0} = 0 \tag{3.14}$$

consistent with eq.(3.10). Thus, the solution of (3.10) in the region $\{\psi \geq 0, t \geq 0\}$ takes the following form:

$$U(\psi, t) = \begin{cases} 0 & \text{if } \psi \leq 2t, \\ \frac{\psi^{1/2}}{(\psi - 2t)^{1/2}} & \text{if } \psi > 2t. \end{cases} \tag{3.15}$$

Note that the function (3.15) has a discontinuity along the characteristic line $\psi = 2t$. Remembering that $\psi = \frac{r^2}{8\kappa}$ we obtain

$$U(r, t) = \begin{cases} 0 & \text{if } 0 \leq r \leq 4\sqrt{\kappa t}, \\ \frac{r}{(r^2 - 16\kappa t)^{1/2}} & \text{if } r > 4\sqrt{\kappa t}. \end{cases} \tag{3.16}$$

As we expected, the fixed point (3.8) (or (3.9)) is not reached in the limit $t \to +\infty$. However, the t-dependence of the solution (3.16) can be absorbed into the

redefinition of the gravitational constant κ: $\kappa^* = \kappa t$. Then, (3.16) can be written in the form:

$$U(r, \kappa^*) = \begin{cases} 0 & \text{if } 0 \leq r \leq 4\sqrt{\kappa^*}, \\ \frac{r}{(r^2 - 16\kappa^*)^{1/2}} & \text{if } r > 4\kappa^*. \end{cases} \tag{3.17}$$

We will omit $*$ later on.

Let us now return to eq.(2.14) connecting the metric of the spherically symmetric solution with the potential $U(r)$. We get that the quantum spherically symmetric gravitational fluctuations lead to the following deformation of the Schwarzschild metric:

$$g(r) = -\frac{2M}{r} + \frac{1}{r}(r^2 - a^2)^{1/2}, \tag{3.18}$$

where $r \geq a = 4\sqrt{\kappa}$.

Analyzing the metric (3.18) we discover that the singularity of the Schwarzschild solution at $r = 0$ is now shifted to the finite radius $r = a$ (in four-dimensional picture it means that singularity now is "spread" over a two-dimensional sphere of radius $r = a$). One can see this when calculating the 4D scalar curvature (2.17)

$$R^{(4)} = \frac{1}{a^2}[2x^2(1 - \frac{1}{\sqrt{1-x^2}}) + x^4(1-x^2)^{-3/2}], \tag{3.19}$$

where $x = \frac{a}{r}$. In the limit $x \to 1$ we have $R^{(4)} \to +\infty$. On the other hand, for large r ($x \to 0$) we obtain

$$R^{(4)}(r) \approx \frac{2}{r^2}(\frac{a}{r})^4. \tag{3.20}$$

One can also get from (3.18) the asymptotic expression for the metric $g(r)$ for large $r >> a$:

$$g(r) \approx 1 - \frac{2M}{r} - \frac{a^2}{2r^2}. \tag{3.21}$$

Eq.(3.21) looks like the metric of a charged body with the mass M and the charge $Q^2 = a^2/2$ but with opposite sign in front of the charge's term. It is interesting that the potential U in (2.14), (3.18) does not lead to an additional contribution to the mass M,which is due to the fact that the U-term in (2.14), (3.18) has the asymptote $\sim \frac{1}{r^2}$ for large r.

It is worth noting that as follows from the general expression (2.17) (and (3.19)) the scalar curvature $R^{(4)}(r)$ does not depend on the mass of a gravitating body, it is rather universal and is determined by the parameters of the gravitational field itself (via the gravitational constant). Later on, the minimal radius a will be supposed to be equal to the Planck radius r_{pl} by assuming that possible factors ~ 10 are non-essential.

We see from (3.18) that the Schwarzschild horizon at $r_h = 2M$ is also shifted $r_h = \sqrt{4M^2 + a^2}$ so that the asymptotically flat metric (3.18) describes the space-time with the same causal structure as the Schwarzschild one. At the same time, the Schwarzshild singularity at $r = 0$ manifests itself both in the metric and in the

curvature ($R^{(4)} \sim \delta(r)$) while the deformed metric (3.18) is regular at $r = a$ and only the scalar curvature $R^{(4)}(r)$ is still singular.

Formally, there exists an extension of the metric (3.18) behind the singularity at $r = a$. To see this, let us change the variable: $r = a \cosh x$. Then, the space-time for $x > 0$ has the metric (3.18) while for $x < 0$ the metric takes the same form (3.18) but with sign $(-)$ in front of the second term. In terms of the variable r it simply means that eqs.(3.17)-(3.18) are extended to the other branch of the square root function. So we obtain that the resulting metric is a two-valued function of the radius r:

$$g^{(\pm)} = -\frac{2M}{r} \pm \frac{1}{r}(r^2 - a^2)^{1/2} \qquad (3.22)$$

At the point $r = a$, both functions $g^{(+)}(r)$ and $g^{(-)}(r)$ (but not their derivatives) are glued continuously. The scalar curvature on the $(-)$-sheet takes the form

$$R^{(4)}_{(-)} = \frac{1}{a^2}[2x^2(1 + \frac{1}{\sqrt{1 - x^2}}) - x^4(1 - x^2)^{-3/2}], \qquad (3.23)$$

where $x = \frac{a}{r}$. In the limit $x \to 1$ we have $R^{(4)}_{(-)} \to -\infty$. The $(-)$-sheet is also asymptotically flat but for large r $R^{(4)}_{(-)}(r)$ has a different type of asymptote than for the $(+)$-sheet (3.20):

$$R^{(4)}_{(-)}(r) \approx \frac{4}{r^2}. \qquad (3.24)$$

Note that for positive M the function $g_{(-)}(r)$ is negative everywhere. So the whole $(-)$-sheet is behind the horizon ($r = r_h$) from the point of view of an observer staying on the $(+)$-sheet. At this stage such a picture of complete space-time seems to be formal since no observer can penetrate through the singularity at $r = a$ and appear in the $(-)$-sheet. However, we will see in the next section that taking into account the conformal anomaly we obtain the space-time with the same structure but with the smooth behaviour at $r = a$. The extended space-time then occurs to be regular everywhere and geodesically complete.

4 Solution with the anomaly

Let us consider now the deformation of the potential $U(r)$ due to the fluctuations of the ghost and matter fields. Taking into account only the spherically symmetric excitations we obtain that these fields contribute to the quantum effective action via the Weyl anomaly from the gravitational integration measure and that of matter fields.

In two-dimensional dilaton gravity there is a well known ambiguity [13] in choosing the 2D metric to construct the Faddeev-Popov ghost determinant. One may use any metric of the form: $g_\alpha = r^\alpha g$, where $r = \exp\frac{\phi}{2}$ is dilaton and α is an arbitrary

constant. However, in our case the measure of integration over the 2D gravitational fields $(r, g_{\alpha\beta}^{(2)})$ is induced by integration over the 4D metric $g_{\mu\nu}^{(4)}$ (in assumption of spherical symmetry) and consequently should be determined with respect to the rescaled metric $\hat{g}_{\alpha\beta} = r^2 g_{\alpha\beta}$, so no ambiguity arises.

Hence, after gauge fixing $g_{\alpha\beta} = e^{2\sigma}\hat{g}_{\alpha\beta}$, the action will be supplied with the usual logarithm of the Faddeev-Popov ghost determinant [13]:

$$S_{FP} = \frac{24}{96\pi} \int d^2 z \sqrt{-\hat{g}} R_{\hat{g}} \Box_{\hat{g}}^{-1} R_{\hat{g}}, \tag{4.1}$$

where $R_{\hat{g}}$ is the 2D scalar curvature determined by the metric $\hat{g}_{\alpha\beta}$.

Considering the spherically symmetric configurations of matter fields we find that they are described by some effective 2D action which in general takes the form: $\mathcal{L}_{mat}^{eff} = \mathcal{L}_{(k)} r^{2k}$, where k runs over positive and negative integers (for example, the $k = 1$ term appears for 4D scalar fields). We will consider here only the simplest case of decoupled dilaton r when $\mathcal{L}_{(0)} = \sum_{i=1}^{N} (\nabla f^i)(\nabla f^i)$ is the action for the 2D conformal fields. The integration measure for the matter fields f^i is determined with respect to non-rescaled metric $g_{\alpha\beta}$.

Thus we come to the following quantum effective action:

$$S = S_{gr} + S_{FP} + N S_{anom},$$
$$S_{anom} = -\frac{1}{96\pi} \int R_g^{(2)} \Box_g^{-1} R_g^{(2)}. \tag{4.2}$$

Using the identity $R_{\hat{g}} = \frac{1}{r^2} \Box_g \ln r^2 + \frac{1}{r^2} R_g$, the action (4.2) can be rewritten in the form

$$S = S_{grav} - \frac{A}{8} \int R_g \Box_g^{-1} R_g + \frac{B}{2} \int (R_g \ln r - \frac{(\nabla r)^2}{r^2}) \sqrt{-g} d^2 z, \tag{4.3}$$

where $A = \frac{N-24}{12\pi}$, $B = \frac{24}{12\pi}$.

In the conformal gauge $g_{\alpha\beta} = e^{2\sigma} g_{\alpha\beta}$ we obtain from (4.3)

$$S = \int d^2 z \sqrt{-\tilde{g}} [\frac{1}{2\psi}(1 - \frac{B}{4\psi})(\nabla \psi)^2 + 2(1 - \frac{B}{4\psi})(\nabla \sigma)(\nabla \psi) + \frac{A}{2}(\nabla \sigma)^2$$

$$-\frac{1}{2}\bar{R}(A\sigma + 2\psi - \frac{B}{2}\ln \psi) - \frac{1}{4\kappa}U(\psi)e^{2\sigma}] + S_{FP}[g] + N S_{anom}[g], \tag{4.4}$$

where $\psi = \frac{r^2}{8\kappa}$ and $\bar{R}$ is the scalar curvature corresponding to the metric $g_{\alpha\beta}$. The action (4.4) again takes the form of the 2D σ-model (3.6) where

$$X^i = (\psi, \sigma); \Phi(X) = 2\psi + A\sigma - \frac{B}{2}\ln \psi; \ T = -\frac{1}{4\kappa} U(\psi)e^{2\sigma};$$
$$G_{ij} = \begin{pmatrix} \frac{1}{\psi}(1 - \frac{B}{4\psi}) & 2(1 - \frac{B}{4\psi}) \\ 2(1 - \frac{B}{4\psi}) & A \end{pmatrix} \tag{4.5}$$

The metric G_{ij} is flat and its determinant is

$$det\ G = -4(1 - \frac{B}{4\psi})(1 - \frac{(A+B)}{4\psi}). \qquad (4.6)$$

Note that $A + B = \frac{N}{12\pi} > 0$ and $det\ G$ (4.6) is zero if $\psi = \frac{B}{4}$ and $\psi = \frac{(A+B)}{4}$. At these points the kinetic term in (4.4) is singular and if $A \neq 0$ changes its sign.

For the tachyon β-function we get

$$\Delta\beta^T = 0, \ \beta^T = -G^{ij}\nabla_i\nabla_j T \qquad (4.7)$$

Analyzing this equation we consider two different cases.

4.1 A=0

We begin with the consideration of the case when the matter fields do not contribute to the effective action (4.3-4), i.e. $A = 0$. One can see that the target space metric then takes the form

$$ds^2_{targ} = (1 - \frac{B}{4\psi})ds^2_{B=0}, \qquad (4.8)$$

where $ds^2_{B=0} = \frac{1}{\psi}d^2\psi + 4d\psi d\sigma$ is the target metric of the σ-model (3.4). Due to the conformal transformation of the 2D Laplacian, we get from (4.7) that

$$\beta^T = \frac{1}{(1 - \frac{B}{4\psi})}\beta^T_{B=0}, \qquad (4.9)$$

where $\beta^T_{B=0}$ is the beta function (3.6).

Thus, we obtain the renormalization group equation for the potential $U(\psi)$

$$\partial_t U = \frac{1}{(1 - \frac{B}{4\psi})}[\frac{1}{\psi}U - 2\partial_\psi U]. \qquad (4.10)$$

To solve this equation, let us introduce the function $F(\psi)$ such that

$$dF = (1 - \frac{B}{4\psi})d\psi. \qquad (4.11)$$

Choosing the integration constant so that $F(\psi = \frac{B}{4}) = 0$ we obtain

$$F(\psi) = \psi - \frac{B}{4} - \frac{B}{4}\ln\frac{4\psi}{B}. \qquad (4.12)$$

As one can see, the second derivative $F''(\psi)$ (4.12) has a discontinuity at the point $\psi = \frac{B}{4}$.

Eq.(4.10) is the transport equation with the characteristic $F(\psi) - 2t = 0$. Initially, the function $U(\psi, t)$ is defined in the region $\{\psi > 0, t \geq 0\}$. However, as one can see from the form of the characteristic the line $\psi = B/4$ is a singular one

since the coefficient in front of $\partial_t U$ in (4.10) is zero. Therefore, eq.(4.10) must be considered separately in the regions $0 < \psi \le B/4$ and $\psi \ge B/4$. It is important that the function $F(\psi)$ (4.12) has single-valued inverse function $F^{-1}(x)$ defined for $x \ge 0$. The value of $U(\psi, t)$ for $F(\psi) - 2t > 0$ is obtained by transporting the initial condition at $t = 0$ along the characteristic. On the other hand, the value of $U(\psi, t)$ for $F(\psi) - 2t < 0$, is obtained by transporting the boundary condition: $U|_{\psi=B/4} = \mu(t)$. Considering eq.(4.10) in the regions $0 < \psi \le B/4$, $\psi \ge B/4$ separately, we may choose different boundary conditions $\mu_1(t), \mu_2(t)$. Actually, we have no concrete choice for $\mu(t)$. Therefore, for simplicity we will assume that $\mu(t) = 0$. The general solution of (4.10) is

$$U(\psi, t) = \sqrt{\psi} f(F(\psi) - 2t). \tag{4.13}$$

From the initial condition

$$U(\psi, t = 0) = 1 \tag{4.14}$$

we get the equation on the function f in the region where $F(\psi) - 2t > 0$

$$f(F(\psi)) = \psi^{-\frac{1}{2}} \tag{4.15}$$

Thus, one obtains that $f(x) = (F^{-1}(x))^{-\frac{1}{2}}$, where $F^{-1}(x)$ is the function inverse to (4.12) single-valued in the regions $0 < \psi \ge B/4$ and $\psi \ge B/4$. Note that $F^{-1}(x)$ is a continuous monotonic function but its derivative is singular at $x = 0$. Thus for $F(\psi) - 2t > 0$ we have the following solution of (4.10) with the initial condition (4.14):

$$U(\psi, t) = \frac{\psi^{\frac{1}{2}}}{(F^{-1}[F(\psi) - 2t])^{\frac{1}{2}}}$$

On the other hand, for $F(\psi) - 2t < 0$ the form of the function $f(..)$ is defined by the boundary condition: $\sqrt{B/4} f(-2t) = \mu(t)$. Since we have chosen $\mu(t) = 0$, $f(..) \equiv 0$ in this region. Thus, the complete solution of eq.(4.10) reads

$$U(\psi, t) = \begin{cases} 0 & \text{if } F(\psi) - 2t < 0, \\ \frac{\psi^{1/2}}{(F^{-1}[F(\psi)-2t])^{1/2}} & \text{if } F(\psi) - 2t > 0 \end{cases} \tag{4.16}$$

'Remembering now that $\psi = \frac{r^2}{8\kappa}$ we express the solution (4.16) in terms of the variables (r, t). Note at first that

$$F(\frac{r^2}{8\kappa}) = \frac{1}{8\kappa} F^*(r^2),$$

$$F^*(r^2) = r^2 - b^2 - b^2 \ln \frac{r^2}{b^2}, \tag{4.17}$$

where $b^2 = 2B\kappa$. From (4.17) we have that $[F^*]^{-1} 8\kappa = 8\kappa F^{-1}$ and consequently (4.16) takes the form

$$U(r, t) = \begin{cases} 0 & \text{if } F^*(r^2) < 16\kappa t, \\ \frac{r}{(F^{*-1}[F^*(r)-16\kappa t])^{1/2}} & \text{if } F^*(r^2) > 16\kappa t. \end{cases} \tag{4.18}$$

As one can easily see the t-dependence of the solution (4.18) again can be absorbed into the redefinition of the constants $\kappa^* = \kappa t$, $B^* = \frac{B}{t}$, with the function F^* being unchanged. Equation $F^*(r^2) - 16\kappa t = 0$ has two roots: $0 < r_{1m} < b$ and $r_{2m} > b$. In terms of new variables the solution (4.18) looks as follows:

$$U(r) = \begin{cases} \overline{\frac{r}{(F^{*-1}[F^*(r^2 - 16\kappa)])^{1/2}}} & \text{if } 0 < r \leq r_{1m}, \\ 0 & \text{if } r_{1m} \leq r \leq r_{2m}, \\ \frac{r}{(F^{-1}[F(r^2) - 16\kappa])^{1/2}} & \text{if } r \geq r_{2m}. \end{cases} \quad (4.19)$$

The potential (4.19) tends to (3.19) when $b \to 0$.

Let $r \geq r_{2m}$. The potential $U(r)$ (4.19) is a continuous function in this region. Near the point $r = r_{2m}$ it takes the form

$$U(r) \approx \frac{r}{b}(1 - \frac{1}{\sqrt{2}b}\sqrt{1 - (\frac{b}{r_{2m}})^2}(|r^2 - r_{2m}^2|)^{1/2}). \quad (4.20)$$

Thus, at $r = r_{2m}$ it has a finite value: $U(r_{2m}) = \frac{r_{2m}}{b}$. However, the derivative of the potential is singular: $\partial_r U(r) \to -\infty$.

The metric function $g = -\frac{2M}{r} + \frac{1}{r}\int_{r_{2m}}^{r} U(\rho)d\rho$ in the vicinity of $r = r_{2m}$ can be written as follows:

$$g \approx -\frac{2M}{r} + \frac{r}{2b} - \frac{1}{3\sqrt{2}b^2}\sqrt{1 - (\frac{b}{r_{2m}})^2}(|r^2 - r_{2m}^2|)^{3/2}, \quad (4.21)$$

and consequently, it is more regular near the point $r = r_{min}$ than (3.18) considered in Section 3. Indeed, we see from (4.21) that $g(r)$ and $g'(r)$ are regular in $r = r_{2m}$ though the second derivative is still singular. It means that the behaviour of the geodesics is regular near this point (since only the first derivatives of the metric enter into equations for geodesics) but the 4D curvature is singular: $R^{(4)} \to +\infty$ if $r \to r_{2m}$. For large $r >> r_{2m}$ the potential (4.19) asymptotically coincides with (3.17):

$$U(r) \approx \frac{r}{(r^2 - 16\kappa)^{1/2}} \quad (4.22)$$

obtained in the previous section. Hence, we have the same asymptote (3.20), (3.21) for the metric $g(r)$ and the curvature $R^{(4)}(r)$ for large $r >> r_{2m}$. Formally, there exists an extension of the metric beyond the point $r = r_{2m}$. Indeed, we may again consider the variable $r = r_{2m}\cosh x$. Then, we come to the same picture as in Section 3. with two sheets sewed on the hypersurface $r = r_{2m}$. The only difference from Section 3. is that the singularity at the minimal radius $r = r_{2m}$ is more mild now.

The potential (4.19) determines also a non-trivial metric defined in the compact region $0 < r \leq r_{1m}$. One can see that this metric describes the space-time which has singularities (of curvature) at the points $r = 0$ and $r = r_{1m}$. The curvature near $r = 0$ has the form: $R^{(4)}(r) \approx \frac{2}{r^2}(1 - e^{-\frac{4}{b}})$, while the behaviour of the metric

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near $r = r_{1m}$ is similar to that near $r = r_{2m}$. This space-time has no asymptotically flat region and is not connected with the space-time defined for $r \geq r_{2m}$. So it is not observable for any observer staying at $r > r_{2m}$. The physical meaning of such a space-time is not clear for us.

Thus, the general picture in the case when the Faddeev-Popov ghosts are taken into account ($A = 0, B \neq 0$) mainly repeats the picture considered in the previous section. We may also conclude that the influence of the ghosts is in smoothering of the singularity at the minimal radius r_{min}.

4.2 $A \neq 0$

The expression for β^T is covariant with respect to the target metric G_{ij}. To find β^T we use the fact that G_{ij} is flat and consequently it can be reduced to the standard diagonal form by means of the coordinate transformation in the target space. Following [14], the target metric (4.5)

$$ds_{targ}^2 = \frac{1}{\psi}(1 - \frac{B}{4\psi})d\psi^2 + 4(1 - \frac{B}{4\psi})d\psi d\sigma + Ad\sigma^2 \qquad (4.23)$$

by introducing the new target "coordinates" (ω, χ)

$$\omega^2 = 4\psi; \quad \chi = \frac{1}{2}\sigma + \frac{\psi}{A} - \frac{B}{4A}\ln 4\psi \qquad (4.24)$$

can be reduced to the form

$$ds_{targ}^2 = 4Ad\chi^2 - \frac{1}{A}\frac{1}{\omega^2}(\omega^2 - B)(\omega^2 - A - B)d\omega^2 \qquad (4.25)$$

Note again that $B > 0, B + A > 0$ for any N.

Let $A + B > B$. We see that the second term in (4.25) is positive if ω lies in the intervals $I_1 = (0, B)$ or $I_3 = (A + B, +\infty)$ and it changes the sign if ω lies in $I_2 = (B, A + B)$.

Let us consider the new variable Ω

$$\Omega = \int^{\omega} \frac{dy}{y}\sqrt{\pm(y^2 - B)(y^2 - A - B)}, \qquad (4.26)$$

where one must take sign $(+)$ for the intervals I_1, I_3 and $(-)$ for the interval I_2.

In the new coordinates (χ, Ω), the metric (4.25) is diagonal

$$ds_{targ}^2 = 4Ad\chi^2 - (\pm\frac{1}{A})d\Omega^2. \qquad (4.27)$$

Then, we get for the tachyon T in terms of the new variables

$$T = -\frac{1}{4\kappa}U(\psi)e^{2\sigma} = -\frac{1}{4\kappa}U(\omega)e^{-\frac{\omega^2}{A}}\omega^{\frac{2B}{A}}e^{4\chi}$$

$$\equiv -\frac{1}{4\kappa}\hat{U}(\Omega)e^{4\chi} \qquad (4.28)$$

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For β^T we obtain

$$\beta^T = -[\frac{1}{4A}\partial^2_\zeta T - (\pm A)\partial^2_\Omega T]. \tag{4.29}$$

Inserting (4.28) into (4.29) we obtain the β-function for the potential $\hat{U}$

$$\beta^{\hat{U}} = (\pm A)\partial^2_\Omega \hat{U} - \frac{4}{A}\hat{U} \tag{4.30}$$

where one must take sign $(+)$ for the intervals I_1, I_3 and $(-)$ for I_2.

As before, we consider now the renormalization group equation

$$\partial_t \hat{U} = (\pm)A\partial^2_\Omega \hat{U} - \frac{4}{A}\hat{U}, \tag{4.31}$$

where $t = ln\frac{\mu}{\mu_0}$. Assuming that for $t = 0$ the potential $U(\omega,t)$ coincides with the "bare" one: $\hat{U}(\omega, t = 0) = 1$, we get that eq.(4.31) should be supplied with the initial condition

$$\hat{U}|_{t=0} = \phi(\Omega) \equiv \epsilon^{-\frac{\omega^2(\Omega)}{A}}(\omega(\Omega))^{\frac{2B}{A}} \tag{4.32}$$

where $\omega(\Omega)$ is the inverse function to (4.26).

We see that $\hat{U}$ satisfies the differential equations of different type in various intervals. Let us assume that $A > 0$ ($N > 24$). Then (4.31) is the standard heat equation being considered in the intervals I_1, I_3, while it is the heat equation "with decreasing time" in I_2. The problem is that the "decreasing time" heat equation is known to be non-correct. Solving it formally by means of the Fourier transform in the interval I_2 we get:

$$\hat{U}(\Omega,t) = \epsilon^{-4t/A}\sum_{k=1}^{+\infty} a_k e^{A(\frac{\pi k}{\Lambda})^2 t}\sin\frac{\pi k}{\Lambda}\Omega, \tag{4.33}$$

where we assumed that in I_2 the variable Ω changes in the interval $(-\Lambda,0)$; $a_k = (\phi(\Omega),\sin\frac{\pi k}{\Lambda\Omega})$ are the Fourier coefficients of the initial condition (4.32). In order the sum (4.33) to be convergent for any finite t, the coefficients a_k must decrease faster than the exponent. It is not obviously the case for the initial condition of the type (4.32). So the solution of (4.31) in the interval I_2 does not exist at least within the quadratically integrable functions (probably (4.31) can be solved in the class of distributions).

If $A < 0$ ($N < 24$) eq.(4.31) is the standard heat equation only in the finite interval $I_2 = (A + B, B)$, while it is again of the "decreasing time" type in semi-infinite intervals I_1, I_3 with the same problems concerning the solution as above (the formal solution takes the same form as (4.33) changing the sum $\sum_k$ by the integral $\int_0^{+\infty} dk$). We have no definite idea about the possible physical interpretation of the solution valid only in the interval $-\Lambda < \Omega < 0$ or equivalently $2\kappa(A + B) < r^2 < 2\kappa B$. So we will consider only the case of positive A.

Let $A > 0$ and consider eq.(4.31) in the interval I_3. We may choose the integration constant in I_3 to arrange that $\Omega(A + B) = 0$. Then, $\Omega(\omega)$ determined by (4.26) varies in the interval $(0, +\infty)$

$$\Omega = \frac{1}{2}\sqrt{(\omega^2 - B)(\omega^2 - A - B)} - \frac{A + 2B}{2}\ln\frac{\sqrt{\omega^2 - B} + \sqrt{\omega^2 - A - B}}{\sqrt{A}} \quad (4.34)$$

$$-\frac{\sqrt{B(A + B)}}{2}\ln\frac{2B(A + B) - (A + 2B)\omega^2 + 2\sqrt{B(A + B)}\sqrt{(\omega^2 - B)(\omega^2 - A - B)}}{-A\omega^2}$$

Thus, the function $\hat{U}$ is defined in the region $\{t \geq 0, \Omega \geq 0\}$. Hence we should have an appropriate boundary condition at $\Omega = 0$. In fact, the results (the behaviour of $g(r)$ and $R^{(4)}(r)$ and possibility of extension on $(-)$-sheet) are not changed if we take an arbitrary condition: $\hat{U}|_{\Omega=0} = \mu(t)$. However, for simplicity, we choose the zero boundary condition:

$$\hat{U}|_{\Omega=0} = 0 \quad (4.35)$$

For the chosen initial and boundary conditions eq.(4.31) has the solution:

$$\hat{U}(\Omega, t) = \frac{e^{-\frac{4t}{A}}}{\sqrt{At}}\frac{1}{2\sqrt{\pi}}\int\limits_0^{+\infty}[e^{-\frac{(\Omega-\xi)^2}{4At}} - e^{-\frac{(\Omega+\xi)^2}{4At}}]\phi(\xi)d\xi, \quad (4.36)$$

where ξ and $\omega(\xi)$ are related by

$$\xi = \int\limits_{\sqrt{A+B}}^{\omega(\xi)}\frac{dy}{y}\sqrt{(y^2 - B)(y^2 - A - B)}. \quad (4.37)$$

We are interested in the potential $U(\omega, t) = e^{\frac{\omega^2}{A}}\omega^{\frac{-2B}{A}}\hat{U}(\Omega, t)$, where ω and Ω are related by (4.26). Note that the t-dependence of the solution (4.36) can be again absorbed into the redefinition of the constants A, B, κ. Indeed, let us consider the "renormalized" constants $A^* = \frac{A}{t}, B^* = \frac{B}{t}, \kappa^* = \kappa t$. Note that $\omega^2 \equiv \frac{r^2}{2\kappa} = t\omega^2(\kappa^*)$, $\Omega(A, B, \kappa) = t\Omega(A^*, B^*, \kappa^*)$. We assume that A^*, B^*, κ^* coincide with their "observable" values ($A^* = \frac{N-24}{12\pi}$, $B^* = \frac{24}{12\pi}$). Then, in terms of the new constants we obtain the potential $U(\omega)$

$$U(\omega) = \frac{1}{\sqrt{A}}e^{-\frac{4}{A}}\frac{1}{2\sqrt{\pi}}\int\limits_0^{+\infty}[e^{-\frac{(\Omega-\xi)^2}{4A}} - e^{-\frac{(\Omega+\xi)^2}{4A}}](\frac{\phi(\xi)}{\phi(\Omega)})d\xi, \quad (4.38)$$

where $\omega^2 = \frac{r^2}{2\kappa}$ and we have omitted $*$. The potential $U(r)$ for different (A, B) is plotted in Fig.1.

We interpret the 4D metric (2.14)

$$g(r) = \frac{-2M}{r} + \frac{1}{r} \int\limits_{r_{min}}^{r} U(\rho)d\rho \qquad (4.39)$$

where $r_{min} = \sqrt{2\kappa(A+B)}$, with the potential $U(r)$ in the form (4.38) as the Schwarzschild metric deformed due to the quantum spherically symmetric excitations of the ghosts and matter fields. The role of quantum fluctuations of the field f is only in vacuum polarization around the gravitating body which leads to the right hand side of the Einstein equations (2.19) in the form (2.20). A different situation would happen when the collapse of the f-field impulse is considered [6-7]. Then, we would have to take into account the back-reaction of the Hawking radiation that needs the static solutions of equations obtained by varying the quantum action (4.3). We do not consider it here.

Analyzing the metric (4.39) we note that here the minimal distance $r_{min} = \sqrt{2\kappa(A+B)}$ again appears. The expression (4.39) is valid only for $r \geq r_{min}$. The essential difference of the metric (4.39) with the potential (4.38) from that of (3.19) or (4.21) is that it is more regular near the point $r = r_{min}$. To see this, note that the solution of the heat equation $\hat{U}(\Omega)$ (4.36) in the vicinity of the point $r = r_{min}$ takes the form $\hat{U}(\Omega) = c\Omega$ $(0 < \Omega << 1)$, where $c > 0$ is an irrelevant constant. Then, we have for $U(\omega)$:

$$U(\omega) = ce^{\frac{\omega^2}{A}}\omega^{-\frac{2B}{A}}\Omega(\omega), \qquad (4.40)$$

where $\omega^2 = \frac{r^2}{2\kappa}$. From (4.40) and (4.34) we obtain for $r \approx r_{min}$ that

$$U(r) \approx c'(r^2 - r_{min}^2)^{3/2}.$$

Thus, the metric function $g(r)$ (4.39) near $r = r_{min}$ can be approximately written in the form

$$g(r) \approx 1 - \frac{2M}{r} + \frac{C'}{r}(r^2 - r_{min}^2)^{5/2}, \qquad (4.41)$$

where C is some constant. So we obtain that $g(r)$ and its derivatives g' and g'' are regular at $r = r_{min}$ and only the third derivative diverges $g'''(r_{min}) = +\infty$. Consequently, the 4D scalar curvature (2.17) takes finite value at $r = r_{min}$ though its derivative diverges $R'^{(4)}(r_{min}) = -\infty$. This non-analyticity implies a rather mild singularity which does not affect the behaviour of the geodesics. The latter is regular near $r = r_{min}$ that implies an analytic extension of the space-time beyond the hypersurface $r = r_{min}$ (in the opposite case we would obtain the manifold with the boundary at $r = r_{min}$ that seems to be unsatisfactory). As we have discussed above, it cannot be the extension to the small radius $r < r_{min}$ since we have no solution of our equations in this region. To construct such an extension, we note that the variable Ω was introduced in such a way that its differential squared $(d\Omega)^2$ gives the second term in (4.25). It is clear that for fixed ω both Ω (4.26) and $-\Omega$ satisfy this condition. We use this fact to consider $\Omega(\omega)$ as a two-fold function in

the interval I_3. Then, we can continue the above obtained expression and valid for $\Omega > 0$ to the negative values of Ω.

We see from (4.36) that $\hat{U}(\Omega)$ continued onto the interval $(-\infty, +\infty)$ is an odd function: $\hat{U}(-\Omega) = -\hat{U}(\Omega)$. As a result, for the fixed r the metric function $g(r)$ takes two values

$$g^{(\pm)}(r) = \frac{-2M}{r} \pm \frac{1}{r} \int\limits_{r_{min}}^{r} U(\rho)d\rho, \qquad (4.42)$$

where $U(\rho)$ is given by (4.38). The corresponding expression for the 4D scalar curvature is

$$R_{(\pm)}^{(4)} = \frac{2}{\kappa r^2}(1 \mp U(r)) \mp \frac{U'}{\kappa r}. \qquad (4.43)$$

Thus, we obtain the same picture as considered in Section 3. The metric (4.42) describes the 4D space-time with two sheets ($g^{(\pm)}$ is the metric on ($\pm$)-sheet) which are glued on the hypersurface of constant radial coordinate $r = r_{min}$. The functions $g^{(+)}(r)$, $g^{(-)}(r)$ and their first and second derivatives are regular and sewed continuously at $r = r_{min}$ and only the third derivatives diverge $g'''^{(\pm)}(r_{min}) = \pm\infty$. As a result, we see that one sheet is the extension of the other (One can see this transparently by using the variable x: $r = r_{min}\cosh x$. Then, $\Omega(-x) = -\Omega(x)$ and the ($-$)-sheet corresponds to $x < 0$. Metric is extended continuously into this region.): the geodesics of one sheet are not ended at $r = r_{min}$ but continuously extended to the other sheet. One can see from (4.38) that $U(r) > 0$ and hence $g^{(-)}(r) < 0$ while $g^{(+)}(r)$ has zero at the single point r_h which is a solution of the equation $2M = \int\limits_{r_{min}}^{r_h} U(\rho)d\rho$. To see the behavior of (4.42) at large distances ($r \gg r_{min}$) it is useful to note that the potential $U(\omega)$ (4.38) for $\omega^2 \gg A + B$ asymptotically coincides with (4.19), (3.19):

$$U(r) \sim \frac{r}{(r^2 - 16\kappa)^{1/2}} \qquad (4.44)$$

and consequently, the metric $g^{(\pm)}(r)$ (4.42) behaves for large $r \gg r_{min}$ as follows:

$$g^{(\pm)}(r) \sim 1 - \frac{2M}{r} \pm \frac{(r^2 - 16\kappa)^{1/2}}{r} \qquad (4.45)$$

Both sheets are asymptotically flat. However, the flatness is reached in a different way that one can see from the scalar curvature (4.43) at large distances:

$$R_{(+)}^{(4)} \sim \frac{2}{r^2}(\frac{a}{r})^4 \; ; \; R_{(-)}^{(4)} \sim \frac{4}{r^2}. \qquad (4.46)$$

This coincides with what we have obtained in Section 3. (3.20), (3.24). The plot of $R_{(+)}^{(4)}(r)$ is shown in Fig.2. We see that the space-time has a horizon located on the ($+$)-sheet at $r = r_h > r_{min}$ and the whole ($-$)-sheet is behind this horizon from the point of view of an observer staying at $r > r_h$ on ($+$)-sheet. The topology of
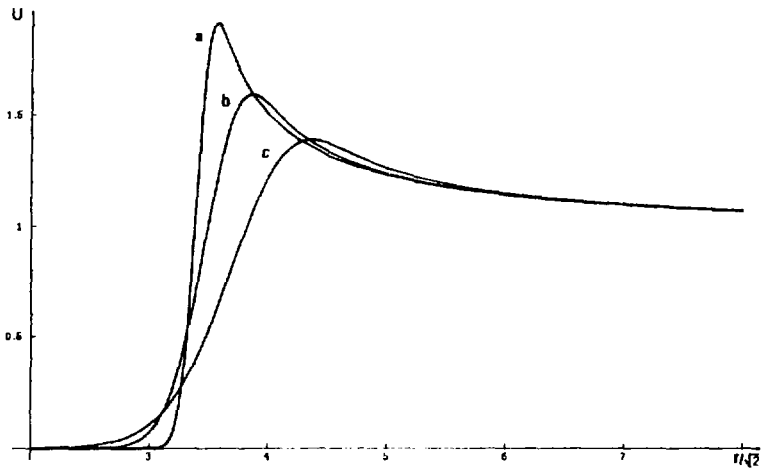
Fil. 1. The shape of the "dilaton" potential $U(r, A, B)$ for: a). $A = 0.1$, $B = 1$: b). $A = 0.3$, $B = 1$; c). $A = 1$, $B = 1$.
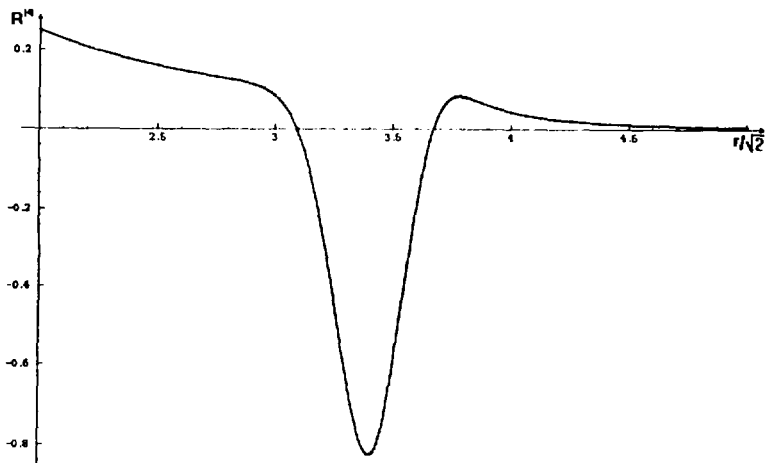

Fig. 2. The shape of the 4D scalar curvature $R^{(4)}(r)$ induced by quantum corrections for $A = 0.1$, $B = 1$.

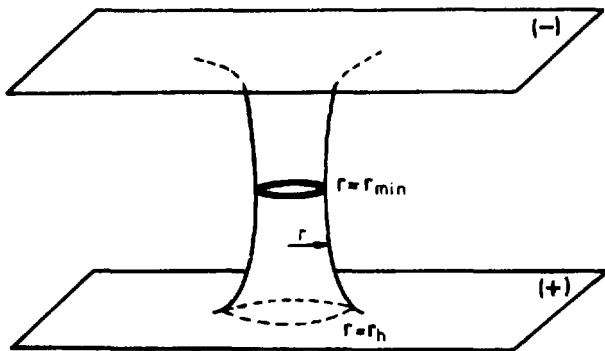Fig.3. The $t = const$ slice of the extended Schwarzschild space-time deformed by quantum corrections. It consists of two asymptotically flat ($\pm$)-sheets glued on the hypersurface of constant radial parameter $r = r_{min}$. The event horizon is located on the hypersurface $r = r_h$ of the (+)-sheet.
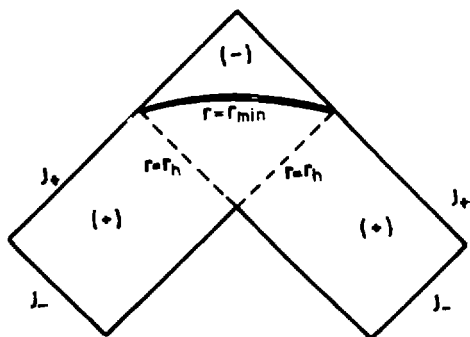

Fig.4. The Penrose diagram of the space-time deformed by quantum corrections for $A, B > 0$. The asymptotically flat (+)-region has the same causal properties as the classical Schwarzschild solution. It is analytically extended beyond the hypersurface $r = r_{min}$ to the other asymptotically flat (−)-region, so that the complete space-time is free from singularity.

$t = const$ slice of the space-time is shown in Fig.3. The Penrose diagram of the space-time can be seen in Fig.4.

It is worth noting that the resulting space-time does not much differ from that of the black hole with the internal de Sitter space instead of singularity. Such space-times have been earlier considered in [15] where it has been shown that such a picture may occur under the condition that limiting curvature exists and the Schwarzschild singularity does not arise. In case of the two-dimensional black holes such solutions were considered in a number of papers [16].

If now ω lies in the interval $I_1 = (0, B)$ (or equivalently the radius r lies in $(0, \sqrt{2\kappa B})$), the function $\Omega(\omega)$ (4.26) takes the form

$$\Omega(\omega) = \int_\omega^{\sqrt{B}} \frac{dy}{y} \sqrt{(B - y^2)(A + B - y^2)}, \qquad (4.47)$$

where we have chosen the integration constant so that $\Omega(B) = 0$. For $0 < \omega < \sqrt{B}$ we get that $\Omega(\omega) \geq 0$ and $\Omega(0) = +\infty$. The solution of eq.(4.31) (with sign $(+)$) takes the form (4.36). For the potential $U(\omega)$ we get the expression (4.38) where now $\Omega(\omega)$ (and $\omega(\xi)$) is given by (4.47). Proceed analogously, we also can consider here the $(\pm)$-sheets with the metric $g^\pm(r)$, respectively:

$$g^{(\pm)}(r) = -\frac{2M}{r} \mp \frac{1}{r} \int_r^{r_0} U(\rho) d\rho, \qquad (4.48)$$

where $r_0 = \sqrt{2\kappa B}$. The curvature is given by (4.43). The space-time is regular at $r = r_0$ and both sheets are sewed at $r = r_0$ continuously in the same manner as above. However, we again obtain the singularity at $r = 0$ since the 4D scalar curvature in the limit $r \to 0$ tends to

$$R_{(\pm)}^{(4)} \sim \frac{2}{r^2}[1 \mp e^{-\frac{4}{A+B}}]. \qquad (4.49)$$

Thus, for $A > 0$ the general picture is similar to that we have obtained for $A = 0$: the solution of eq.(4.32) describes two non-connected space-times. The first one, located at $r \geq r_{min}$, is free from the singularities and is asymptotically flat, while the second, located in the region $0 \leq r \leq r_0$, is singular at $r = 0$. However, only the first space-time has an obvious physical meaning and realizes our preliminary assumption that quantum corrections might lead to drastic deformation of the Schwarzschild solution and avoid space-time singularity.

5   Conclusion

We have studied the problem of deformation of the Schwarzschild solution due to quantum corrections in the approximation when only spherically symmetric excitations are taken into account.

One of our predictions concerns the behavior of the metric function $g(r)$ and the corresponding curvature $R^{(4)}(r)$ outside the gravitating body with the mass M at distances much larger than the Planck scale, $r >> a = 4\sqrt{\kappa}$:

$$g(r) \approx 1 - \frac{2M}{r} - \frac{1}{2}(\frac{a}{r})^2 - \frac{1}{8}(\frac{a}{r})^4 \; ; \; R^{(4)}(r) \approx \frac{2}{r^2}(\frac{a}{r})^4. \qquad (5.1)$$

It is worth noting that this expression is rather universal and does not depend on whether gravitational ghosts and matter contributions are included in the consideration or not. One can see from (5.1) that the space-time outside the gravitating body is no more Ricci flat as it follows from the classical Einstein equations, though the scalar curvature $R^{(4)}(r)$ rapidly tends to zero and becomes too negligible to be observed in present gravitational experiments.

The other important point is the behavior of the space-time near the Schwarzschild singularity. We have shown that quantum corrections lead to the shift of the singularity at $r = 0$ to the finite distance $r_{min} \sim r_{pl}$ and make it smoother. The scalar curvature $R^{(4)}(r)$ takes the finite value at $r = r_{min}$, so the space-time looks regularly near this minimal radius and allows the analytic extension beyond it. The complete space-time is free from singularities and consists of two asymptotically flat sheets glued on hypersurface of constant radial parameter $r = r_{min}$, so that one sheet is behind the horizon with respect to an observer staying on the other sheet. This is the result of the deformation of the Kruskal extension of the classical Schwarzschild metric due to quantum corrections. We see that these corrections indeed make the singular classical space-time more regular as it was originally assumed.

The method developed can be applied to the study of the other known classically singular solutions of general relativity: the Reissner-Nordstrom and cosmological ones. This work is in progress.

The next problem of interest is how the corrections found may change the gravitational collapse. It is known that the collapse in the classical Einstein gravity ended by formation of the singularity. Probably our results mean that the real singularity is not formed and at the end of gravitational collapse one obtains the regular space-time. However, at the present stage we can not make any definite conclusion since the Hawking radiation and its back-reaction were not taken into account. The previous study of 2D black holes tells us that the space-time points such as $r = r_{min}$ at which the $D = 2$ σ-model becomes degenerate (see eq.(4.6)) are the possible places for new singularities to be formed [17]. This problem needs further investigation.

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Казаков Д.И., Солодухин С.Н.   E2-93-371
О квантовой деформации решения Швардшильда

Мы рассматриваем деформацию решения Швардшильда общей относительности вследствие учета сферически-симметричных квантовых флуктуаций метрики и полей материи. В этом случае четырехмерное действие Эйнштейна редуцируется к эффективной двумерной дилатонной гравитации. Мы нашли, что сингулярность Швардшильда при $r = 0$ сдвигается на конечный радиус $r_{min} \sim r_{PL}$, где скаляр кривизны оказывается конечным. Полное пространство-время оказывается регулярным и состоит из двух асимптотически плоских листов, склеенных на гиперповерхности постоянного радиуса.

Kazakov D.I., Solodukhin S.N.   E2-93-371
On Quantum Deformation of the Schwarzschild Solution

We consider the deformation of the Schwarzschild solution in general relativity due to spherically symmetric quantum fluctuations of the metric and the matter fields. In this case, the 4D theory of gravity with Einstein action reduces to the effective two-dimensional dilaton gravity. We have found that the Schwarzschild singularity at $r = 0$ is shifted to the finite radius $r_{min} \sim r_{PL,}$ where the scalar curvature is finite, so that the space-time looks regular and consists of two asymptotically flat sheets glued at the hypersurface of constant radius.

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