IC/94/67
INTERNAL REPORT
(Limited Distribution)

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

LARGE AMPLITUDE LOCALIZED STRUCTURES IN A RELATIVISTIC ELECTRON-POSITRON ION PLASMA

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ABSTRACT

The nonlinear propagation of circularly polarized electromagnetic (CPEM) waves with relativistically strong amplitude in an unmagnetized cold electron-positron ion plasma is investigated. The possibility of finding soliton solutions in such a plasma is explored. In one- and two-dimensions it is shown that the presence of a small fraction of massive ions in the plasma leads to stable localized solutions.

MIRAMARE - TRIESTE
March 1994

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Recently, the nonlinear propagation of electromagnetic waves in electron-positron (e-p) plasma has attracted considerable attention [1]. These plasmas, found (for example) near the polar caps of pulsars, in the active galactic nuclei, in the early universe, will always be created in system whose temperature exceeds twice the electron rest mass ($\sim 1.2\,\mathrm{MeV}$) [2]. Propagation of intense short laser pulses in a plasma can also lead to pair production resulting in a three-component electron-positron-ion (e-p-i) plasma [3]. In fact such three-component plasmas have been seen in laboratory experiments [4-5] intending to use positrons as probes to study transport in tokamaks. In addition to several other applications [like the pulsar magnetosphere modeling [6]], an investigation of the e-p/e-p-i plasmas is likely to further our understanding of the early universe [7-8], in particular, of the MeV epoch in the evolution of the universe; it may, indeed, be possible that a deeper insight into the behavior of an interacting plasma fluid in this era may provide valuable clues to its later evolution. A stable localized solution with density excess may, coupled with gravity, create templates for confining matter and creating inhomogeneities necessary to understand the observed structure of the visible universe.

The importance of the three-component admixture plasma has led to several theoretical investigations. Rizzato [9] studied the localization of weakly nonlinear circularly polarized electromagnetic [CPEM] waves in a cold plasma made up of electrons, positrons, and ions. In Ref. [10] the propagation of intense electromagnetic radiation in an admixture of unmagnetized three-component plasma is investigated analytically, and it is found that such a plasma may be localized with the generation of a humped ambipolar electrostatic potential, and that this potential could be used to accelerate charged particles. It is also noted in [10], that the procedure of series expansion, is not valid for the case when $\alpha \ll 1$, where α is the ratio of the unperturbed ion to electron densities.

In this paper we abandon the small amplitude approximation, and study the nonlinear propagation of ultrarelativistic intense electromagnetic (EM) waves in a plasma of unmagnetized electrons, positrons, and massive ions, we aim to find localized stable structures sustained by this plasma.

The equilibrium state of the three-component system is characterized by an overall charge neutrality $n_o^- = n_o^+ + N_{oi}$, where n_o^-, n_o^+ and N_{oi} are the unperturbed number densities of the electrons, positrons, and ions, respectively. Due to their relatively large inertia, the ions do not respond to the dynamics under consideration and just provide a neutralizing background.

To describe the propagation of electromagnetic waves in such a plasma, we start with Maxwell equations expressed in terms of the vector ($\mathbf{A}$), and the scalar (ϕ) potentials:

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} - \Delta \mathbf{A} + \frac{\partial}{\partial t} \nabla \phi + \left(n^- \mathbf{v}^- - (1-\alpha)\, n^+ \mathbf{v}^+ \right) = 0 \tag{1}$$

and

$$\Delta \phi = \left(n^- - (1-\alpha)\, n^+ - \alpha \right) . \tag{2}$$

The system is closed by invoking the hydrodynamic equations consisting of the equation of motion

$$\frac{\partial \mathbf{P}^\pm}{\partial t} + \nabla (1 + (\mathbf{P}^\pm)^2)^{1/2} = \mp \frac{\partial \mathbf{A}}{\partial t} \mp \nabla \phi , \tag{3}$$

and the continuity equation:

$$\frac{\partial n^\pm}{\partial t} + \nabla (n^\pm \mathbf{v}^\pm) = 0 , \tag{4}$$

for each of the mobile components. Equations (1)-(4), written in the gauge $\nabla \cdot \mathbf{A} = 0$, are dimensionless with the following normalizations: The time and space variables are measured in units of the electron plasma frequency $\omega_e \; (= (4\pi n_o^- e^2/m_e)^{1/2})$, and the collisionless skin depth (c/ω_e), the vector and scalar potentials are normalized to $m_e c^2/e$, the relativistic momentum $\mathbf{P}$ to $m_e c$, and n^- and n^+ to their respective equilibrium densities n_o^- and n_o^+. The coefficient $\alpha = N_{oi}/n_o^-$ is the ratio of ion equilibrium density to electron equilibrium density. In terms of $\mathbf{P}$, the dimensionless quiver velocity is given by

$$\mathbf{v}^\pm = \frac{\mathbf{P}^\pm}{(1 + (\mathbf{P}^\pm)^2)^{1/2}} . \tag{5}$$

For the one-dimensional propagation ($\partial/\partial z \neq 0$, $\partial/\partial x = 0 = \partial/\partial y$) of a CPEM wave with a mean frequency ω_o and a mean wavenumber k_o along the z-axis, the vector potential can be represented as:

$$\mathbf{A} = \frac{1}{2} (\mathbf{x} + i\mathbf{y}) A(z,t) \exp\{ i k_o z - i\omega_o t \} + \text{c.c.} \tag{6}$$

and can be readily shown [through Eq. (3)] to be proportional to the transverse momentum $\mathbf{P}_\perp^\pm$,

$$\mathbf{P}_\perp^\pm = \mp \mathbf{A} . \tag{7}$$

Notice that $A(z,t)$ is a slowly varying function of z and t. The longitudinal motion of plasma is determined by

$$\frac{\partial P_z^\pm}{\partial t} + \frac{\partial}{\partial z} \left(1 + |A|^2 + (P_z^\pm)^2 \right)^{1/2} = \mp \frac{\partial \phi}{\partial z} , \tag{8}$$

the z-components of the equations of motion, and the continuity equations rewritten as:

$$\frac{\partial n^\pm}{\partial t} + \frac{\partial}{\partial z} \left[n^\pm \frac{P_z^\pm}{(1 + |A|^2 + (P_z^\pm)^2)^{1/2}} \right] = 0 \tag{9}$$

where Eq. (7) has been used to eliminate $P_\perp^\pm$.

It is now convenient to introduce new variables, $\xi = z - v_g t$, and $\tau = t$, where $v_g = k_o/\omega_o$ is the group velocity of the electromagnetic wave packet, and $v_g \partial/\partial\xi \gg \partial/\partial\tau$. The wave frequency ω_0 satisfies the dispersion relation: $\omega_o^2 = k_o^2 + (2-\alpha)$ implying $v_g \lesssim 1$ for a transparent plasma for which $\omega_o \gg 1$ ($\omega_o \gg \omega_e$ in the dimensional form). Equations (8) and (9) are easily integrated to obtain $P_z^\pm$ and $n^\pm$ assuming that the vector potential tends to zero at infinity. Substituting the obtained values of $P_z^\pm$ and $n^\pm$ together with Eq. (6) into Eqs. (1) and (2), and assuming that $\omega_o \gg (1 + |A|^2)^{1/2}$ [placing an upper limit on the allowed wave amplitude], we obtain:

$$2i\omega_o \frac{\partial A}{\partial \tau} + \frac{(2-\alpha)}{\omega_o^2} \frac{\partial^2 A}{\partial \xi^2} + A \left[\frac{\phi}{(1-\phi^2)} \right] [\alpha - (2-\alpha)\phi] = 0 , \tag{10}$$

and

$$\frac{\partial^2 \phi}{\partial \xi^2} = \frac{1}{2}\left\{\frac{(1+|A|^2)}{(1+\phi)^2} - \frac{(1-\alpha)\,(1+|A|^2)}{(1-\phi)^2} - \alpha\right\} . \tag{11}$$

Equations (10)–(11) constitute a closed set describing the nonlinear propagation of powerful CPEM waves of arbitrary [as long as $|A| < \omega_o$] amplitude in an unmagnetized, transparent cold electron-positron-ion plasma. It was shown in Ref. [11] that a pure electron-positron plasma ($\alpha = 0$) cannot sustain an electrostatic field ϕ. As a result the CPEM waves cannot be localized in a pure e-p plasma. An investigation of Eq. (11) [12] for a fixed $|A|$, however, reveals that it is possible to create wakefields by a coherent, short electromagnetic wave packet moving in unmagnetized three component plasmas.

In this paper, we seek a localized solution of the system of Eqs. (10) and (11). We are interested in the case of small but nonzero α so that we can have a finite ϕ. If the characteristic length (L) of the wave satisfies the condition $L \gg (1+|A|^2)^{-1/2}$, then from (11) it follows that

$$\phi \cong \frac{1}{2}\frac{\alpha}{(2-\alpha)}\left[\frac{|A|^2}{(1+|A|^2)}\right] , \tag{12}$$

explicitly displaying that ϕ is proportional to α, i.e., $\phi \ll 1$ for $\alpha \ll 1$.

Substituting (12) into (10) and neglecting terms of ϕ^3 and higher orders, we obtain:

$$2i\omega_o \frac{\partial A}{\partial \tau} + \frac{(2-\alpha)}{\omega_o^2}\frac{\partial^2 A}{\partial \xi^2} + \beta^2 A\left[1 - \frac{1}{(1+|A|^2)^2}\right] = 0 , \tag{13}$$

where $\beta = .5\alpha/(2-\alpha)^{1/2} \ll 1$. For stationary solitons, the ansätz (λ is a constant corresponding to a nonlinear frequency shift)

$$A = A(\xi)\exp\left\{\frac{i\lambda^2\tau}{2\omega_o}\right\} \tag{14}$$

reduces Eq. (13) to

$$\frac{d^2 A}{d\eta^2} - \Omega^2 A + A\left[1 - \frac{1}{(1+A^2)^2}\right] = 0 , \tag{15}$$

with $\eta = [\omega_o\beta/(2-\alpha)^{1/2}]\xi$, and $\Omega = \lambda/\beta$. Invoking the boundary conditions appropriate to a localized solution, i.e., $A = 0 = dA/d\eta$ as $|\eta| \to \infty$, Eq. (15) can be readily integrated and

allows soliton-like solutions for $\Omega^2 < 1$. There are several ways in which the exact implicit solution of Eq. (15) can be displayed. The most revealing perhaps, is the form

$$|\eta| = \frac{\cos^{-1}[(1-\Omega^2)(1+A^2)]^{1/2}}{(1-\Omega^2)^{1/2}} + \frac{1}{2\Omega}\,\ell n\,\frac{\Omega(1+A^2)^{1/2} + [1-(1-\Omega^2)(1+A^2)]^{1/2}}{\Omega(1+A^2)^{1/2} - [1-(1-\Omega^2)(1+A^2)]^{1/2}} . \tag{16}$$

For all values of Ω^2, Eq. (16) can be satisfied at $|\eta| = 0$ if $(1-\Omega^2)(1+A(0)^2) = 1$ leading to $A^2(0) \equiv A_m^2 = \Omega^2/1-\Omega^2$, where the amplitude A_m is the maximum value A can attain. Clearly $A_m \to 0$ as $\Omega \to 0$ and A_m becomes large as $\Omega \to 1$. Remembering that A is exactly equal to the particle momentum measured in $m_e c$, a large A_m corresponds to a highly relativistic plasma, the principal regime of interest for this paper.

Let us begin the analysis of Eq. (16) by determining the asymptotic behavior of A. As long as Ω is not extremely close to unity, it is only the second term which can provide the balance as $|\eta| \to \infty$. Thus for sufficiently large $|\eta|$, Eq. (16) leads to the exponentially decaying solution [for all Ω]

$$A_{asy} \simeq \Omega\,\mathrm{sech}\,\Omega|\eta| . \tag{17}$$

Having demonstrated that we have indeed found localized solutions for all Ω, we shall now derive approximate formulas to describe the main (not the asymptotic) part of the soliton. Structure of Eq. (16) clearly suggests that in the two limiting cases of interest: $\Omega \to 0$ (nonrelativistic) and $\Omega \to 1$ (high relativistic), the right-hand side is dominated by the second and the first-term respectively. Naturally in the nonrelativistic limit, the asymptotic shape (17), which is the usual soliton solution of the nonlinear Schrödinger equation pertains for all $|\eta|$.

The highly relativistic large amplitude wave ($\Omega \to 1$, $A_m \gg 1$), is new and considerably more interesting. Barring the exponentially decaying tail, the main body of the soliton is well approximated by

$$A = A_m \cos\left(\frac{\eta}{A_m}\right) \tag{18}$$

and will be termed a 'cosine' soliton. The general shape of the large amplitude soliton (obtained numerically) is displayed in Fig. 1, and is barely distinguishable from (18) in the nonasymptotic region. In Fig. 2, we plot the soliton width L_m as a function of A_m and find that, for $A_m > 1$, L_m is linearly proportional to A_m as predicted by (18).

The total plasma density variation associated with the soliton,

$$\delta N = \delta n^+ + \delta n^- \approx A^2 \tag{19}$$

is large for $A^2 \gg 1$; the solitons with ultrarelativistic amplitudes create large concentrations of plasma density.

The stability of the soliton solution can be investigated using the well-known stability criterion of Vakhitov and Kolokolov [13]. According to this criterion the soliton is stable if

$$\frac{\partial I}{\partial \Omega^2} > 0 \,, \tag{20}$$

where I represents the "number of photons"

$$I = \int d\eta\, A^2 \,. \tag{21}$$

From a direct integration of the defining equations, one finds

$$I = A_m(1 + A_m^2)^{1/2} + \frac{1}{2}(1 + A_m^2)^{3/2} \arccos\left(\frac{1 - A_m^2}{1 + A_m^2}\right) \,, \tag{22}$$

and it is trivial to see that $\partial I/\partial\Omega^2 = (\partial I/\partial A_m^2)\partial A_m^2/\partial\Omega^2 = (1-\Omega^2)^{-2}\partial I/\partial A_m^2 > 0$, proving the stability of the one-dimensional soliton for all Ω.

We conclude that it is possible to obtain a large amplitude soliton solution in an unmagnetized cold plasma consisting of electrons, positrons, and a small fraction of massive ions. We assert the fact that the presence of even a very small fraction of massive ions is crucial to the soliton formation; a pure electron-positron plasma cannot sustain this disturbance. The electromagnetic wave pulse with arbitrary amplitude, under certain given conditions, will always be spread out in a pure electron-positron plasma [11]. The addition of a small fraction of massive ions, stops the pulses from spreading out; the solitons will emerge from the modulational interactions of these pulses. We note in passing that such soliton potentials propagating with $v_g \simeq c$, could readily cause acceleration of resonant particles [14].

We now generalize our results by allowing a transverse variation of the fields. If we assume that A depends weakly on the transverse coordinates $[A = A(\xi, x, y, t)]$, i.e., $(\partial A/\partial\xi) \gg \nabla_\perp A$, Eq. (13) can be rewritten with an additional term $\Delta_\perp A$. Assuming $\Delta_\perp A \gg \omega_o^{-2}(\partial^2 A/\partial\xi^2)$, Eq. (13) modifies to

$$2i\omega_o \frac{\partial A}{\partial \tau} + \Delta_\perp A + \beta^2 A\left[1 - \frac{1}{(1+|A|^2)^2}\right] = 0 \tag{23}$$

which, with the substitution (14), yields

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} - \Omega^2 A + A\left[1 - \frac{1}{(1+A^2)^2}\right] = 0 \tag{24}$$

for the cylindrically symmetric configuration.

We will concentrate on the so-called "ground state solution" [13] of Eq. (25). This solution is positive, radially symmetric, is monotonically decreasing with increasing r, and satisfies the boundary conditions $(dA/dr)_r = 0$, $A(\infty) = 0$,. It is easy to see that such a solution exists only if $\Omega^2 < 1$.

We solve this nonlinear eigenvalue problem numerically. A typical result of numerical calculations is displayed in Fig. 3, where A is plotted as a function of r. In this example the eigenvalue $\Omega^2 = 0.2881$ $(A(0) = A_m \approx 1)$. In Fig. 4 we present the dependence of Ω^2 on A_m, the amplitude of the localized solution. The relationship $\Omega^2 = \Omega^2(A_m)$ can be considered as a type of "nonlinear" dispersion relation. We see that for the ultrarelativistically strong EM waves (i.e. $A_m \gg 1$), $\Omega^2 \to 1$.

For the ultrarelativistic case, it is also possible to find a nearly analytic solution of Eq. (24). Indeed, for the region where $A_m \geq A \gg 1$, the solution of Eq. (24) is simply the zeroth-order Bessel function:

$$A = A_m \cdot J_o(kr) \tag{25}$$

where $k = (1-\Omega^2)^{1/2}$. In the asymptotic region, the solution must decay, and Eq. (25) is solved by the modified Bessel function

$$A \sim K_o(\Omega r) \sim \frac{1}{(\Omega r)^{1/2}} \cdot \exp(-\Omega r) \,, \tag{26}$$

revealing the characteristic exponential decay. The numerical solution of Eq. (25) (solid line) along with the analytical expression (26) (dashed line) is displayed in Fig. 5. One can see that the main part of the solution (which contains most of the EM wave energy) is well described by the Bessel solution (26), the two-dimensional analog of the "cosine" soliton [Eq. (18)]. Note that, as in the one-dimensional case, the soliton width ($d \sim k^{-1}$) is an increasing function of the amplitude $A_m > 1$.

The stability of two dimensional solution can be determined by using the condition (20). For the cylindrical symmetric case,

$$I = \int_0^\infty dr \; r \; A^2 \,, \tag{27}$$

which, for the large amplitude case ($A_m \gg 1$), will be dominated by contributions from the region in which the Bessel function solution holds. Simple algebra leads us to

$$I = C_1 \frac{A_m^2}{k^2} \,, \tag{28}$$

$$C_1 = \int_0^{C_2} dx \; x \; J_o^2(x) > 0$$

where C_2 is a constant of order unity. From (28), and from the condition $\partial A_m/\partial\Omega > 0$ [see Fig. 4] we get that $\partial I/\partial\Omega^2 > 0$. This proof is clearly not formal, but we believe that it is quite adequate for the large amplitude solitons. Using detailed computer simulations, we found that the stability criterion $\partial I/\partial\Omega^2 > 0$ is satisfied for arbitrary amplitude soliton solutions.

The stability of the localized structures in the electron-positron-ion plasmas distinguishes them fundamentally from the inherently unstable solitonic solutions obtained for pure e-p plasmas. Since unstable nonlinear solutions are, generally, not accessible, it would seem that the stable e-p-i solitons are more likely to lead to observable physical consequences.

In conclusion, we have shown that in electron positron plasmas with a small fraction of ions, it is possible to have localized stable structures with relativistically strong amplitudes of EM radiation and with large density bunching. Astrophysical objects, like radio galaxies, quasars or radio pulsars could radiate ultrarelativistic EM waves, which, in the ever present e-p-i plasmas in their vicinity may lead to the formation of stable solitons. As emphasized earlier, it is these stable solutions which should be preferentially used to explain, for example, the "micropulsations" in pulsar radiation [15].

We also believe that these stable localized structures as sources of large density inhomogeneities may provide templates for structure formation in the early universe. Detailed applications of this theory to cosmological, as well as laboratory plasmas, is left for future publications.

Acknowledgments

One of the authors (V.I.B.) would like to thank Professor Abdus Salam, the International Atomic Energy Agency, UNESCO and the International Centre for Theoretical Physics, Trieste, for support.

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Figure Captions

Fig.1. A typical large amplitude structure; A versus η. Barring the exponentially decaying tail ($|\eta| > 10$), the rest of the soliton is very well approximated by the 'Cosine' formula of Eq. 18.

Fig.2. The soliton width L_m as a function of the amplitude A_m. For $A_m > 1$, the relationship is linear as predicted by Eq. 18.

Fig.3. A moderate amplitude 2D soliton; A versus r, the radial coordinate.

Fig.4. The nonlinear dispersion relation, the effective eigenvalue Ω^2 as a function of A_m, the amplitude. As A_m goes to infinity, Ω^2 approaches unity.

Fig.5. A comparison of the numerical 2D solution with the Bessel function approximation (Eq. 25). Again there exists excellent agreement in for the bulk of the structure.

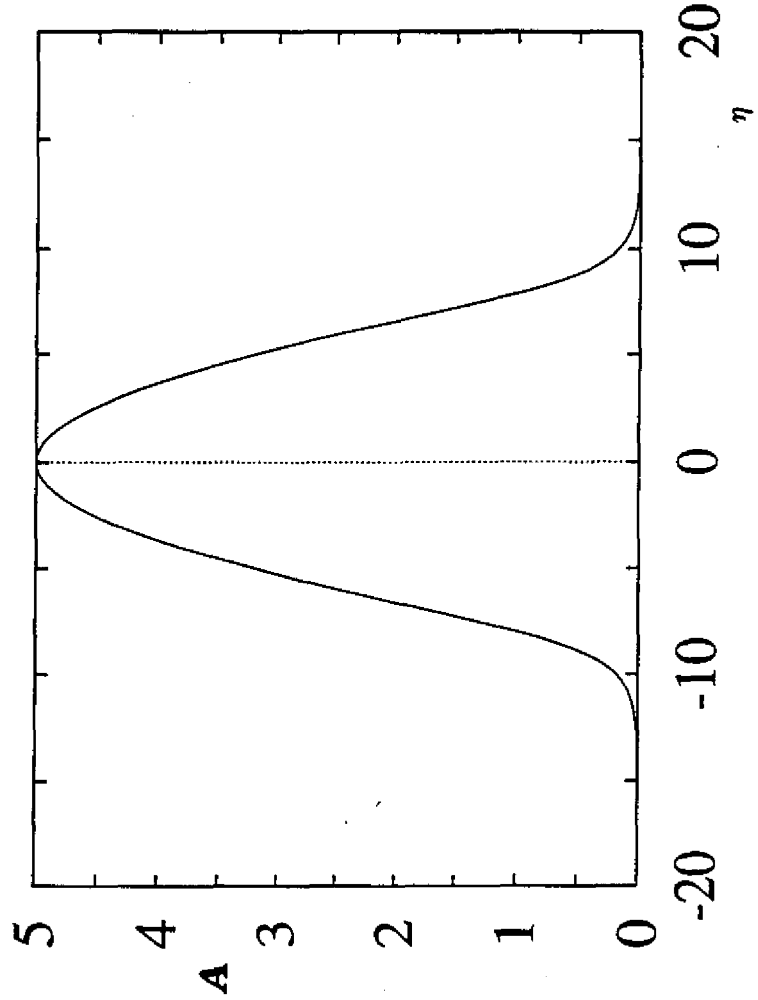


Fig. 1

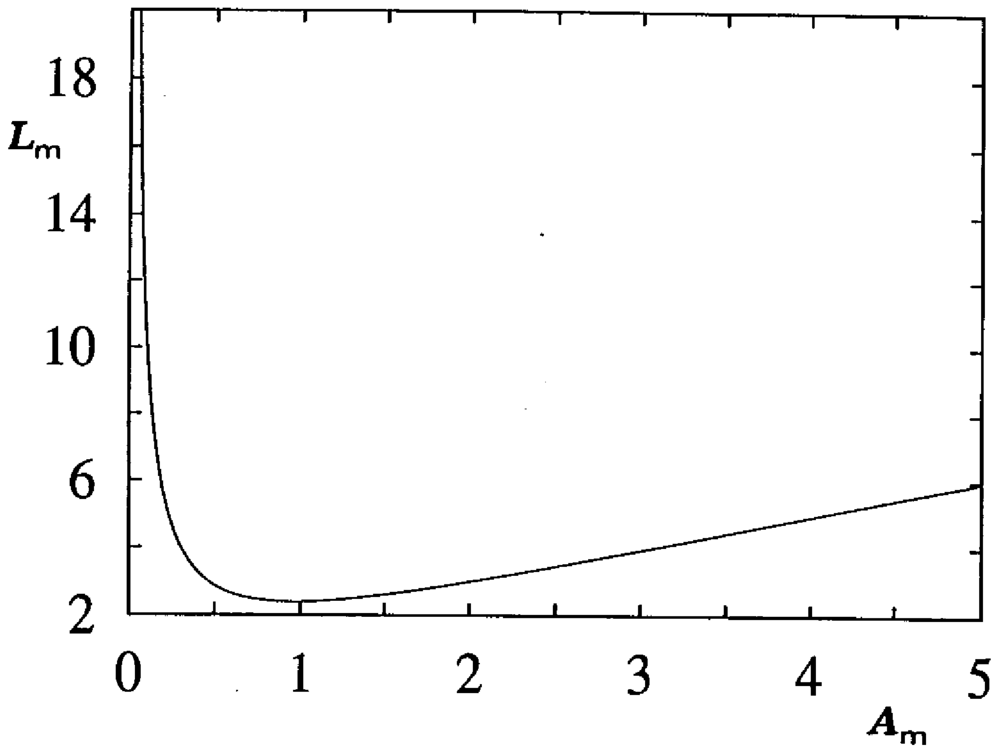


Fig. 2

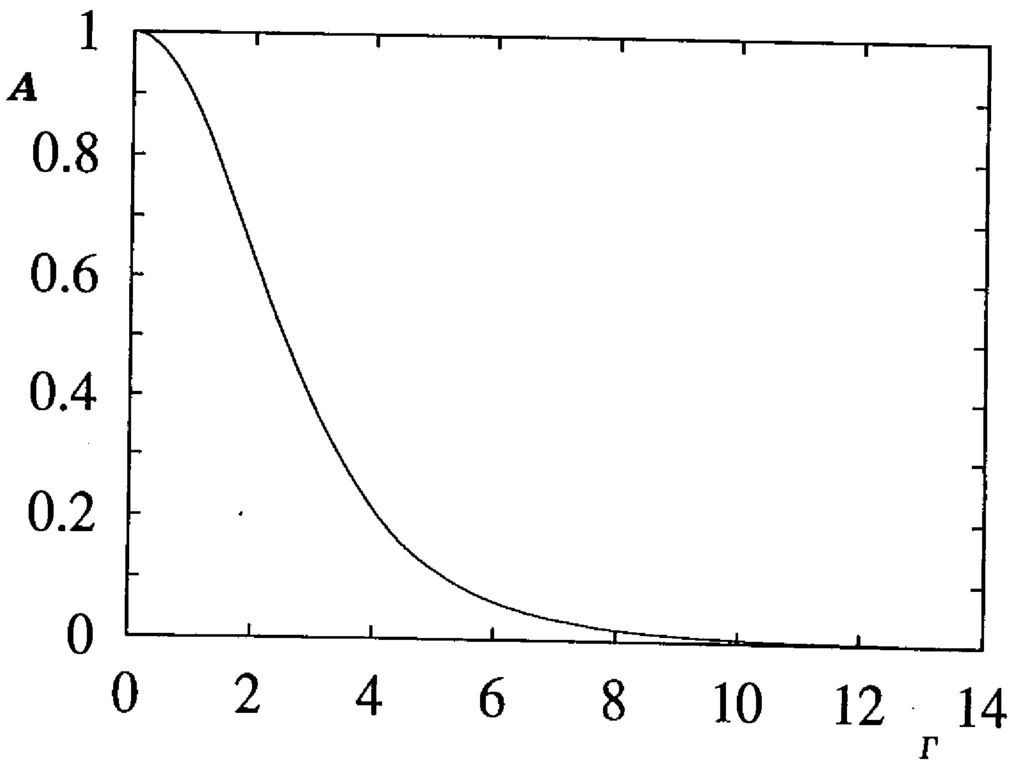
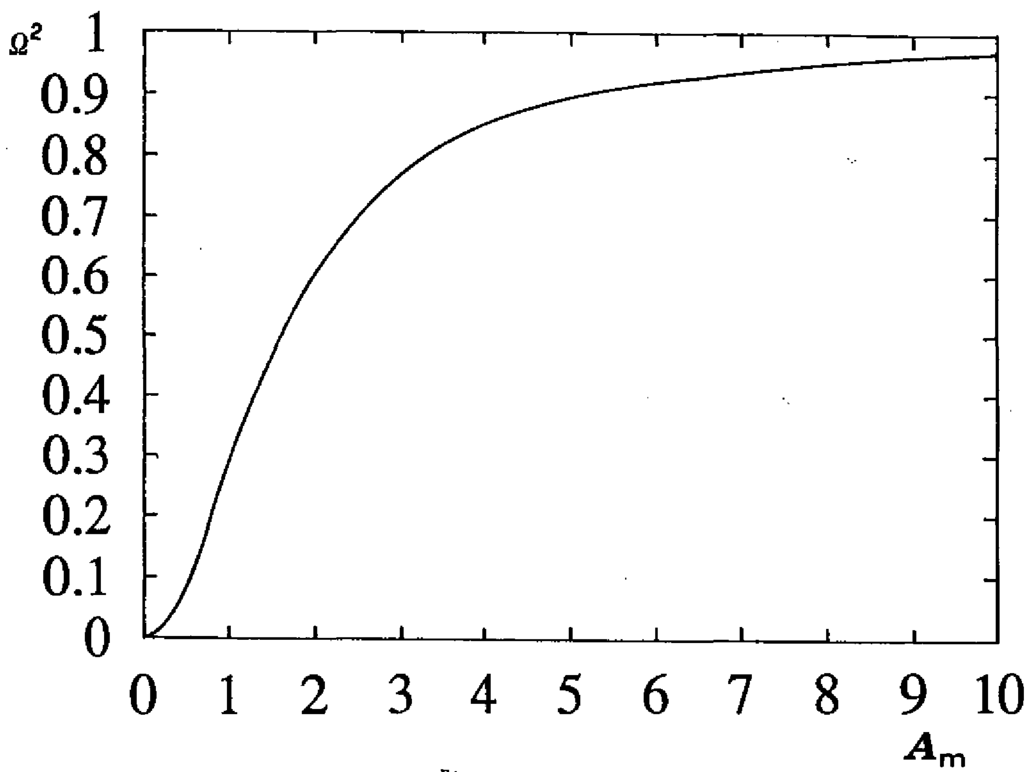


Fig. 3


Ω^2
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
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0
0 1 2 3 4 5 6 7 8 9 10
A_m

Fig. 4

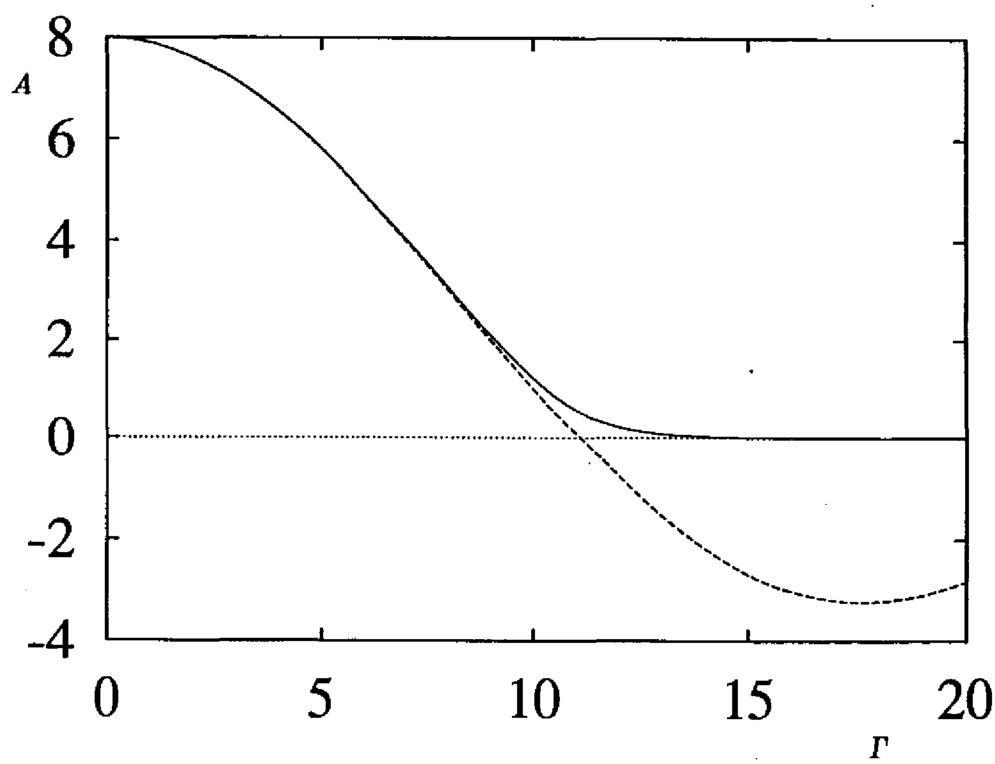
8
A
6
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Fig. 5