IC/93/344

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

REMARKS ON MULTIPLE SOLUTIONS FOR ASYMPTOTICALLY LINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

Chang Kung Ching

Li Shujie

and

Liu Jiaquan

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

MIRAMARE-TRIESTE

IC/93/344

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

REMARKS ON MULTIPLE SOLUTIONS FOR ASYMPTOTICALLY LINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS [1]

Chang Kung Ching [2], Li Shujie [3] and Liu Jiaquan [4]
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

Some new results on the existence of multiple solutions for asymptotically linear elliptic boundary value problems via critical gorups are obtained.

MIRAMARE – TRIESTE
October 1993

Key words and phrases: multiple solutions, elliptic boundary value problems, critical groups.

---

[1]Partially supported by Chinese National Science Foundation

[2]Permanent address: Institute of Mathematics, Peking University, Beijing 100871, People's Republic of China.

[3]Permanent address: Institute of Mathematics, Academia Sinica, Beijing 100080, People's Republic of China.

[4]Permanent address: Department of Mathematics, Graduate School, Academia Sinica, USTC, Beijing 100039, People's Republic of China.

1. Introduction

It is known that the critical groups are useful in distinguishing critical points, cf. [Cha]. We shall present here a few examples from semilinear elliptic boundary value problems in showing how do they work in the study of multiple solutions. Let us consider the following problem

$$\left.\begin{aligned} -\Delta u &= g(x,u) \\ u|_{\partial\Omega} &= 0 \end{aligned}\right\} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$. Let λ_j be the j–th eigenvalue of $-\Delta$ with 0–Dirichlet boundary data. We assume,

(g_1) $g \in C^1(\bar{\Omega} \times \mathbb{R}^1, \mathbb{R}^1)$, $\quad g(x,0) = 0$

(g_2) $g'(x,0) < \lambda_1 \quad \forall\, x \in \bar{\Omega}$

(g_3) $\displaystyle \lim_{|t|\to\infty} \frac{g(x,t)}{t} \triangleq g_\infty > \lambda_2$

g_∞ satisfies one of the following three conditions:

(i) $g_\infty \notin \sigma(-\Delta)$, the spectrum of $-\Delta$.

(ii) $g_\infty \in \sigma(-\Delta)$ and $\phi(x,u) \triangleq g(x,u) - g_\infty u$ is bounded and satisfies the Landesman–Lazer condition

$$\int_\Omega \Phi\left(x, \sum_{j=1}^m t_j\ \varphi_j(x)\right) dx \to \infty \quad \text{as} \quad \sum_{j=1}^m t_j^2 \to \infty$$

where $\Phi(x,t) = \int_0^t \varphi(x,s)ds$, $\quad span\{\varphi_1, \varphi_2, \cdots \varphi_m\} = Ker(-\Delta - g_\infty I)$.

(iii) $g_\infty \in \sigma(-\Delta)$ and ϕ satisfies the strong resonance condition:

$$\forall\, \xi_j \in \mathbb{R}^m, |\xi_j| \to \infty \quad \forall\, u_j \to u \quad \text{in} \quad H_0^1(\Omega) \quad \text{and} \quad \forall\, v \in H_0^1(\Omega);$$

we have

$$\lim_{j\to\infty} \int \phi\left(x, u_j(x) + \sum_{i=1}^m \xi_j^i\ e_i(x)\right)\ v(x)dx = 0$$

and

$$\lim_{j\to\infty} \int_\Omega \Phi\left(x, u_j(x) + \sum_{i=1}^m \xi_j^i\ e_i(x)\right)\ v(x)dx = 0$$

where $\{e_i(x)\}_1^m$ is an orthonormal basis of the eigenspace $Ker(-\Delta - g_\infty I)$, and $\xi_j = (\xi_j^1, \xi_j^2, \cdots \xi_j^{m_0})$.

Our first result is

Theorem A. Assume g satisfies $(g_1) - (g_3)$, then (1.1) has at least three nontrivial solutions.

For the second result we assume that g satisfies,

(g_4) $\lambda_1 < g'(x,0) < \lambda_k < g_\infty \quad \forall\, x \in \bar{\Omega}$, where g_∞ satisfies one of the conditions (i), (ii), (iii) given in (g_3).

Then we have

Theorem B. Assume g satisfies $(g_1), (g_4)$. And assume that there exists to $\neq 0$ such that $g(x, t_0) = 0 \;\; \forall\, x \in \bar{\Omega}$, then (1.1) has at least three nontrivial solutions. Moreover, if we substitute (g_4) by the following

$$(g_4)' \qquad \lambda_2 < g'(x,0) < \lambda_k < g_\infty, \quad \forall\, x \in \bar{\Omega},$$

then (1.1) has at least four nontrivial solutions.

Corollary. Assume g satisfies $(g_1), (g_4)$. And assume that there exist $t_1 < 0$, $t_2 > 0$ such that $g(x, t_i) = 0 \;\; \forall\, x \in \bar{\Omega}, i = 1, 2$, then (1.1) has at least five nontrivial solutions.

Remark. Many authors have made contributions in this problem. The results for at least one nontrivial solution were obtained in [AmZ], [LiL], [FM], for at least two nontrivial solutions in [AmbM], [Ah] and [H] under the assumption: $g_\infty < \lambda_1$. Further results have been studied by many authors (see e.g. [Cha] and references therein).

Our theorems deal with the case $g_\infty > \lambda_2$. Theorem A is quite similar to the superlinear case (see [Wa]). But Theorem B and its Corollary are more delicate.

2. Proof of Theorem A

Set

$$f(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - G(x,u)\right] dx$$

for $u \in X \triangleq H_0^1(\Omega)$, where $G(x,t) = \int_0^t g(x,\tau)d\tau$. It is well known that $f \in C^2(X, \mathbb{R})$ satisfies the Palais–Smale condition. Any critical point of f corresponds to a (weak) solution of (1.1). Without loss of generality, we assume that f has only finite number of critical points.

Theorem A is proved by the following two steps.

Step 1. (1.1) has two nontrivial solutions, one is positive, another is negative.

Set

$$g_1(x,t) = \begin{cases} g(x,t) & t \geq 0 \\ 0 & t \leq 0 \end{cases} \tag{2.1}$$

and consider the modified problem

$$\left.\begin{aligned} -\Delta u &= g_1(x,u) \\ u|_{\partial\Omega} &= 0\,. \end{aligned}\right\} \tag{2.2}$$

We define

$$f_1(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - G_1(x,u)\right] dx$$

where $G_1(x,t) = \int_0^t g_1(x,\tau)d\tau$. We claim that f_1 satisfies (PS). Let (u_n) be a sequence such that

$$|f_1(u_n)| < c$$

and

$$\nabla f_1(u_n) \to \theta \quad \text{as} \quad n \to \infty\,. \tag{2.3}$$

From (g_3) and (2.1) we get

$$g_1(x,t) = g_\infty t + 0(t) \quad \text{as} \quad t > 0 \quad \text{large}\,. \tag{2.4}$$

(2.3) implies that $\forall\, \varphi \in X$

$$\int_\Omega [\nabla u_n \, \nabla\varphi - g_1(x,u_n)\varphi]dx \to 0 \quad \text{as} \quad n \to \infty\,. \tag{2.5}$$

Set $\varphi = u_n$, we have

$$\begin{aligned} \|u_n\|^2 &\leq \int_\Omega g_1(x,u_n)u_n \, dx + 0(\|u_n\|) \\ &\leq C + C\|u_n\|^2_{L^2} + 0(\|u_n\|)\,. \end{aligned}$$

If $\|u_n\|_{L_2}$ is bounded, then so is $\|u_n\|$. Otherwise, $\|u_n\|_{L^2} \to +\infty$. Let $v_n = \dfrac{u_n}{\|u_n\|_{L^2}}$, then $\|v_n\|_{L^2} = 1$ and $\|v_n\|$ is bounded. A subsequence of v_n converges to v with $\|v\|_{L^2} = 1$, strongly in L^2 and weakly in H_1^0. From (2.5) it follows

$$\int_\Omega [\nabla v \, \nabla\varphi - g_\infty v^+ \varphi]dx = 0, \quad \forall\, \varphi \in H_1^0, \tag{2.6}$$

where

$$v^+ = \max\{0, v\}\,.$$

The regularity theory implies

$$\begin{cases} \Delta v + g_\infty \, v^+ = 0 & \text{in} \quad \Omega \\ v = 0 & \text{on} \quad \partial\Omega\,. \end{cases} \tag{2.7}$$

By the maximum principle $v = v^+ \geq 0$. But $g_\infty \neq \lambda_1$ and hence $v \equiv 0$ which contradicts with $\|v\|_{L^2} = 1$. A standard argument shows that (u_n) has a convergent subsequence. (PS) is true for f_1.

From (g_2) there exist $\rho > 0, \delta > 0$ such that

$$f_1(u) \geq \delta \quad \forall\, u \in S_\rho = \{u \in X \,|\, \|u\| = \rho\}$$

and from $g_\infty > \lambda_2$ we can take t large such that

$$f_1(t\,\varphi_1) < 0$$

where φ_1 is the first eigenfunction of $-\Delta$ with 0–Dirichlet boundary data. Consequently, by the mountain pass lemma (2.2) has a weak solution u_1. By means of maximum principle and regularity of solution of elliptic BVP we know that the solution u_1 of (2.2) is classical and $u_1 > 0$ for $x \in \Omega$ and the outward directional derivative $\frac{\partial u(x)}{\partial n} < 0$ for $x \in \partial\Omega$. Therefore u_1 is a solution of (1.1).

Similarly, we get a negative solution u_2 of (1.1).

Using Chapter II, Theorem 1.6 of [Cha], we have

$$rank\ C_q(f_1, u_1) = \delta_{q1} \tag{2.8}$$

where $C_q(f,u)$ denotes the q^{th} critical group of f at u. By Chapter III, Theorem 1.1 and Corollary 1.2 of [Cha], we have

$$rank\ C_q(f,u_1) = \ rank\ C_q(f|_{C_0^1(\bar\Omega)}, u_1) = \ rank\ C_q(f_1|_{C_0^1(\bar\Omega)}, u_1) = \delta_{q1} \quad \forall\ q = 0,1,2,\dots \tag{2.9}$$

By the same method, we have

$$rank\ C_q(f,u_2) = \delta_{q1} \quad \forall\ q = 0,1,2,\dots \tag{2.10}$$

Step 2. The existence of third solution.

Let $X^-(X^+)$ be the negative (positive) subspace of $(-\Delta - g_\infty I)$ (respectively), then there exists $R > 0$ such that

$$\underset{u \in X^-\ \|u\| \geq R}{Sup} \quad f(u) \quad < \quad \inf_{v \in X^+} \quad f(v)\ .$$

According to [Liu] we know that f possesses a critical point u satisfying

$$rank\ C_m(f,u) \neq 0 \tag{2.11}$$

where $2 \leq m = \dim X^-$. Since θ is a local mimimizer, we have

$$rank\ C_q(f,\theta) = \delta_{q0}\ . \tag{2.12}$$

Combining (2.9), (2.10), (2.11) and (2.12) we proved that u is the third nontrivial solution of (1.1). Theorem A is proved.

Remark 2.1 When g_∞ satisfies condition (iii), f satisfies the $(PS)_c$ condition $\forall\ c \neq 0$. The first deformation theorem can be extended to study the fake critical set (see [ChaL]).

3. Proof of Theorem B

The proof is divided into four steps. We suppose $t_0 > 0$, the proof is similar for $t_0 < 0$.

Step 1. Let us define

$$\tilde g(x,t) = \begin{cases} 0 & t < 0 \\ g(x,t) & t \in [0,t_0] \\ 0 & t > t_0\ . \end{cases}$$

and

$$\tilde{f}(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - \tilde{G}(x,u)\right] dx$$

where $\tilde{G}(x,t) = \int_0^t \tilde{g}(x,\tau)d\tau$. Since $\tilde{f}$ is bounded below and satisfies (PS), there is a minimizer u^1 of $\tilde{f}$. According to the maximum principle, we obtain: either $u^1 \equiv 0$ or $0 < u^1(x) < t_0 \ \ \forall\, x \in \Omega$, $\frac{\partial u^1}{\partial n}\Big|_{\partial\Omega} < 0$. But by the assumption $g'(x,0) > \lambda_1$, θ is not a minimizer, i.e., $u^1 \not\equiv \theta$. Thus u^1 must be a local minimizer of the functionals f and f_1 (the latter was defined previously in Theorem A), in the $C_0^1(\bar{\Omega})$ topology. However, according to Chapter III, Theorem 1.1 and Corollary 1.2 of [Cha] (as well as [BrN]), one concludes that u^1 is also a local minimizer of f in $H_0^1(\Omega)$ topology. Thus

$$rank\ C_q(f,u^1) = \delta_{q0}\ . \tag{3.1}$$

Step 2. Since f_1 is unbounded below, and u^1 is a local minimizer of f_1, we obtain a positive mountain pass point $u^2 \not\equiv \theta$ of f_1, (cf. Chapter II, Remark [Cha]). By the same reason, u^2 is also a critical point of f, and we have

$$\begin{aligned} C_q(f,u^2) &= C_q(f|_{C_0^1(\bar{\Omega})}, u^2) \\ &= C_q(f_1|_{C_0^1(\bar{\Omega})}, u^4) \\ &= C_q(f_1,u^2)\ . \end{aligned}$$

Therefore

$$rank\ C_q(f,u^2) = \delta_{q1} \tag{3.2}$$

the last equality follows from Chapter II, Theorem 1.6 [Cha]. (3.1) and (3.2) imply that $u^1 \neq u^2$.

Step 3. As in the proof of Theorem A, we obtain a critical point u^3 satisfying

$$rank\ C_q(f,u^3) \neq 0 \tag{3.3}$$

with $m = \dim\ X^- \geq 2$.

We only want to show $u^3 \neq \theta$. Indeed,

$$rank\ C_q(f,\theta) = 0 \tag{3.4}$$

$\forall\, q > \dim \oplus_{1\leq i\leq k-1} ker\ (-\Delta - \lambda_i I)$, because $g'(x,0) < \lambda_k$.

From $\lambda_k < g_\infty$, we obtain

$$C_m(f,\theta) = 0\ . \tag{3.5}$$

Therefore $u^3 \neq \theta$.

Step 4. Under $(g_4)'$ we can get one more solution by the mountain pass lemma.

In Step 1 we have obtained a $u^1 \in C_X$ which is a local minimizer of $\tilde{f}$, where $C_X = C \cap C_0^1(\bar{\Omega})$ and $C = \{u \in H_0^1(\Omega) | t\varphi_1 \leq u \leq t_0, a.e.\ \text{for } t \text{ small}\}$. Therefore $d^2\tilde{f}(u^1)$ is nonnegative and we have

$$-\Delta - g'(x,u^1) = d^2\tilde{f}(u^1) \geq 0\ . \tag{3.6}$$

Let

$$v = u - u^1, g_1(x,v) = g(x, v+u^1) - g(x,u^1) \ ,$$

we have

$$g_1'(x,\theta) = g'(x,u^1) \tag{3.7}$$

and (1.1) is equivalent to

$$\left.\begin{array}{ll} -\Delta v & = g_1(x,v) \\ v|_{\partial\Omega} & = 0 \ . \end{array}\right\} \tag{3.8}$$

Let

$$\tilde{g}_1(x,t) = \begin{cases} g_1(x,t) & t \geq 0 \\ 0 & t \leq 0 \end{cases} \tag{3.9}$$

and define

$$\begin{aligned} f_1(v) &= \int_\Omega \left[\frac{1}{2}|\nabla v|^2 - G_1(x,v)\right] dx \ , \\ \tilde{f}_1(v) &= \int_\Omega \left[\frac{1}{2}|\nabla v|^2 - \tilde{G}(x,v)\right] dx \end{aligned}$$

where $G_1(x,t) = \int_0^t g_1(x,\tau)d\tau, \quad \tilde{G}_1(x,t) = \int_0^t \tilde{g}_1(x,\tau)d\tau.$

From (3.6) and (3.7) we get

$$-\Delta - g_1'(x,\theta) \geq 0 \ . \tag{3.10}$$

From (3.10) we know that θ is a local minimizer of $\tilde{f}_1$. Using the mountain pass lemma we immediately get a critical point v^+ of $\tilde{f}_1, v^+ > \theta$ and

$$rank\ C_p(\tilde{f}_1, v^+) = \delta_{p1} \ . \tag{3.11}$$

Let $u^+ = v^+ + u^1$ then $u^+ > u^1$. Note that

$$f_1(v) = f(u) + const \tag{3.12}$$

and thus

$$\begin{aligned} C_p(f,u^+) &= C_p\left(f\big|_{C_0^1(\bar{\Omega})}, u^+\right) &= C_p\left(f_1|_{C_0^1(\bar{\Omega})}, v^+\right) \\ &= C_p\left(\tilde{f}_1|_{C_0^1(\bar{\Omega})}, v^+\right) &= C_p\tilde{f}_1, v^+) \ . \end{aligned}$$

Therefore

$$rank\ C_p(f,u^+) = \delta_{p1} \ . \tag{3.13}$$

By the similar way we get a negative solution v^- of (3.8). Let $u^- = v^- + u^1$, then $u^- < u^1$ and u^- is a solution of (1.1) which satisfies

$$rank\ C_p(f,u^-) = \delta_{p1} \ . \tag{3.14}$$

By $\lambda_2 < g'(x,0)$ we have

$$rank\ C_p(f,\theta) = \delta_{pk} \quad k \geq 2 \tag{3.15}$$

(3.14) and (3.15) implies $u^- \neq \theta$. In combining with Step 3, then we get four nontrivial solutions. Theorem B. is proved.

Proof of Corollary. By the assumption $\exists\, t_1 < 0$ such that $g(x,t_1) = 0$ we can do in the same way by cut off function and obtain two other negative solutions: a local minimizer, and a mountain pass point.

Acknowledgments

The authors would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

REFERENCES

[Ah] S. Ahmad, "Multiple nontrivial solutions of resonant and nonresonant asymptotically linear problems", Proc. Amer. Math. Soc. **96** (1987) 405–409.

[AmZ] H. Amann and E. Zehnder, "Nontrivial solutions for a class of nonresonance problem and applications to nonlinear differential equations", Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **7** (1980) 539–603.

[AmbM] A. Ambrosetti and G. Mancini, "Sharp nonuniqueness results for some nonlinear problems", Nonlinear Anal. **5** (1979) 635–645.

[BrN] H. Brezis and L. Nirenberg, "H^1 versus C^1 local minimizers", (to appear).

[Cha] K.C. Chang, *Infinite Dimensional Morse Theory and Multiple Solution Problems* (Birkhäuser, Boston, 1993).

[ChaL] K.C. Chang and J.Q. Liu, "A strong resonance problem", Chinese Ann. Math. **11**, B.2 (1990) 191–210.

[FM] D.G. De Figueiredo and O.H. Miyagaki, "Semilinear elliptic equations with the primitive of the nonlinearity away from the spectrum", Nonlinear Anal. **17** (1991) 1201–1219.

[H] N. Hirano, "Multiple nontrivial solution of semilinear elliptic equations", Proc. Amer. Math. Soc. **103** (1988) 468–472.

[LiL] S.J. Li and J.Q. Liu, "Nontrivial critical points for asymptotically quadratic function", J. Math. Anal. and Applications **165** (1992) 333–345.

[Liu] J.Q. Liu, "A Morse index of a saddle point", System Sci. and Math. Sci. **2** (1989) 32–39.

[Wa] Z.Q. Wang, "On a superlinear elliptic equation", Ann. Inst. Henri Poincaré, Vol.8, No.1 (1991) 43–57.