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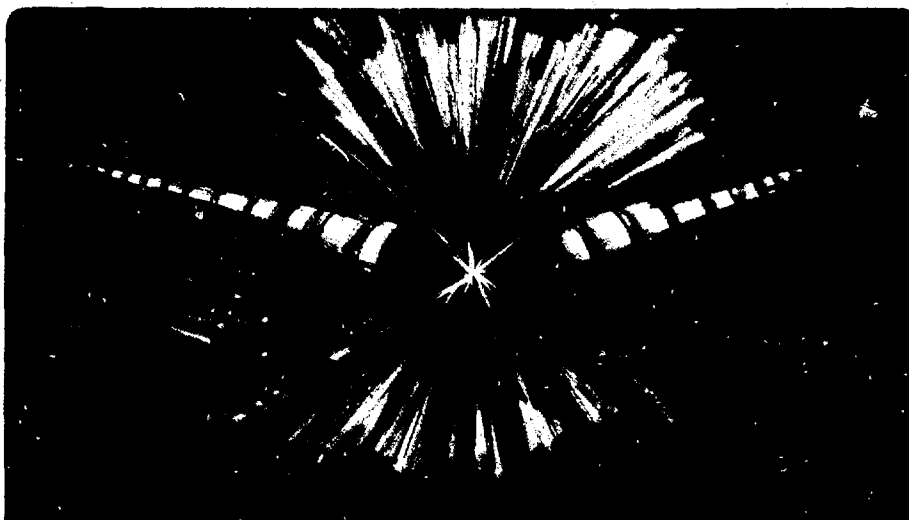
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Solution of K-V Envelope Equations

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SOLUTION OF K-V ENVELOPE EQUATIONS

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The envelope equations for a KV beam with space charge have been analyzed systematically by an expansion followed by integrations. The focusing profile as a function of axial length is assumed to be symmetric but otherwise arbitrary. Given the beam current, emittance, and peak focusing field, we find the envelopes $a(s)$ and $b(s)$ and obtain $\langle a \rangle$, a_{max} , σ , and σ_0 . Explicit results are presented for various truncations of the expansion. The zeroth order results correspond to those from the well-known smooth approximation; the same convenient format is retained for the higher order cases. The first order results, involving single correction terms, give 3 to 10 times better accuracy and are good to $\sim 1\%$ at $\sigma_0 = 70^\circ$. Third order gives a factor of 10-30 improvement over the smooth approximation and derived quantities accurate to $\sim 1\%$ at $\sigma_0 = 112^\circ$. The first order expressions are convenient design tools. They lend themselves to variable energy problems and have been applied to the design, construction, and testing of ESQ accelerators at LBL.

I. K-V ENVELOPE EQUATIONS

A non-relativistic beam with a uniform density (K-V distribution) transported by a series of linear symmetric quadrupoles is described by the paraxial equations for the envelopes a and b :

$$a'' = -K(z)a + \frac{\epsilon^2}{a^3} + \frac{2Q}{a+b} \quad (1)$$

$$b'' = -K(z)b + \frac{\epsilon^2}{b^3} + \frac{2Q}{a+b} \quad (2)$$

where $K(z)$ represents the alternating quadrupole gradient and ϵ is the emittance (we assume $\epsilon_x = \epsilon_y$). Q is the normalized perveance, defined nonrelativistically by $Q = (4\pi\epsilon_0)^{-1}(m/2q)1/21V^{-3/2}$, with I the beam current and qV the beam energy.

A. Review of Smooth Approximation Formulas

Before presenting our own results, we recollect the well known smooth approximation results [1], [2], [3] and introduce more of our notation. Equations (31) and (37) in Ref. [1] are

$$\frac{1}{2L} \int_0^{2L} K(z)\rho(z)dz = \frac{\sigma_0^2}{(2L)^2} \quad (3)$$

and

$$\frac{\sigma_0^2}{(2L)^2} = \frac{\epsilon^2}{A^4} + \frac{Q}{A^2},$$

where we have converted to our notation: $\mu_0 \rightarrow \sigma_0$, $S \rightarrow 2L$, $\bar{R} \rightarrow A$, $K \rightarrow Q$, $\delta_x(s) \rightarrow K(z)$, $\delta_x(s) \rightarrow \rho(z)$. In Eq. (3), it is not clear in Ref. [1] how $\rho(z)$ is to be calculated, but if we use our Eq. (21) [below] and change the order of integrations, the left side is

$\langle [\int dz' K(z')]^2 \rangle$, where $\langle \dots \rangle$ means averaging over a cell of length $2L$. We use this quantity often and name it K_{eff} , the effective focusing force:

$$\left\langle \left[\int dz' K(z') \right]^2 \right\rangle = K_{eff} \quad (4)$$

The zero-order matching equation is

$$K_{eff} = \frac{\epsilon^2}{A^4} + \frac{Q}{A^2} \quad (5)$$

The above result is also given in Ref. [3], Eq. (10.92), which explicitly defines to lowest order $\sigma_0^2 / (2L)^2 = \langle [\int dz' K(z')]^2 \rangle$. Therefore, in this paper

$$\sigma_{0smooth} = 2LK_{eff}^{1/2} \quad (6)$$

is called the smooth approximation for σ_0 .

The depressed tune s is (Ref. [2], Eq. (6)),

$$\sigma_{0smooth} = 2L \frac{\epsilon}{A^2}, \quad (7)$$

where A^{-2} is obtained from the zero-order equation (5).

The smooth-approximation formulas are popular design tools for AG systems because they are simple and explicit. However, they become seriously inaccurate for applications with large focusing fields and large phase advances. Our simple generalizations derived below improve the accuracy by a factor of 3 to 10 or more (Table I, Section III) while retaining the above simple explicit formats.

II. SYSTEMATIC SOLUTION

We allow the form of $K(z)$ to be arbitrary except for the assumption of symmetry. We write $K(z) = Kh(z)$ with $K = K(0)$ and $h(0) = 1$ and define the cell length as $2L$. We assume the quadrupole form factor h is symmetric about $z = 0$, periodic over a cell length and antiperiodic over a half-cell length:

$$h(-z) = h(z), h(z - 2L) = h(z), h(z - L) = -h(z). \quad (8)$$

It follows that: $h(L/2) = 0$, h is antisymmetric about $L/2$, $h(z)$ ranges between $+1$ and -1 , and $\langle h(z) \rangle = 0$. In the following, we start all our integrations at $z = 0$, where $h = 1$.

A. Expansion about Mean Radius

We assume the beam is matched so that $\langle a \rangle = \langle b \rangle = A$ and expand about the mean radius: $a(z) = A + \tilde{a}(z) = A(1 + \rho)$, where we define the ripple ratio

$$\rho(z) = \frac{\tilde{a}(z)}{A}$$

The assumption of quadrupole symmetry means that $\bar{b}(z) \approx -\bar{a}(z)$, so that $a+b \approx 2A$. (Actually, there is a correction term which we drop without affecting the results, as discussed below.) In any case, the coupling between Eqs. (1) and (2) is eliminated:

$$a'' = -K(z)a + \frac{\epsilon^2}{a^3} + \frac{Q}{A}. \quad (9)$$

(After solving for a , the solution for b is obtained by changing the sign of terms containing odd powers of K .) Substituting $A(1+\rho)$ for $a(z)$ in (9), expanding, and dividing by A , we have

$$\rho'' = -Kh(z) - Khp + \frac{\epsilon^2}{A^4} (1 - 3\rho + 6\rho^2 \dots) + \frac{Q}{A^2}. \quad (10)$$

Averaging,

$$0 = -K\langle hp \rangle + \frac{\epsilon^2}{A^4} (1 + 6\langle \rho^2 \rangle \dots) + \frac{Q}{A^2}. \quad (11)$$

Subtracting,

$$\rho'' = -Kh(z) - K\langle hp \rangle - \frac{3\epsilon^2}{A^4} (\rho - 2\langle \rho^2 \rangle \dots), \quad (12)$$

where the operator $\langle \dots \rangle$ gives just the oscillatory part of a function:

$$\langle \phi \rangle \equiv \phi - \langle \phi \rangle. \quad (13)$$

If we assume that a never vanishes, then $\rho < 1$ and the above Taylor expansion converges; (11) and (12) taken together have exactly the same content as the original equation, (9). Reference [4] extends Eq. (10) through ρ^6 .

B. Systematic Solution: Periodic Part

We follow Courant and Snyder in their treatment of the Hill equation [5]. They regard the focusing coefficient $K(z)$ as "small in some sense," put $K(s) = 1/2 \epsilon g(s)$, and expand their beta function ($\sim a^2$) in powers of the "smallness parameter" ϵ . Our treatment differs in that we include space charge and must work with $a = A(1+\rho)$ instead of their beta function:

$$\rho(z) \equiv \epsilon f_1(z) + \epsilon^2 \phi_2(z) + \epsilon^3 \phi_3(z) + \dots \quad (14)$$

In Ref. [4] we show how to feasibly include terms through $\epsilon^7 \phi_7$, but here we use only the three terms shown in (14).

As in [5], our basic small quantity is the focusing strength K , and we write

$$K \equiv \epsilon k. \quad (15)$$

From Eqs. (4) and (5), $K^2 L^2 (\text{Const} \approx \epsilon^2 / A^4 + Q/A^2)$. The terms on the right can be no larger than $\epsilon^2 K^2 L^2$, so we give them ϵ^2 ordering and define α and q by:

$$\frac{3\epsilon^2}{A^4} \equiv \epsilon^2 \alpha, \quad (16)$$

$$\frac{Q}{A^2} \equiv \epsilon^2 q. \quad (17)$$

(We assume that either of these terms could dominate, i.e., s is in the range $0 < \sigma < \sigma 0$.)

We insert (14)–(17) into (12). Through ϵ^3 ,

$$\epsilon f_1'' + \epsilon^2 f_2'' + \epsilon^3 f_3'' = -\epsilon k h(z) - \epsilon^2 k \langle hf_1 \rangle - \epsilon^3 k \langle hf_2 \rangle - \epsilon^3 \alpha f_1.$$

Equating like powers of ϵ , $f_1'' = -kh(z)$, $f_2'' = -k\langle hf_1 \rangle$, $f_3'' = -\alpha f_1 - k\langle hf_2 \rangle$. Integrating,

$$\begin{aligned} f_1 &= -k \int \int h \\ f_2 &= k^2 \int \int \langle hg \rangle \\ f_3 &= +\alpha k \int \int g - k^3 \int \int \langle h\delta \rangle, \end{aligned}$$

where

$$g \equiv \int \int h, \quad (18)$$

and the small term

$$\delta(z) \equiv \int \int \langle hg \rangle \quad (19)$$

has double the fundamental frequency of the focusing lattice. The operator $\int \int$ gives just the oscillatory part of the repeated integral

$$\int \int \psi \equiv \left\{ \begin{array}{l} z \\ \int dz' \int \psi(z') dz'' \\ 0 \quad 0 \end{array} \right\}.$$

Note: For an operand such as $g(z)$ with the symmetries of Eq. (8),

$$\int \int \psi \equiv \left\{ \begin{array}{l} z \quad z' \\ \int dz' \int \psi(z'') dz'' \\ L/2 \quad 0 \end{array} \right\}.$$

For an operand such as $\langle hg \rangle$ which lacks this symmetry, one could construct the appropriate lower limit, but more easily subtract the average as in (13).

A feature of our ordering is that f_1 , f_3 , etc., turn out to have only odd harmonics and odd powers of k , while f_2 , etc., have only the even cases. Thus

$$\rho = \frac{\bar{a}(z)}{A} = -\epsilon k g + \epsilon^2 k^2 \delta + \epsilon^3 \alpha k \int \int g - \epsilon^3 k^3 \int \int \langle h\delta \rangle + \dots,$$

$$= \frac{\bar{b}(z)}{A} = +\epsilon k g + \epsilon^2 k^2 \delta + \epsilon^3 \alpha k \int \int g - \epsilon^3 k^3 \int \int \langle h\delta \rangle + \dots \quad (20)$$

Defining the leading-order ripple,

$$\rho_0(z) \equiv -\epsilon k g = -K g = -K \int \int h \quad (21)$$

and using (15) and (16), we have, finally

$$\rho = \frac{\bar{a}(z)}{A} = +\rho_0 + K^2 \delta(z) - \frac{3\epsilon^2}{A^4} \int \int \rho_0 - K^3 \int \int \langle h\delta \rangle, \quad (22)$$

$$= \frac{\bar{b}(z)}{A} = -\rho_0 + K^2 \delta(z) - \frac{3\epsilon^2}{A^4} \int \int \rho_0 - K^3 \int \int \langle h\delta \rangle \quad (23)$$

Note: For large focusing strengths, the double-frequency term $K2d$ becomes significant: e.g., if $\sigma_0 = 120^\circ$, $K^2\delta \approx 0.025$. Then, noting from (22) and (23) that $a + b = 2A(1 + K^2\delta)$, one might think it necessary to include the correction factor $(1 - K^2\delta)$ on the Q term in Eq. (9). In Ref. [4] this done and shown to affect the results only in higher order: for σ_0 as large as 120° , the correction contributes at most 0.04% to the maximum radius and nothing at all to the matching equation.

The matching equation, derived below, gives the mean radius A needed for (22) and (23).

C. Systematic Solution: Average Part, Matching Equation

We insert (15)–(17) into (11) to get

$$\epsilon k \langle hp \rangle = \epsilon^2 \frac{\alpha}{3} + 2\epsilon^2 \alpha \langle \rho^2 \rangle + \epsilon^2 q, \quad (24)$$

where $\rho(z)$ is given by (20). To order ϵ^4 ,

$$2\epsilon^2 \alpha \langle \rho^2 \rangle = 2\epsilon^4 \alpha k^2 \langle g^2 \rangle. \quad (25)$$

By above-mentioned k parity, $\langle hf^2 \rangle = 0$, so

$$\begin{aligned} \epsilon k \langle hp \rangle &= -\epsilon^2 k^2 \langle h \parallel h \rangle + \epsilon^4 \alpha k^2 \langle h \parallel g \rangle - \epsilon^4 k^4 \langle h \parallel \{h\delta\} \rangle \\ &= +\epsilon^2 k^2 \langle [j h]^2 \rangle + \epsilon^4 \alpha k^2 \langle g^2 \rangle - \epsilon^4 k^4 \langle [j \{hg\}]^2 \rangle. \end{aligned}$$

We reordered integrations using the $h(z)$ symmetries, Eq. (8). For example, $-\langle h \parallel h \rangle = \langle [j h]^2 \rangle$, with notation

$$j \psi \equiv \int_0^z \psi(z') dz'.$$

The $\langle g^2 \rangle$ term cancels half of (25), and the matching equation through ϵ^4 is

$$\begin{aligned} \epsilon^2 k^2 \langle [j h]^2 \rangle + \epsilon^4 k^4 \langle [j \{hg\}]^2 \rangle &= \epsilon^2 \frac{\alpha}{3} + \\ \epsilon^4 \alpha k^2 \langle g^2 \rangle + \epsilon^2 q. & \quad (26) \end{aligned}$$

Using Fourier expansion of $h(z)$ for the small second term on the left side, we find [4]

$$\langle [j \{hg\}]^2 \rangle = 1-8 \langle g^2 \rangle (1 + 20/27 c_3 + \dots) \langle [j h]^2 \rangle,$$

where c_3 is 3 times the ratio of the 3rd harmonic to the 1st harmonic of $h(z)$. (For FOFO with occupancy $\eta = 0.5$, $c_3 = 1$.) We define the LHS of (26) as

$$\begin{aligned} K_I^{\text{eff}} &= K^2 \langle [j h]^2 \rangle (1 + 1/8 H_3 \langle \rho_0^2 \rangle) \\ &= K^{\text{eff}} (1 + 1/8 H_3 \langle \rho_0^2 \rangle), \quad (27) \end{aligned}$$

where $H_3 \equiv 1 + 20/27 c_3$, and where we use the abbreviation

$$\langle \rho_0^2 \rangle = K^2 \langle g^2 \rangle = K^2 \langle [j h]^2 \rangle, \quad (28)$$

the mean-square ripple to leading order. On the right side of (26), if we define

$$\epsilon_1^2 \equiv \epsilon^2 (1 + 3 \langle \rho_0^2 \rangle), \quad (29)$$

our matching equation is

$$K_I^{\text{eff}} = \frac{\epsilon_1^2}{A^4} + \frac{Q}{A^2}, \quad (30)$$

easily solvable for the mean radius A , required for ρ and σ . We have retained the convenient form of the smooth approximation (5) by introducing the one-term corrections denoted by subscripts I . (In Ref. [4] we go to higher order and calculate 6 correction terms.)

Envelopes: For matched beam, $a(z) = A + \tilde{a}(z)$ and $b(z) = A + \tilde{b}(z)$, where one uses (30), (22) and (23).

D. Phase Advances

Depressed Tune: From the well-known phase-amplitude result [5], the phase advance per quadrupole cell of

length $2L$ is $\sigma = \epsilon \int_0^{2L} a^{-2} dz$. Using the definition

$a(z) \equiv A[1 + \rho(z)]$, expanding, and noting that the 2ρ term has zero average, one obtains $\sigma = 2L\epsilon A^{-2}(1 + 3\langle \rho^2 \rangle \dots)$. To leading order, $\langle \rho^2 \rangle = \langle \rho_0^2 \rangle$, and

$$\sigma = 2L \frac{\epsilon}{A^2} (1 + 3\langle \rho_0^2 \rangle), \quad (31)$$

where A^{-2} is calculated from (A56). Except for the correction term, this has the form of the smooth approximation, Eq. (7), but A^{-2} is calculated more accurately here.

Undepressed Tune: We set $Q = 0$ so that (30) becomes

$$K_I^{\text{eff}} = \frac{\epsilon_1^2}{A^4} = \frac{\epsilon^2}{A^4} (1 + 3\langle \rho_0^2 \rangle)$$

with K_I^{eff} from (27). Combining this with (31) for $Q = 0$, we eliminate ϵA^{-2} to get

$$\sigma_0 = 2L (K_I^{\text{eff}})^{1/2} (1 + 3\langle \rho_0^2 \rangle). \quad (32)$$

[Cf. Eq. (6).] We will see below that these simple formulas for σ and σ_0 , with single correction terms, give 3 to 10 times greater accuracy than the smooth approximations.

We emphasize that all the above results apply to a general symmetric lattice [4] with or without discontinuities in $K(z)$.

III. SPECIAL CASE: FODO LATTICE

A. Solution of Ripple Equation

For the FODO cell, it is not hard to obtain the lowest-order ripple function $\rho_0(z)$ —see Ref [4]. Here, we just quote its maximum value. With h the occupancy factor,

$$\begin{aligned} \epsilon f_1(0) = \rho_0^{\text{max}} &= K \int_0^{L/2} dz \int_0^z h(z') dz' \\ &= \frac{1}{8} \eta (2 - \eta) K L^2. \quad (33) \end{aligned}$$

For $\epsilon^2 f_2 = K^2 \langle [j \{hg\}]^2 \rangle$, it is convenient to Fourier analyze because most harmonics make negligible

contributions to the results. Writing $h(z) = h_1[\cos(\pi z/L) + c_3 \cos(3\pi z/L) + \dots]$,
 $\epsilon^2 f_2 \approx 1/8 \rho_m^2 (1 + 10/27 c_3) \cos 2\pi z/L$; (34)

$c_3 = 1$ for $h = 0.5$; $\pi \equiv h_1 KL^2/\pi^2$. Neglected terms in (34) would contribute less than 0.06% to the final result for a_{\max} .

For $\epsilon^3 f_3$ we integrate the first term but use Fourier representation for the second. We quote just maximum value [4]:

$$\epsilon^3 f_3(0) = \frac{1}{16} \eta \left(1 - \frac{\eta^2}{2} + \frac{\eta^3}{8} \right) \frac{\epsilon^2}{A^4} KL^4 + \frac{1}{16} \rho_m^3 \left(1 + \frac{10}{27} c_3 \right) \quad (35)$$

The maximum radius is

$$a_{\max} = A [1 + \epsilon f_1(0) + \epsilon^2 f_2(0) + \epsilon^3 f_3(0)] \quad (36)$$

adding up the results of Eqs. (34) – (36).

B. Matching Equation, Phase Advance, Transportable Current

For the FODO model, we obtain

$$K^2 \langle [h]^2 \rangle = 1/12 \eta^2 (3 - 2\eta) K^2 L^2 \quad (37)$$

which is used to calculate K_I^{eff} on the left side of the matching equation (30). On both sides of (30) there are correction terms involving (ρ_0^2) ; from Eq. (24) we find

$$\langle \rho_0^2 \rangle = \frac{1}{120} \eta^2 \left(\frac{5}{2} - \frac{5}{2} \eta^2 + \eta^3 \right) K^2 L^4 \quad (38)$$

for the hard-edge model. With these results, Eq. (30) can be solved for the transportable current Q or the matched beam radius A . We also use (38) in (31) and (32) to get the phase advances σ_0 and σ shown in Table 1.

4.3. Discussion of Table 1.

In Table 1, the lattice parameters, quadrupole voltage V_Q , beam current I , and normalized emittance ϵ_N are given quantities. First-order results for A , a_{\max} , σ , and σ_0 are calculated from Eqs. (30), (36), (31) and (32), respectively, along with (37) and (38). The lattice parameters shown at the top of Table 1 are the same as for the MFE prototype ESQ accelerator [6], except that here the occupancy h is taken to be 0.5.

The analytic results in Table 1 are compared with exact values obtained by numerical integration of Eqs. (1) and (2). For the A , a_{\max} and σ tables, the constant 20-kV focusing voltage V_Q produces a phase shift so of 83.37°; the beam parameters (I, ϵ) are adjusted to keep the beam radius roughly constant while σ varies widely. Table 1 also gives smooth approximation results for A , σ , and σ_0 and the lowest-order result for a_{\max} .

Third-order results from Ref. [4] are also shown, with accuracy usually within a few parts per thousand.

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