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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

GEODESIC SPACES AND HARMONIC MAPS

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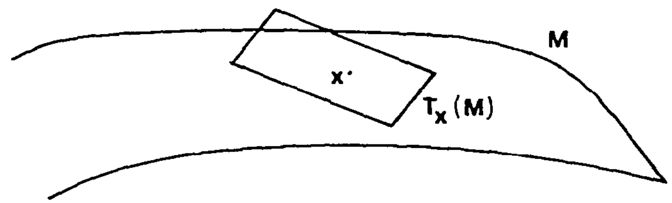
1 Smooth geometry

(1) The basic model and ambient space is a *real vector space* V (of finite dimension). That is an elementary domain for the differential and integral calculus.

(2) Such calculus is required on more general spaces – the natural one being a *smooth manifold M modeled on V*. Thus we are primarily interested in those properties of the calculus which are invariant under coordinate representation.

Examples: Directional derivatives of maps, integrals of volume forms. By way of contrast, *convexity* does not make sense on M.

(3) A key feature of a manifold is its tangent space $T_x(M)$ to M at $x \in M$:



That has a natural vector space structure ($\dim T_x(M) = \dim V$). And $T_x(M)$ varies smoothly as x moves smoothly on M.

2 Metric geometry

(1) *Euclidean structure on* V: A positive definite inner product $\langle\ ,\ \rangle$. With it we can define notions of *lengths* (of vectors and of paths in V) and *volumes.*

(2) We would like to define Euclidean space structure $\langle\ ,\ \rangle_x$ smoothly in every $T_x(M)$. If we have such a metric structure we say that M is a *Riemannian manifold.*

(3) With paths γ and functions f on a manifold

$$\mathbb{R} \xrightarrow{\gamma} M \xrightarrow{f} \mathbb{R}$$

we can define velocity vectors $\gamma'(t) \in T_{\gamma(t)}(M)$ and lengths of paths $L(\gamma) = \int_I |\gamma'(t)|dt$. And the differential $df(x)$ of f at $x \in M$. However, neither the parallel acceleration γ'' of γ nor the second differential $d^2f(x)$ makes sense, in general.

In case M is a Riemannian manifold we can indeed formulate such second order concepts. That is the content of the *fundamental theorem of Riemannian geometry* (= the existence of the canonical connexion of Levi-Civita).

(4) A characteristic feature of second order differentiation processes is that they do not generally commute. In fact, the *curvature* K_M of a Riemannian manifold measures the extent to which differentiation is not commutative.

(5) With that, we have a complete calculus - the *Riemannian tensor calculus* – on M. In particular, we have the tools required to calculate higher order derivatives. The tensor calculus has a metric – and very geometric - character. As such, it has been instrumental in mathematical physics (the mathematical framework of relativity and of quantum physics). And, of course, of great importance in differential topology.

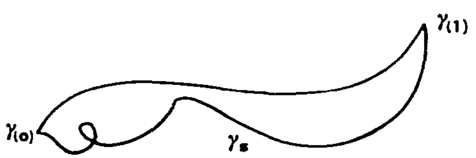
3 First applications of tensor calculus

(1) Geodesic paths. Given $\gamma : I \to M$, its *energy*

$$E(\gamma) = \frac{1}{2}\int_I |\gamma'(t)|^2 dt$$

where

$$|\gamma'(t)|^2 = \langle\gamma'(t),\ \gamma'(t)\rangle_{\gamma(t)}.$$



For a deformation (γ_s) of $\gamma = \gamma_0$ with fixed endpoints,

$$\left.\frac{dE(\gamma_s)}{ds}\right|_{s=0} = \frac{1}{2}\int \frac{d}{ds}\left.\langle\gamma_s'(t),\ \gamma_s'(t)(t)\rangle\right|_{s=0} dt$$
$$= -\int \langle \frac{D}{dt}\gamma'(t),\ \left.\frac{d\gamma(t)}{ds}\right|_0 \rangle dt .$$

Here $\frac{D\gamma'}{dt}$ denotes the acceleration of γ.

Say that γ *is a geodesic segment* (parametrized proportionally to arc length) if its acceleration $\equiv 0$.

(2) *Harmonic functions.* Given $f : M \to I\!R$, its *energy* (- Dirichlet integral)

$$E(f) = \frac{1}{2}\int_M |df(x)|^2 dx .$$

Here M is supposed compact; and dx denotes the volume element in $T_x(M)$.

Again, taking a deformation (f_s) of $f = f_0$,

$$\left.\frac{d}{ds}E(f_s)\right|_0 = \frac{1}{2}\int \frac{d}{ds}\langle df_s(x),\ df_s(x)\rangle\Big\|_{s=0} dx$$
$$= -\int \langle d^*df(x),\ df_s(x)\big|_0\rangle dx .$$

Here d^*df is the *Laplacian of f on M*. Say that f *is harmonic if* $d^*df \equiv 0$.

(3) In both these examples, we have arrived just at the point where curvature is about to appear – and is often very useful in drawing global geometrical conclusions.

Here is a common generalization:

Let $\varphi : M \to N$ be a map between Riemannian manifolds; and assume M compact. There its *energy*

$$E(\varphi) = \frac{1}{2}\int_M |d\varphi(x)|^2 dx .$$

Smooth critical points of E are called *harmonic maps*, [Eells, Sampson].

4 Geometry in geodesic spaces

(1) Although the tensor calculus provides a powerful format for geometric problems, (i) its infinitesimal aspects are very delicate; (ii) the curvature tensor is too hard for us to understand.

About 45 years ago H. Busemann and A.D. Alexandrov proposed/developed a synthetic geometry which (a) keep much of the first order theory; and (b) extracts those aspects of the curvature tensor K_M which have constant sign.

That theory has been greatly enriched by M. Gromov, and by Alexandrov's St. Petersburg school. And recently, many fine applications have been made.

Let us look at some of these now, [Nikolaev].

(2) A metric space $(Y > d)$ is a *geodesic space* if any two points y_0, y_1 can be joined by a continuous path $\gamma : I \to Y$ such that

$$d(y_0, y_1) = L(\gamma)$$
$$= \sup \left\{ \sum_{i=0}^{m-1} d(\gamma(t_i), \gamma(t_{i+1})) : a \leq t_0 < \ldots < t_m = b \right\} ,$$

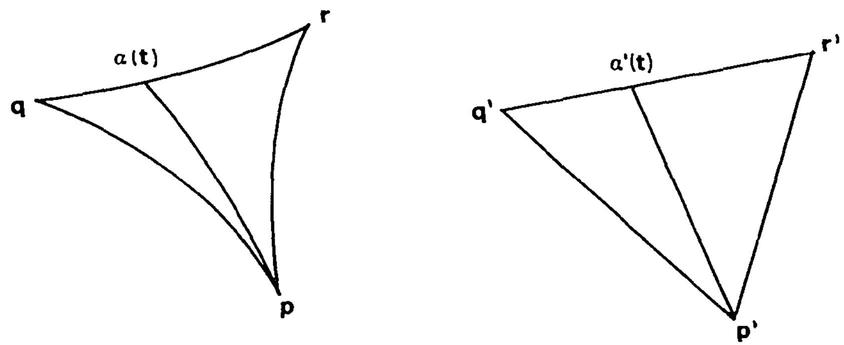
the supremum taken over all finite partitions of $I = [a, b]$. Say that γ is a *geodesic segment* if $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in I$.

(3) Take a real number $K \leq 0$. Let $\mathbb{R}^2(K)$ denote the 2-dimensional simply connected space form with curvature K.

Consider a triangle pqr in (Y, d) with geodesic sides and compare it with the corresponding triangle $p'q'r'$ in $\mathbb{R}^2(K)$:

$$d(p,q) = |p' - q'|, \quad d(p,r) = |p' - r'|, \quad d(q,r) = |q' - r'| ,$$

where $|\ \ |$ refers to the metric in $\mathbb{R}^2(K)$,



Say (Y, d) *has curvature* $\leq K$ if every $y \in Y$ has a neighbourhood V in which every geodesic triangle satisfies

$$d(\alpha(t), p) \leq |\alpha'(t) - p'|$$

for all $\alpha(t) \in qr$.

(4) **Example 1:** Any smooth Riemannian manifold M with curvature $K_M \leq K$ is a geodesic space with curvature $\leq K$.

Example 2: Any finite dimensional locally finite polyhedron can be metrized as a geodesic space with curvature ≤ 0.

5 Harmonic maps $\varphi : M \to Y$

(1) Roughly speaking, we can define a.e. the integrand $|d\varphi|^2 : M \to \mathbb{R}$ as the limit (as $\varepsilon \to 0$) of the spherical means

$$\int_{S(x,\varepsilon)} \frac{d^2(\varphi(x), \varphi(\eta))}{\varepsilon^2} \, d\eta / \mathrm{vol}\ S(x, \varepsilon)$$

Then we have the *energy*

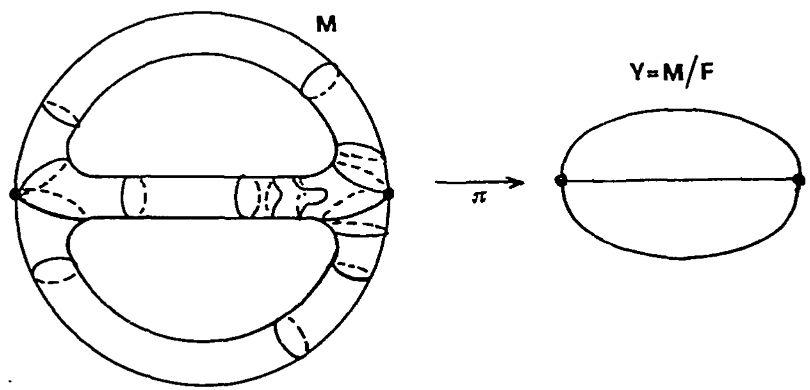
$$E(\varphi) = \frac{1}{2} \int_M |d\varphi(x)|^2 dx$$

as before. And the notion of *harmonicity of* φ.

(2) **Theorem:** *Suppose M, Y compact with curve $Y \leq 0$. Then any map $\varphi_0 : M \to Y$ can be deformed to an E-minimum φ-which is Lipschitz.*

([Eells, Sampson in case Y is a Riemannian manifold. [Gromov, Schoen]; [Korevaar, Schoen]; [Jost] in the present form.)

(3) **Application:** Let M be a closed Riemannian surface, and $\mathcal{F}$ a measured foliation of M with closed leaves:



The leaf space $M/\mathcal{F}$ is a 1-dimensional metric polyhedron Y with curve ≤ 0. The projection map $\pi : M \to Y$ can be deformed to a harmonic map φ. General principles assert that $(\varphi^* d)^{2,0}$ is a holomorphic quadratic differential whose real part determines a measured foliation equivalent to $\mathcal{F}$. (Thus [M. Wolf] proved the theorem of Hubbard–Masur.)

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