MAGNETIC FIELD INHOMOGENEITY EFFECTS IN WEAKLY RELATIVISTIC PLASMAS

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Abstract

We have obtained expressions for an effective dielectric tensor for weakly relativistic magnetoactive plasmas with magnetic field inhomogeneity in the perpendicular direction. The effective dielectric tensor ($\varepsilon_{ij}^{\text{eff}}$) satisfies the required symmetry conditions and describes correctly the energy exchange between wave and particles in a stationary, non-homogeneous plasma, when utilized in a dispersion relation which is formally equal to the homogeneous one. We illustrate with some numerical examples for electron-cyclotron absorption.

1. Introduction

In order to discuss the dielectric properties of magnetized inhomogeneous media, we consider the case of a stationary, weakly inhomogeneous magnetoactive plasma, with perpendicular magnetic field gradients. It is known that in such a weakly inhomogeneous medium, the wave amplitude can be modified not only due to wave-particle interaction, but also due to changes in the group velocity and, less important, due to mode conversion and partial reflections.

Assuming that the inhomogeneity is sufficiently small for the last two phenomena to be neglected, *Beskin et al., 1987* have devised a procedure which separates the relevant wave-particle interaction from the change in the group velocity, when discussing the changes in wave amplitude along wave propagation. It has been shown that the correct dielectric tensor is an effective tensor, obtained by the addition of corrections due to the inhomogeneity to a tensor obtained with the use of a plane-wave approximation (ε_{ij}^{0}). In the case of inhomogeneities in density, temperature, and drift velocity, explicit expressions for the dielectric tensor have been obtained, by the addition of first order corrections in the plasma gradients (*Caldela F° et al., 1989*; *Caldela F°, 1990*; *Cavalcanti et al., 1991, 1993*; *Ziebell et al., 1993*). In the case of inhomogeneous magnetic field, infinite series of corrections has to be added, and the effective dielectric tensor can be obtained from ε_{ij}^{0} according to the following (*Beskin et al., 1987*)

$$\varepsilon_{ij}^{\text{eff}}(\boldsymbol{r},\boldsymbol{k},\omega) = (2\pi)^{-3}\int\int d^3k' d^3\eta\, \varepsilon_{ij}^{0}(\boldsymbol{r}+\boldsymbol{\eta}/2,\boldsymbol{k}',\omega)\, e^{i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{\eta}} \,. \quad (1)$$

where $\boldsymbol{k}$ is the wave vector, ω is the angular frequency and $\boldsymbol{r}$ is the position.

The tensor $\varepsilon_{ij}^{\text{eff}}$ is constructed in order to describe correctly the energy exchange between particles and fields, and satisfies the relevant simmetry conditions like the Onsager relation.

$$\varepsilon_{ij}^{\text{eff}}(\boldsymbol{r},-\boldsymbol{k},\omega,-\boldsymbol{B}_0;F(p_\perp,-p_\parallel)) = \varepsilon_{ji}^{\text{eff}}(\boldsymbol{r},\boldsymbol{k},\omega,\boldsymbol{B}_0;F(p_\perp,p_\parallel)), \quad (2)$$

where $\boldsymbol{B}_0$ is the ambient magnetic field, and $F(p_\perp,p_\parallel)$ is the distribution function of the plasma particles. These symmetry relations are general relations, derived by the linear response theory of non-equilibrium statistical mechanics. The effective tensor satisfies a dispersion relation which is formally the same as that for a homogeneous plasma,

$$\det(k_i k_j - k^2\delta_{ij} + \omega^2\varepsilon_{ij}^{\text{eff}}/c^2) = 0\,. \quad (3)$$

With the evaluation of ε_{ij}^{0} and the evaluation of the integral appearing in Eq. (1), explicit expressions for $\varepsilon_{ij}^{\text{eff}}$ can be obtained. We have considered the case of plasmas with inhomogeneities perpendicular to the ambient magnetic field. General expressions have been obtained and shall appear in a forthcoming publication (*Gaelzer et al., 1993*). In the present paper we make no attempt to any particular application, although the geometry considered is relevant for many actual cases, both in laboratory experiments and in space plasmas. We consider as an example to illustrate the use of the effective dielectric tensor the case of ordinary mode waves propagating parallel to the direction of the inhomogeneity. Since this direction is perpendicular to the direction of the magnetic field, in the case of distributions which are symmetric along the parallel component of the velocity the dispersion relation factorizes and for the ordinary mode only one component of the dielectric tensor is required.

In section 2 some details of the evaluation of the dielectric tensor are briefly described, and in section 3 some numerical analysis for the ordinary mode are presented.

2. The effective dielectric tensor

The medium has an ambient magnetic field in the z direction and a constant gradient in the x direction.

$$\begin{aligned} \boldsymbol{B}_0 &= B_0(1 + k_B x)\hat{z} \\ k_B &\equiv \frac{1}{B_0}\left.\frac{dB_0(x)}{dx}\right|_{x=0} . \end{aligned} \tag{4}$$

After linearization of the Vlasov equation, the equation for the perturbed distribution function can be solved by the method of characteristics, which implies time integration along the unperturbed trajectories given by the single particle equations of motion in the ambient magnetic field. It is assumed that at $t' \to -\infty$ all perturbations vanish and at $t' \to t$ the particle has momentum and position $\boldsymbol{p}$ and $\boldsymbol{r}$, respectively.

To integrate the non-linear set of motion equations, we have used a perturbative method, expanding in powers of k_B all momenta and coordinates, and retaining only terms up to order k_B. It is shown that, in order to avoid non-physical secular terms, the cyclotron frequency has to be corrected, resulting equations which satisfy the initial conditions and are perfectly coherent among themselves (*Gaelzer et al., 1993*). However, the most important corrections due to the inhomogeneity are the macroscopical drift and the nonlinear correction to the frequency, which is essential to avoid secularities. These corrections are retained, and all the other terms of order $\mathcal{O}(k_B)$ are neglected. The orbit equations are therefore given by:

$$x'_\alpha(\tau) - x = \frac{p_\perp}{m_\alpha \Omega_\alpha}[\sin\varphi - \sin(\varphi - \omega_\alpha \tau)] \tag{5.a}$$

$$y'_\alpha(\tau) - y = \frac{p_\perp}{m_\alpha \Omega_\alpha}[\cos(\varphi - \omega_\alpha \tau) - \cos\varphi] + \frac{k_B p_\perp^2}{2\gamma_\alpha m_\alpha^2 \Omega_\alpha}\tau \tag{5.b}$$

$$z'_\alpha(\tau) - z = \frac{p_\parallel}{m_\alpha \gamma_\alpha}\tau \tag{5.c}$$

$$p'_{\alpha x}(\tau) = p_\perp \cos(\varphi - \omega_\alpha \tau) \tag{5.d}$$

$$p'_{\alpha y}(\tau) = p_\perp \sin(\varphi - \omega_\alpha \tau) \tag{5.e}$$

$$p'_{\alpha z}(\tau) = p_\parallel . \tag{5.f}$$

where $\tau = t' - t$, $\Omega_\alpha = q_\alpha B_0 / m_\alpha c$ is the cyclotron frequency of the α-th species, $\gamma_\alpha = \sqrt{1 + p^2/m_\alpha^2 c^2}$, $p_\perp$ and $p_\parallel$ are the perpendicular and parallel momenta, φ is the phase angle and

$$\omega_\alpha = \frac{\Omega_\alpha}{\gamma_\alpha}(1 + k_B x) + k_B \frac{p_\perp \sin\varphi}{\gamma_\alpha m_\alpha} .$$

The maintenance of the terms proportional to k_B in ω_α is essential for the correct description of the wave-particle interaction (*Antonsen & Manheimer, 1978; Cairns et al., 1991*). The additional term in (5.b) describes the macroscopic $\nabla B_0 \times \boldsymbol{B}_0$ drift of the particles and must be retained in the integration.

After Fourier transforming the electromagnetic fields and the current density we arrive at a tensor ε_{ij}^0 which does not describe the energy exchange between particles and fields. The correct description will be achieved after the transformations (1), giving the ε_{ij}^{eff} tensor, which we show here for waves that propagate in the $x-z$ plane:

$$\overline{\varepsilon}^{eff}=\overline{1} - \sum_{\alpha} i\frac{4\pi q_\alpha^2}{m_\alpha \omega} \sum_{n=-\infty}^{\infty} \int_0^{\infty} d\tau \int d^3u\, u_\perp \mathcal{L} f_{0\alpha}(u_\perp^2, u_\parallel) \boldsymbol{\Pi}_{n\alpha}^{-} \boldsymbol{\Pi}_{n\alpha}^{+} e^{iD_{n\alpha}\tau} \tag{6}$$

$$- \hat{z}\hat{z} \sum_{\alpha} \frac{4\pi q_\alpha^2}{m_\alpha \omega^2} \int d^3u\, \frac{u_\parallel}{\gamma} \left(\frac{u_\parallel}{u_\perp}\frac{\partial}{\partial u_\perp} - \frac{\partial}{\partial u_\parallel}\right) f_{0\alpha}(u_\perp^2, u_\parallel) .$$

where $D_{n\alpha} = \gamma\omega - k_\parallel u_\parallel c - n\Omega_\alpha(1+k_B x)$, $\boldsymbol{u} = \boldsymbol{p}/m_\alpha c$, and the operator $\mathcal{L}$ and the vector $\boldsymbol{\Pi}_{n\alpha}^{\pm}$ are defined by:

$$\boldsymbol{\Pi}_{n\alpha}^{\pm} = \frac{nJ_n(b_\alpha \pm a_n\tau/2)}{b_\alpha \pm a_n\tau/2}\hat{\boldsymbol{x}} \pm iJ_n'(b_\alpha \pm a_n\tau/2)\hat{\boldsymbol{y}} + \frac{u_\parallel}{u_\perp}J_n(b_\alpha \pm a_n\tau/2)\hat{\boldsymbol{z}}$$

$$\mathcal{L} = \left(1 - \frac{N_\parallel u_\parallel}{\gamma}\right)\frac{\partial}{\partial u_\perp} + \frac{N_\parallel u_\perp}{\gamma}\frac{\partial}{\partial u_\parallel} .$$

with $b_\alpha = k_\perp u_\perp c/\Omega_\alpha$, $a_n = k_B n u_\perp c$, $N_\parallel = k_\parallel c/\omega$ and $J_n(z)$ is the Bessel function of order n. The general expression is given in *Gaelzer et al., 1993*.

We immediatly see that when $k_B = 0$, the tensor (6) reduces to the well-known dielectric tensor for homogeneous magnetoplasma. It satisfies also the Onsager reciprocity relations, eq. (2) and conserves, by construction, the whole energy of the wave-particle system (*Beskin et al. 1987*).

3. Ordinary mode absorption near cyclotron frequency

In the case of electron cyclotron waves, the effect of ions can be neglected in the dispersion relation. Assuming a Maxwellian distribution function for the electrons,

$$f_0(u_\perp^2, u_\parallel) = n_e \left(\frac{\mu}{2\pi}\right)^{3/2} e^{-\mu u^2/2} .$$

where n_e is the electron density and $\mu = m_e c^2/T_e$, where T_e is the electron temperature: considering a weakly relativistic regime, $\gamma \simeq 1 + u^2/2$, and waves in perpendicular propagation, we arrive from eq. (6), to the following ε_{33}^{eff} component:

$$\varepsilon_{33}^{eff} = 1 + \mu X \mathcal{H}_{00} + \mu X \sum_{n=1}^{\infty} \sum_{s=\pm 1} \mathcal{H}_{ns} . \tag{7}$$

$$\mathcal{H}_{ns} = ie^{-2\epsilon_n^2} \int_0^{\infty} dt \frac{\exp\left[i(\delta_{ns} - 2\epsilon_n^2)t + 2\epsilon_n^2/(1-it)\right]}{(1-it)^{5/2}} \mathcal{G}_n\left(\frac{\beta^2 - \epsilon_n^2 t^2}{1-it}\right)$$

$$\mathcal{G}_n(z) = e^{-z}I_n(z) .$$

where $X = 4\pi n_e q_e^2/m_e\omega^2$, $2\epsilon_n = N_B n \mu^{1/2}$, $\delta_{ns} = \mu[1 - snY(1 + k_B x)]$, $\beta = N_O^2/Y\mu^{1/2}$, $N_B = k_B c/\omega$, $Y = \Omega_e/\omega$ and $I_n(z)$ is the modified Bessel function of the first kind. Again for $k_B = 0$, $\mathcal{H}_{ns}$ reduces to the generalized relativistic plasma dispersion function (*Robinson, 1986*). We have then the dispersion function for ordinary mode waves,

$$N_O^2 = \varepsilon_{33}^{eff}(N_O^2) . \tag{8}$$

As a numerical example, we suppose ordinary mode waves propagating perpendicular to $\boldsymbol{B}_0$ in a tokamak with a magnetic field profile given by eq. (4). Choosing the position at the center of the torus ($x = 0$), we examine as both the absorption and refraction change as a function of the parameter $r_N = N_B/|\overline{N_O}|$, where $|\overline{N_O}|$ is the modulus of the refractive index for a frequency equal to $\omega = 0.9\Omega_e$. Figure 1 shows the real and imaginary parts of N_O^2 for several values of r_N.

It is seen that, unlike the case of density and temperature inhomogeneities, magnetic field inhomogeneity is effective near the cyclotron frequency (*Caldela et al., 1990*), and can substantially

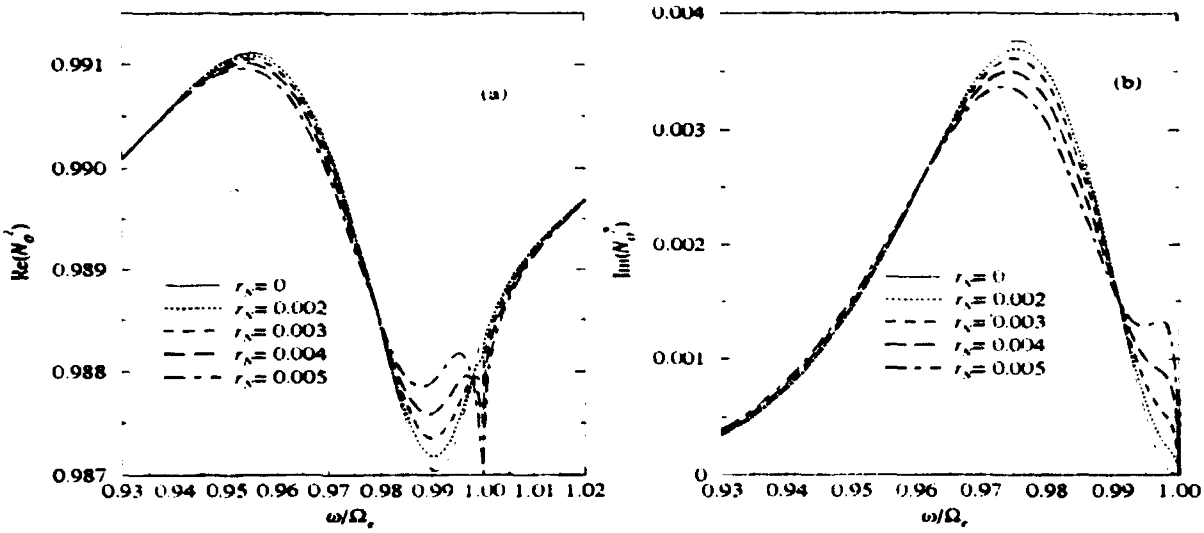

Figure 1: (a) Real and (b) imaginary parts of N_0^2 as a function of ω/Ω_e for $r_N = 0$, 2×10^{-3}, 3×10^{-3}, 4×10^{-3} and 5×10^{-3}, for a maxwellian plasma with $T_e = 1.25$keV.

modify electron cyclotron wave absorption. This fact may be relevant for plasma heating and current generation in tokamaks.

There are many other situations where interesting plasma phenomena occur in inhomogeneous magnetic field. Such is the case of the drift instabilities. However, the study of these instabilities requires examination of waves propagating perpendicular to the inhomogeneity, a situation which was not considered here. It is our intention to pursue our studies on the subject.

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