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New series of 3D lattice integrable models

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Abstract

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In this paper we present a new series of 3-dimensional integrable lattice models with N colors. ~~The case $N = 2$ generalizes the elliptic model of paper~~ [8]. The weight functions of the models satisfy modified tetrahedron equations with N states and give a commuting family of two-layer transfer-matrices. The dependence on the spectral parameters corresponds to the *static limit* of the modified tetrahedron equations and weights are parameterized in terms of *elliptic functions. The models contain two free parameters: elliptic modulus and additional* parameter η. Also we briefly discuss symmetry properties of weight functions of the models.

Аннотация

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В этой работе мы представляем новую серию трехмерных интегрируемых решеточных моделей с N цветами. ~~Случай $N = 2$ обобщает эллиптическую модель работы~~ [8]. Весовые функции моделей удовлетворяют модифицированным уравнениям тетраэдра с N состояниями и дают коммутирующее семейство двухслойных трансфер-матриц. Зависимость от спектральных параметров соответствует статическому пределу модифицированных уравнений тетраэдра, и веса параметризуются в терминах эллиптических функций. Модели содержат два свободных параметра: эллиптический модуль и добавочный параметр η. ~~Также мы кратко обсуждаем симметрийные свойства весовых функций моделей.~~

1. Introduction

The purpose of this work is an enlarging of still poor zoo of 3D integrable lattice models. We say the model to be integrable if it possesses a family of commuting transfer matrices. The existence of such family is ensured, for example, by the construction of solutions of the tetrahedron equation, which is a three dimensional generalization of the Yang - Baxter equation [1–5]. As an example we can mention the N - color trigonometric model by Bazhanov - Baxter [6]. Although its integrability has been proved by a different method, the Boltzmann weights of this model are the solutions of the tetrahedron equation [7]. This solution, as well as the first known Zamolodchikov's one [1, 2] (which is a particular case of the Bazhanov-Baxter model when $N = 2$) can be parameterized in terms of trigonometric functions depending on tetrahedron angles. In our previous paper [8] we have constructed an elliptic two - color solution of a modified system of the tetrahedron equations, which provides the commutativity of two - layer transfer-matrices. This solution in a sense is not "full" and corresponds to the case of the so called "static" limit of tetrahedron equations.

In difference from Bazhanov – Baxter model, for which the solution of tetrahedron equations contains six angular variables (five of them are independent), the static solution of the modified tetrahedron equations from [8] can be parameterized by three angle – like variables and one additional parameter (the modulus of elliptic functions).

In this paper we generalize this elliptic solution to the case of an arbitrary number N of spin values and obtain one more parameter, on which weight functions depend. This additional parameter is the same for all weights and new solutions are still static. Boltzmann weights of the new models like the weight functions of Bazhanov – Baxter model have the so called Body – Centered – Cube (BCC) form, invented by Baxter in [9]. This fact allows us to use the technique of the Star – Square relations, developed in [7].

The paper is organized as follows. In Section 2 we recall main definitions and notations, give the form of the Boltzmann weights and write out the modified tetrahedron equations. In Section 3 we formulate conditions for the BCC ansatz for weight functions to obey the modified tetrahedron equations. In Section 4 a natural parameterization of the obtained solution is given in terms of elliptic functions. In Section 5 we discuss symmetry properties of weight functions of the model. At last, Appendix contains a detailed consideration of $N = 2$ case and an explanation of the transition to the weights of the model from paper [8].

2. Body-Centered-Cube (BCC) ansatz for weight functions

In this section we recall some definitions from [7, 10] and give the explicit form of weight functions. We also write out modified tetrahedron equations, which provide a commutativity of two-layer transfer matrices [8].

Consider a simple cubic lattice $\mathcal{L}$ consisting of two types of elementary cubes alternating in checkerboard order in all directions (see Fig. 1).

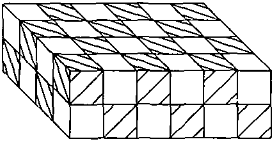
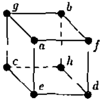


Fig. 1

At each site of $\mathcal{L}$ place a spin variable taking its values in Z_N, for any integer $N \geq 2$ (elements of Z_N are given by N distinct numbers $0, 1, \ldots, N-1$ considered *modulo* N). To each "white" cube we assign a weight function $W(a|efg|bcd|h)$ (see Fig. 2).



$$= \quad W(a|e,f,g|b,c,d|h)$$

Fig. 2

and to each "dashed" cube - a weight function $\overline{W}(a|efg|bcd|h)$, where as usual $a, \ldots, h$ – spin variables placed in the vertices of each elementary cube.

Then the partition function of the model reads

$$Z = \sum_{spins} \prod_{\substack{\text{"white"} \\ \text{cubes}}} W(a|efg|bcd|h) \prod_{\substack{\text{"dashed"} \\ \text{cubes}}} \overline{W}(a|efg|bcd|h). \qquad (2.1)$$

Following [8] suppose that weight functions W and $\overline{W}$ satisfy the following equations:

$$\sum_{d} \quad W(a_4|c_2, c_1, c_3|b_1, b_3, b_2|d)\overline{W}'(c_1|b_2, a_3, b_1|c_4, d, c_6|b_4)$$
$$\times \quad W''(b_1|d, c_4, c_3|a_2, b_3, b_4|c_5)\overline{W}'''(d|b_2, b_4, b_3|c_5, c_2, c_6|a_1)$$

$$= \sum_d \; W'''(b_1|c_1,c_4,c_3|a_2,a_4,a_3|d)\overline{W}''(c_1|b_2,a_3,a_4|d,c_2,c_6|a_1)$$
$$\times \; W'(a_4|c_2,d,c_3|a_2,b_3,a_1|c_5)\overline{W}(d|a_1,a_3,a_2|c_4,c_5,c_6|b_4), \qquad (2.2)$$

where W, W', W'', W''' and $\overline{W}$, $\overline{W}'$, $\overline{W}''$, $\overline{W}'''$ are four independent pairs of weight functions. Suppose that the dual variant of (2.2) is also valid (with all W's replaced by $\overline{W}$'s and vice versa). We call this pair of equations as a system of modified tetrahedron equations. Note that if put $\overline{W} = W$ in (2.2), we come to the standard version of tetrahedron equations (Eqs. (2.2) of [3]). In the Bazhanov - Baxter model all weights entering in tetrahedron equations are parameterized in terms of six angles $\theta_1, \ldots, \theta_6$ satisfying one quadrilateral constraint. The explicit dependence on spectral parameters can be exhibited as ($\overline{W} = W$)

$$W \to W(\theta_2,\theta_1,\theta_3), \quad W' \to W(\pi-\theta_6,\theta_1,\pi-\theta_4),$$
$$W'' \to W(\theta_5,\pi-\theta_3,\pi-\theta_4), \quad W''' \to W(\theta_5,\theta_2,\theta_6). \qquad (2.3)$$

In the static limit three spectral parameters $\theta_1,\theta_2,\theta_3$ in weight functions $W(\theta_1,\theta_2,\theta_3)$, $\overline{W}(\theta_1,\theta_2,\theta_3)$ are constrained by relation $\theta_1+\theta_2+\theta_3=\pi$. Then formulas (2.3) are transformed as follows:

$$W \to W(\theta_2,\theta_1,\pi-\theta_1-\theta_2),\; W' \to W(\theta_2+\theta_5,\theta_1,\pi-\theta_1-\theta_2-\theta_5),$$
$$W'' \to W(\theta_5,\theta_1+\theta_2,\pi-\theta_1-\theta_2-\theta_5),\; W''' \to W(\theta_5,\theta_2,\pi-\theta_2-\theta_5) \; (2.4)$$

and the same for $\overline{W}$ weights. Note that in this case the quadrilateral constraint between tetrahedron angles is satisfied automatically.

As it is shown in [8] the system of modified tetrahedron equations provides a commuting family of two-layer transfer-matrices. Let us construct a horizontal transfer-matrix $T(W,\overline{W})$ from alternating weights W and $\overline{W}$ as in Fig. 1. Here we imply that ingoing indices of the transfer-matrix correspond to the spins of the lower layer, outgoing indices – to the spins of the upper one and the summation over the spins of the middle layer is performed. As usual partition function (2.1) can be rewritten as

$$Z = Tr\{T(W,\overline{W})]\}^M, \qquad (2.5)$$

where $2M$ is a number of horizontal layers of the lattice and we imply periodic boundary conditions. When equation (2.2) and its dual variant are satisfied, we have the following commutativity relation:

$$T(W,\overline{W})T(W',\overline{W'}) = T(W',\overline{W'})T(W,\overline{W}). \tag{2.6}$$

Now let us specify the explicit form of weight functions. To make it, denote

$$\omega = \exp(2\pi i/N), \qquad \omega^{1/2} = \exp(\pi i/N). \tag{2.7}$$

Further, taking x, y, z to be complex parameters constrained by the Fermat equation

$$x^N + y^N = z^N \tag{2.8}$$

and l to be an element of Z_N, define

$$w(x,y,z|l) = \prod_{s=1}^{l} \frac{y}{z - x\omega^s}. \tag{2.9}$$

In addition, define the function with one more argument

$$w(x,y,z|k,l) = w(x,y,z|k-l)\Phi(l), \quad k,l \in Z_N, \tag{2.10}$$

where

$$\Phi(l) = \omega^{l(l+N)/2}. \tag{2.11}$$

Let us mention also two formulas for w functions, which are useful for calculations:

$$w(x,y,z|l+k) = w(x,y,z|k)w(x\omega^k,y,z|l), \tag{2.12}$$

$$w(x,y,z|k,l) = \omega^{kl}/w(z,\omega^{1/2}y,\omega x|l,k). \tag{2.13}$$

Now introduce the set of homogeneous variables x_i, $i = 1,\ldots,8$ and x_{13}, x_{24}, x_{58}, x_{67} satisfying

$$x_{13}^N = x_1^N - x_3^N, \quad x_{24}^N = x_2^N - x_4^N, \quad x_{58}^N = x_5^N - x_8^N, \quad x_{67}^N = x_6^N - x_7^N. \tag{2.14}$$

Using all these notations we define the weight function $W(a|efg|bcd|h)$ as

$$W(a|e,f,g|b,c,d|h) = \sum_{\sigma\in Z_N} \frac{w(x_3,x_{13},x_1|d,h+\sigma)w(x_4,x_{24},x_2|a,g+\sigma)}{w(x_8,x_{58},x_5|e,c+\sigma)w(x_7/\omega,x_{67},x_6|f,b+\sigma)}. \tag{2.15}$$

In fact, the Boltzmann weights of form (2.15) generalize the weight functions of the Bazhanov -Baxter model [6, 10]. Up to inessential gauge factors the latter corresponds to the choice

$$x_5 = x_1, \quad x_6 = x_2, \quad x_7 = x_3, \quad x_8 = x_4. \tag{2.16}$$

Following paper [6] we will call formula (2.15) as a Body-Centered-Cube (BCC) ansatz for weight functions.

3. Star-square relation and the proof of modified tetrahedron equations

In paper [7] tetrahedron equations for the Bazhanov - Baxter model with N states were proved using the so called Star-Square and "inversion" relations. We will follow the method of this paper for weight function (2.15).

First recall "inversion" relation for functions $w(x, y, z|k, l)$:

$$\sum_{k \in Z_N} \frac{w(x, y, z|k, l)}{w(x, y, \omega z|k, m)} = N \delta_{l,m} \frac{(1 - z/x)}{(1 - z^N/x^N)}, \tag{3.1}$$

where $l, m \in Z_N$, x, y, z satisfy (2.8) and $\delta_{l,m}$ is the Kronecker symbol on Z_N.

To write down the Star-Square relation introduce a non-cyclic analog of w function, defined recurrently as follows:

$$w(x|0) = 1, \quad \frac{w(x|l)}{w(x|l-1)} = \frac{1}{(1 - x\omega^l)}, \quad l \in Z, \tag{3.2}$$

where Z is the set of all integers. It is obvious that

$$w(x, y, z|l) = (\frac{y}{z})^l w(x/z|l), \quad l \in Z_N, \tag{3.3}$$

where index l, being considered modulo N, is interpreted as an element of Z_N. Then we have the following identity

$$\Big\{ \sum_{\sigma \in Z_N} \frac{w(x_1, y_1, z_1|a + \sigma) w(x_2, y_2, z_2|b + \sigma)}{w(x_3, y_3, z_3|c + \sigma) w(x_4, y_4, z_4|d + \sigma)} \Big\}_0 = \frac{(x_2 z_1/x_1 z_2)^{a-b} (x_1 y_2/x_2 z_1)^b (z_3/y_3)^c (z_4/y_4)^d}{\Phi(a - b) \omega^{(a+b)/2}}$$
$$\times \frac{w(\omega x_3 x_4 z_1 z_2/x_1 x_2 z_3 z_4 | c + d - a - b)}{w(\frac{x_4 z_1}{x_1 z_4}|d - a) w(\frac{x_3 z_2}{x_2 z_3}|c - b) w(\frac{x_3 z_1}{x_1 z_3}|c - a) w(\frac{x_4 z_2}{x_2 z_4}|d - b)}, \tag{3.4}$$

where the lower index "0" after the curly brackets indicates that the l.h.s. of (3.4) is normalized to unity at zero exterior spins, and the following constraint is imposed

$$\frac{y_1}{z_1}\frac{y_2}{z_2}\frac{z_3}{y_3}\frac{z_4}{y_4}=\omega. \qquad (3.5)$$

We call relation (3.4) as the Star-Square one. Note that the separate w's in the r.h.s. of (3.4) are not single-valued functions on Z_N, while the whole expression is cyclic on the exterior spins a, b, c, d.

The proof of (3.1), (3.4) was given in [7] and we refer the interested reader to this paper.

Now we turn to relation (2.2). Instead of weights W, W', W'', W''' let us substitute into (2.2) explicit formula (2.15) with corresponding sets of parameters:

$$\begin{aligned} W, W', W'', W''' &\to W(x_i, x_{ij}), W(x'_i, x'_{ij}), W(x''_i, x''_{ij}), W(x'''_i, x'''_{ij}), \\ \overline{W}, \overline{W}', \overline{W}'', \overline{W}''' &\to W(\overline{x}_i, \overline{x}_{ij}), W(\overline{x}'_i, \overline{x}'_{ij}), W(\overline{x}''_i, \overline{x}''_{ij}), W(\overline{x}'''_i, \overline{x}'''_{ij}). \end{aligned} \qquad (3.6)$$

Let us multiply both sides of (2.2) by the following product of w weights:

$$\begin{aligned} &\frac{w(x''_7, x''_{67}, x''_6 | c_4, a_2 + l_2)}{w(\overline{x}'''_3/\omega, \overline{x}'''_{13}, \overline{x}'''_1 | c_6, a_1 + l_1)} \frac{w(\overline{x}'_7, \omega\overline{x}'_{67}, \overline{x}'_6 | a_3, c_4 + l_3)}{w(x_4, x_{24}, \omega x_2 | a_4, c_3 + l_4)} \quad \times \\ \times \quad &\frac{w(\overline{x}''_8, \overline{x}''_{58}, \overline{x}''_5/\omega | b_2, c_2 + m_2)}{w(x'''_4, x'''_{24}, \omega x'''_2 | b_1, c_3 + m_1)} \frac{w(\omega x'_8, x'_{58}, x'_5 | c_2, b_3 + m_3)}{w(x_3/\omega, x_{13}, x_1 | c_6, b_4 + m_1)} \end{aligned} \qquad (3.7)$$

and sum over a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4. Note that due to "inversion" relation (3.1) we do not lose any information after such transition from spins a_i, b_i to l_i, m_i.

The functions w in expression (3.7) are chosen in such a way that using relation (3.1) we can calculate the sums over the spins a_1, a_2, a_3, a_4 in the l.h.s. and those over b_1, b_2, b_3, b_4 in the r.h.s. of the obtained equation and cancel the summations over the spins σ's, which come from expression (2.15) for the functions W's and $\overline{W}$'s. Now let us consider applicability conditions (3.5) of Star-Square relation (3.4) for the sums over a_1, a_2, a_3, a_4 in the r.h.s. and over b_1, b_2, b_3, b_4 in the l.h.s. of the obtained equation. We obtain eight conditions on x's and $\overline{x}$'s. Applying relation (3.4) eight times and calculating the sums over d spin in the l.h.s. and r.h.s (the spin structure of the sums over d has the form of relation (3.1) and we demand that corresponding variables x, y, z

entering in the arguments of functions $w(x, y, z|k, l)$ are constrained in such a way that relation (3.1) can be used), we come to the equation without any summation. The l.h.s. and r.h.s. of this equation consist of the products of ten w functions with the same spin structure and expressions like $x_1^{m_2}$ coming from relation (3.4). Let us impose all necessary constraints on parameters x_i, x_{ij}, $\bar{x}_i$ and $\bar{x}_{ij}$ to satisfy this equation. On obeying these constraints one can show that spin independent multipliers coming from relations (3.1), (3.4) coincide.

We also have to satisfy the dual variant of (2.2). Hence, we must add a dual set with x replaced by $\bar{x}$ and vice versa to all obtained constraints on parameters x and $\bar{x}$. We will not write out here all these relations. A detailed analysis shows that we have two solutions. The first one corresponds to the choice

$$\bar{x}_i = x_i,\ \bar{x}_{13} = x_{13},\ \bar{x}_{24} = x_{24},\ \bar{x}_{58} = x_{58},\ \bar{x}_{67} = x_{67},\ i = 1, \dots, 8 \tag{3.8}$$

and the same for x', x'', x'''. This choice corresponds to the Bazhanov – Baxter model, considered in [7].

But there is also another possibility. It will be convenient to fix a normalization of all parameters x in W functions as

$$x_3 = 1, \quad x_4 = 1, \quad x_7 = 1, \quad x_8 = 1 \tag{3.9}$$

for all sets x, x', x'', x''' and $\bar{x}$, $\bar{x}'$, $\bar{x}''$, $\bar{x}'''$. Then all parameters $\bar{x}$ can be expressed in terms of x as:

$$\begin{gathered}\bar{x}_1 = 1/x_2, \quad \bar{x}_2 = 1/x_1, \quad \bar{x}_5 = 1/x_6, \quad \bar{x}_6 = 1/x_5, \\ \bar{x}_{13} = \omega^{-1/2}\frac{x_{24}}{x_2},\ \bar{x}_{24} = \omega^{1/2}\frac{x_{13}}{x_1},\ \bar{x}_{58} = \omega^{1/2}\frac{x_{67}}{x_6},\ \bar{x}_{67} = \omega^{-1/2}\frac{x_{58}}{x_5}.\end{gathered} \tag{3.10}$$

Let us introduce the following notations:

$$\begin{gathered}s = \frac{x_1 x_2}{x_5 x_6}, \quad t_1 = \frac{x_5}{\omega x_1}, \quad t_2 = \frac{x_6}{x_1}, \quad t_3 = \frac{x_{58} x_{67}}{x_{13} x_{24}}, \\ j_1 = \frac{x_6}{x_1}\frac{x_{13} x_{58}}{x_{67} x_{24}}, \quad j_2 = \omega\frac{x_5}{x_1}\frac{x_{13} x_{67}}{x_{58} x_{24}}, \quad j_3 = x_5 x_6\end{gathered} \tag{3.11}$$

and the same for the sets of x', x'', x'''. Then all constraints on parameters take the form

$$\begin{gathered}t_1 = t_2''', \quad t_2 = t_2', \quad t_3 = t_2'', \quad t_1' = t_3''', \quad t_3' = t_3'', \quad t_1'' = t_1''', \\ j_1 = j_2''', \quad j_2 = j_2', \quad j_3 = j_2'', \quad j_1' = j_3''', \quad j_3' = j_3'', \quad j_1'' = j_1''', \\ s = s' = s'' = s''', \quad x_{24} x_{24}'' = x_{24}' x_{24}'''.\end{gathered} \tag{3.12}$$

Relations (3.9-3.12) together with *consistency relations* (2.14) are sufficient conditions for weight functions (2.15) to satisfy modified tetrahedron equations (2.2) and their dual variant. In the next section we will obtain a natural parameterization for relations (2.14), (3.11-3.12) in terms of elliptic functions.

4. Parameterization

Recall that consistency relations (2.14) connect N-th powers of x_i and x_{ij}. Hence, it is convenient to consider N-th powers of (3.11-3.12) and introduce *new parameters*

$$S^2 = s^N, \quad T_i^2 = -t_i^N/S, \quad J_i^2 = j_i^N, \quad i = 1,\ldots,3. \tag{4.1}$$

From (3.11-3.12) we see that parameter S is the same for all weights entering in relation (2.2).

Parameters S, T_i, J_i, $i = 1,2,3$ are defined by four independent variables x_1, x_2, x_5, x_6. Therefore, there are three additional constraints between these values. After simple calculations we obtain

$$J_3 = \frac{T_1J_1 - T_2J_2}{T_1J_2 - T_2J_1}, \quad T_3 = \frac{(T_1^2 - T_2^2)(1 + SJ_1J_2T_1T_2)}{(1 - T_1^2T_2^2)(T_1J_2 - T_2J_1)}, \tag{4.2}$$

and

$$P \equiv \frac{S + T_i^2 - SJ_i^2(1 + ST_i^2)}{T_iJ_i}, \quad i = 1,2, \tag{4.3}$$

where we introduce a new parameter P.

Note that definitions (4.1) contain only the second powers of variables S, T_i, J_i and in (4.2-4.3) we have chosen some signs in a convenient way. Using (4.2) it is easy to obtain that we can add the case $i = 3$ to relations (4.3). The validity of formulas (4.2-4.3) can be checked by substitution of relations (3.11), (4.1) and consistency conditions (2.14).

Introduce an elliptic curve

$$P = \frac{S + x^2 - Sy^2(1 + Sx^2)}{xy}. \tag{4.4}$$

Points (T_i, J_i), $i = 1,2,3$ belong to this curve. We can uniform it in terms of elliptic functions (see, for example, [11]). Note that formulas (4.2) take the

…m of addition theorems for elliptic functions. Then we obtain

$$T_j = im^{1/2}\mathrm{sn}(u_j), \quad d_j = \frac{\mathrm{sn}(\eta - u_j)}{\mathrm{sn}(\eta)}, \qquad j = 1, \ldots, 3,$$
$$P = -2im^{1/2}\mathrm{sn}(\eta)\mathrm{cn}(\eta)\mathrm{dn}(\eta), \quad S = m\,\mathrm{sn}^2(\eta), \tag{4.5}$$

where sn, cn, dn are elliptic functions of modulus m and

$$u_3 = u_1 + u_2. \tag{4.6}$$

In such a way we can obtain the parameterization for x_i^N and x_{ij}^N. When we take roots of N-th powers, a phase ambiguity appears. Making an appropriate choice of these phases so that (3.11-3.12) be valid we obtain the following formulas

$$x_1 = \omega^{-1/2}\left\{\frac{\mathrm{sn}(\eta - u_3)\mathrm{sn}(\eta)}{\mathrm{sn}(u_1)\mathrm{sn}(u_2)}\right\}^{1/N},$$
$$x_2 = \omega^{1/2}\left\{m^2\mathrm{sn}(\eta - u_3)\mathrm{sn}(\eta)\mathrm{sn}(u_1)\mathrm{sn}(u_2)\right\}^{1/N},$$
$$x_5 = \omega^{1/2}\left\{\frac{\mathrm{sn}(u_1)\mathrm{sn}(\eta - u_3)}{\mathrm{sn}(u_2)\mathrm{sn}(\eta)}\right\}^{1/N}, \quad x_6 = \omega^{-1/2}\left\{\frac{\mathrm{sn}(u_2)\mathrm{sn}(\eta - u_3)}{\mathrm{sn}(u_1)\mathrm{sn}(\eta)}\right\}^{1/N}, \tag{4.7}$$

and

$$x_{13} = \omega^{-1/2}\left\{\frac{\Theta(u_3)\Theta(0)H(\eta - u_1)H(\eta - u_2)}{\Theta(\eta - u_3)\Theta(\eta)H(u_1)H(u_2)}\right\}^{1/N},$$
$$x_{24} = \omega^{1/2}\left\{\frac{\Theta(u_3)\Theta(0)\Theta(\eta - u_1)\Theta(\eta - u_2)}{\Theta(\eta - u_3)\Theta(\eta)\Theta(u_1)\Theta(u_2)}\right\}^{1/N},$$
$$\text{[illegible]} = \omega^{1/2}\left\{\frac{H(u_3)\Theta(0)H(\eta - u_1)\Theta(\eta - u_2)}{\Theta(\eta - u_3)H(\eta)\Theta(u_1)H(u_2)}\right\}^{1/N},$$
$$x_{67} = \omega^{-1/2}\left\{\frac{H(u_3)\Theta(0)\Theta(\eta - u_1)H(\eta - u_2)}{\Theta(\eta - u_3)H(\eta)H(u_1)\Theta(u_2)}\right\}^{1/N}, \tag{4.8}$$

where $H(u)$ and $\Theta(u)$ are Jacobi elliptic functions (see [11]). Suppose that real parameters u_1, u_2, η satisfy

$$0 < u_{1,2,3} < \eta < 2K, \tag{4.9}$$

where K is the complete integral of the first kind of the modulus m. These conditions guarantee that all values of the elliptic functions in (4.7-4.8) are non-negative, and we choose the positive values of all roots of N-th power in formulas (4.7-4.8).

Now we can rewrite the Boltzmann weight as a function of four new parameters: $W(u_1, u_2|m, \eta)$ (we omit the dependence from spin variables). It is on easy to check that dual weights $\overline{W}$ can be obtained by the shift of parameter η:

$$\overline{W}(u_1, u_2|m, \eta) = W(u_1, u_2|m, \eta + i\mathcal{M}'), \tag{4.10}$$

where $\mathcal{M}'$ is the complete integral of the first kind of the complementary modulus $m' = \sqrt{1 - m^2}$. Equations (3.12) relate the sets of new parameters in different weights. Then we obtain

$$\begin{aligned} W &\to W(u_2, u_1), \quad W' \to W(u_2 + u_5, u_1), \\ W'' &\to W(u_5, u_1 + u_2), \quad W''' \to W(u_5, u_2). \end{aligned} \tag{4.11}$$

Parameters m and η are the same for all weight functions and we omit them in (4.11). Also note that we have chosen three independent parameters in (4.11) as u_1, u_2 and u_5 to emphasize the connection of our parameterization with static limit one (2.4).

5. Symmetry Properties

In this section we discuss symmetry properties of weight functions (2.15) of the model with respect to the group G of transformations of a three-dimensional cube (see [10, 12]). Recall briefly some definitions. Group G has two generating elements ρ and τ. Any element α of G can be expressed as a composition of these two elements. We define the action of elements τ and ρ on the set of spins $\{a|e, f, g|b, c, d|h\}$ as follows

$$\tau\{a|e, f, g|b, c, d|h\} = \{a|f, e, g|c, b, d|h\} \tag{5.1}$$

and

$$\rho\{a|e, f, g|b, c, d|h\} = \{g|c, a, b|f, h, e, |d\}. \tag{5.2}$$

Further it will be convenient to remove normalization conditions (3.9) and to restore the homogeneity of the parameters x's.

It is easy to see that weight function (2.15) is invariant under the action of element τ, if we make the following transformation of parameters:

$$\begin{gathered}(x_3, x_{13}, x_1) \to (x_3, x_{13}, x_1), \quad (x_1, x_{24}, x_2) \to (x_4, x_{24}, x_2), \\ (x_8, x_{58}, x_5) \to (x_7/\omega, x_{67}, x_6), \quad (x_7, x_{67}, x_6) \to (\omega x_8, x_{58}, x_5). \end{gathered} \tag{5.3}$$

The action of element ρ is less trivial and can be obtained with the help of the Fourier transformation (see [10]). Introduce two more additional parameters u and v which are defined as

$$u^N = \frac{x_3^N x_5^N - x_1^N x_7^N}{x_1^N}, \quad v^N = \frac{x_2^N x_8^N - x_4^N x_6^N}{x_2^N}. \tag{5.4}$$

The phases of parameters u and v can be chosen in an arbitrary way. Then one can show that the following identity is valid:

$$\begin{aligned} &\Big\{ \sum_{\sigma \in Z_N} \frac{w(x_3, x_{13}, x_1 | d, h+\sigma) w(x_4, x_{24}, x_2 | a, g+\sigma)}{w(x_8, x_{58}, x_5 | e, c+\sigma) w(x_7/\omega, x_{67}, x_6 | f, b+\sigma)} \Big\}_0 = \\ &= \omega^{dh+ch-bf-bg} \frac{w(x_4 x_6, x_2 v, x_2 x_8 | a+b, f+g)}{w(x_1 x_7, x_1 u, x_3 x_5 | c+h, d+e)} \times \\ &\times \Big\{ \sum_{\sigma \in Z_N} \frac{w(x_7 x_{13}, x_1 u, x_3 x_{57} | e, d+\sigma) w(x_2 x_{68}, x_2 v, x_6 x_{24} | g, b+\sigma)}{w(x_5 x_{13}, \omega x_1 u, \omega x_1 x_{57} | c, h+\sigma) w(x_4 x_{68}, x_2 v, x_8 x_{24} | a, f+\sigma)} \Big\}_0, \end{aligned} \tag{5.5}$$

where the lower index "0" after the curly brackets implies that the expression in the curly brackets is divided by itself with all exterior spin variables equated to zero.

The arrangement of spins in the sum of the r.h.s. in (5.5) corresponds to the ρ-transformed set of the indices (see (5.2)). Hence, we can define the action of ρ transformation on the parameters as

$$\begin{gathered}(x_3, x_{13}, x_1) \xrightarrow{\rho} (x_7 x_{13}, x_1 u, x_3 x_{57}), \quad (x_1, x_{24}, x_2) \xrightarrow{\rho} (x_2 x_{58}, x_2 v, x_6 x_{24}), \\ (x_7, x_{57}, x_5) \xrightarrow{\rho} (x_5 x_{13}, \omega x_1 u, \omega x_1 x_{57}), \\ (x_8, x_{68}, x_6) \xrightarrow{\rho} (\omega x_4 x_{68}, x_2 v, x_8 x_{24}). \end{gathered} \tag{5.6}$$

Using (5.5) we can choose gauge factors before formula (2.15) in such a way that the whole W function will be invariant under the transformations from the group G. We will not write this symmetric form of weight functions here. Note only that after ρ transformation (5.6) W function is transformed into

$\mathcal{W}$ and vice versa. Using relations (4.7-4.8) from the previous section we can write the following transformation rules for parameters u_1, u_2, η with respect to the elements τ and ρ from the group G:

$$u_1 \xrightarrow{\tau} u_2, \quad u_2 \xrightarrow{\tau} u_1, \quad \eta \xrightarrow{\tau} \eta, \tag{5.7}$$

$$u_1 \xrightarrow{\rho} i\mathcal{M}' - u_1, \quad u_2 \xrightarrow{\rho} u_1 + u_2 - i\mathcal{M}', \quad \eta \xrightarrow{\rho} \eta + i\mathcal{M}'. \tag{5.8}$$

We hope to use these symmetries for the calculation of the partition function for our models.

Appendix. A detailed consideration for the case $N = 2$

The BCC form of the weight is useful for proving the modified tetrahedron equations. But for other purposes a symmetrical form of the Boltzmann weights, which differs from the BCC one by some gauge transformation, is more preferable. In this section we find this gauge multipliers for the case when $N = 2$ and obtain a table of Boltzmann weights analogous to Baxter's table from [9] and endowed by obvious symmetrical properties. Also we show that fixing $\eta = \mathcal{M}$ we obtain the particular case considered in [8].

We start from a simple gauge transformation of our BCC weight (2.15) and consider the weight W':

$$W'(a|e,f,g|b,c,d|h) = (-1)^{bf+ec+a+h_{\,;}-af+ch-dh+ag+cg-df} \times$$
$$\times \frac{Y_{e,d,h,c} Y_{g,e,c,a} Y_{c,h,b,g}}{Y_{a,f,b,g} Y_{b,h,d,f} Y_{e,d,f,a}} W(a|e,f,g|b,c,d|h), \tag{A.1}$$

where

$$Y_{a,f,b,g} = \exp\{-i\frac{\pi}{2} fg\} \exp\{i\pi ab(f-g)\}\cdot \tag{A.2}$$

It is convenient to introduce spins $(-1)^a = \pm 1$ instead of $a = 0, 1$. Further we shall use only the multiplicative spins and write a instead of $(-1)^a$, etc.

Absolute values of these weights (both W and W') depend only on three Baxter's sign variables $abeh$, $acfh$, $adgh$, $abcd$. Therefore up to signs there exist sixteen different weights (A.1). In terms of multiplicative spins we can write the following parameterization:

$$\begin{aligned} W' = \xi W_S \times \\ \times\ \exp\{(M_1+N_1)agbf + (M_1-N_1)cedh+ \\ +(M_2+N_2)agce + (M_2-N_2)bfdh+ \\ +(M_3+N_3)aefg + (M_3-N_3)cgbf+ \\ +(M_0+N_0)abcd + (M_0-N_0)efgh\}, \end{aligned} \qquad (A.3)$$

where, calculating all sign factors, we parameterize W_S as follows:

$abeh$	$acfh$	$adgh$	$W_S(a\|efg\|bcd\|h)$
+	+	+	P_0
−	+	+	R_1
+	−	+	R_2
+	+	−	R_3
+	−	−	abP_1
−	+	−	acP_2
−	−	+	adP_3
−	−	−	$abcdS_C$

Parameter ξ in (A.3) is a normalization factor and we choose it so that $P_0 = 1$. Now we can obtain $\exp(16M_i)$, $\exp(16N_i)$ and W_S^2 just by multiplying and dividing BCC weights for several sets of the spins.

It is useful to express the answer in terms of elliptic functions of another modulus k instead of the used functions of the modulus m

$$k = \frac{1-m}{1+m}. \qquad (A.4)$$

Denoting the complete elliptic integrals of the first kind of the moduli m and $m' = \sqrt{1-m^2}$ as $\mathcal{M}$ and $\mathcal{M}'$, and of the moduli k and $k' = \sqrt{1-k^2}$ as $\mathcal{K}$ and $\mathcal{K}'$, we have the connection between them

$$\mathcal{K}' = (1+m)\mathcal{M}, \quad 2\mathcal{K} = (1+m)\mathcal{M}'. \qquad (A.5)$$

One can easily obtain the reparameterization of x_i^2 and x_{ij}^2 using

$$im^{1/2}\mathrm{sn}(v,m) = k'\frac{\mathrm{sn}(x,k)}{\mathrm{cn}(x,k)\mathrm{dn}(x,k)}, \text{ where } \frac{2x}{1+m} = iv. \qquad (A.6)$$

To calculate expressions for W_S, $\exp(2M_i)$ and $\exp(2N_i)$, we need three more complicated identities:

$$\begin{aligned}
&\frac{\mathrm{sn}(v_1+v_2)-\mathrm{sn}(v_1)-\mathrm{sn}(v_2)-m\mathrm{sn}(v_1)\mathrm{sn}(v_2)\mathrm{sn}(v_1+v_2)}{\mathrm{sn}(v_1+v_2)+\mathrm{sn}(v_1)+\mathrm{sn}(v_2)-m\mathrm{sn}(v_1)\mathrm{sn}(v_2)\mathrm{sn}(v_1+v_2)} \\
&\qquad = \frac{\mathrm{sn}(x_1)\mathrm{sn}(x_2)}{\mathrm{cd}(x_1)\mathrm{cd}(x_2)}, \\
&\frac{\mathrm{sn}(v_1)+\mathrm{sn}(v_2)+\mathrm{sn}(v_1+v_2)+m\mathrm{sn}(v_1)\mathrm{sn}(v_2)\mathrm{sn}(v_1+v_2)}{\mathrm{sn}(v_1)+\mathrm{sn}(v_2)+\mathrm{sn}(v_1+v_2)-m\mathrm{sn}(v_1)\mathrm{sn}(v_2)\mathrm{sn}(v_1+v_2)} \\
&\qquad = \frac{\mathrm{cd}(x_1+x_2)}{\mathrm{cd}(x_1)\mathrm{cd}(x_2)}, \\
&\frac{\mathrm{sn}(v_1)+\mathrm{sn}(v_2)-\mathrm{sn}(v_1+v_2)-m\mathrm{sn}(v_1)\mathrm{sn}(v_2)\mathrm{sn}(v_1+v_2)}{\mathrm{sn}(v_1)+\mathrm{sn}(v_2)+\mathrm{sn}(v_1+v_2)-m\mathrm{sn}(v_1)\mathrm{sn}(v_2)\mathrm{sn}(v_1+v_2)} = \\
&\qquad = -k^2\mathrm{sn}(x_1)\mathrm{sn}(x_2)\mathrm{cd}(x_1+x_2),
\end{aligned} \qquad (A.7)$$

where in the LHS we imply the modulus m, and in the RHS - k, the arguments v_i and x_i being connected by (A.6).

We would like to obtain a model with real weights W_S. To do this we have to consider a regime when u_1, u_2, u_3 are pure imaginary satisfying $0 < \mathrm{Im}u_{1,2,3} < \mathcal{M}'$ and $\eta = \mathcal{M} + i\epsilon$, a small ϵ being real. Defining further the variables $z_{1,2,3}, \lambda$ by

$$\frac{2z_{1,2}}{1+m} = -iu_{1,2}, \; z_1+z_2+z_3 = K, \; \frac{2\lambda}{1+m} = i\eta; \qquad (A.8)$$

and using repeatedly (A.7), we obtain the exponentials

$$\begin{aligned}
\exp 2N_1 &= e^{i\pi/4}\{-ik\mathrm{sn}(\lambda+z_1)\mathrm{cd}(\lambda+z_1)\}^{1/4}, \\
\exp 2N_2 &= \{-ik\mathrm{sn}(\lambda+z_2)\mathrm{cd}(\lambda+z_2)\}^{1/4}, \\
\exp 2N_3 &= \{-ik\mathrm{sn}(\lambda-z_3)\mathrm{cd}(\lambda-z_3)\}^{1/4}, \\
\exp 2N_0 &= \{-ik\mathrm{sn}(\lambda)\mathrm{cd}(\lambda)\}^{1/4};
\end{aligned} \qquad (A.9)$$

$$\exp 2M_1 = \{i\frac{\mathrm{cd}(\lambda + z_1)}{\mathrm{sn}(\lambda + z_1)}\}^{1/4}, \quad \exp 2M_2 = \{i\frac{\mathrm{cd}(\lambda + z_2)}{\mathrm{sn}(\lambda + z_2)}\}^{1/4},$$
$$\exp 2M_3 = \{i\frac{\mathrm{cd}(\lambda - z_3)}{\mathrm{sn}(\lambda - z_3)}\}^{1/4}, \quad \exp 2M_0 = \{-i\frac{\mathrm{sn}(\lambda)}{\mathrm{cd}(\lambda)}\}^{1/4}. \qquad (A.10)$$

In these formulae z_i are real, $0 < z_i < K$, and $\lambda = iK'/2 + \epsilon'$, ϵ' being small and real. The expressions contained in the curly brackets belong to the first and fourth quadrants of the complex plane, the fourth power roots of these values are defined so that they belong to the same quadrants. As to W_S, it coincides with the elliptic solution of ref. [8]:

$$P_0 = 1, \quad P_i = \sqrt{\frac{\mathrm{sn}(z_j)\mathrm{sn}(z_k)}{\mathrm{cd}(z_j)\mathrm{cd}(z_k)}},$$
$$S_0 = k\sqrt{\mathrm{sn}(z_1)\mathrm{sn}(z_2)\mathrm{sn}(z_3)}, \quad R_i = \sqrt{\frac{\mathrm{sn}(z_i)}{\mathrm{cd}(z_j)\mathrm{cd}(z_k)}}, \qquad (A.11)$$

where i, j, k are any permutation of $1, 2, 3$. Here the square roots of the positive values are supposed to be also positive. Our weights coincide with those of [8] when

$$\lambda = i\frac{K'}{2} \Leftrightarrow \eta = \mathcal{M}. \qquad (A.12)$$

The dual weights W' can be obtained from these ones by the changing $M_i \to -M_i$, $i = 0, ..., 3$, and $S_0 \to -S_0$, so that the terms $M_{1,2,3}$ and $N_{1,2,3}$ are the pure gauge of the system of the modified tetrahedron equations and the only difference with the case (A.12) consists in the M_0 and N_0 terms.

Note that the extraction of the λ - dependent terms $\exp(M_0)$ and $\exp(N_0)$ as common multipliers of the λ - independent weight W_S is the property of the choice $N = 2$, when $N \neq 2$ this property fails.

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