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ABSTRACT

We study electronic transport in a one-dimensional ordered chain in the preence of either absorption or amplification at each site (the site-potential having an imaginary positive or negative part) within a single-band tightbinding Hamiltonian. The spectrum in either case for the isolated (closed) quantum system is found to become broader compared to the regular Bloch case where there is no absorption or amplification at any site. Interestingly for the transport through an infinitely long ordered chain (open quantum system), the reflectance saturates to a value greater (lesser) than unity in the amplifying (absorbing) case and the transmittance decays to zero in either case. This fact implies that the transmittance does not grow indefinitely even for an ordered, amplifying (active or lasing) medium and that it is not necessary to have any disorder or interaction induced confining mechanism on the transmitted wave, so as to achieve an amplification in the backscattered wave.

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While the study of transport and localization of classical and fermionic waves in random disordered media has a long history and is reasonably well-understood [1], that in coherently absorbing or amplifying media has picked up only recently [2-5] after a relatively quiet period of about twenty years since the first such work [6]. As discussed in the ref.3 and 4, the study is interesting even for practical purposes particularly in the case of a classical (optical) wave going through a lasing medium where coherent amplification takes place. Pradhan and Kumar [3] concluded that the enhancement in the reflected intensity takes place because of "a synergy between wave confinement by Anderson localization and coherent amplification by the active medium". In order to be able to study if this synergy is really necessary for the amplification of the reflectance, we would first like to study in this paper the situation more systematically in the ordered case, i.e., where the Anderson localization cannot take place, relegating the study of the fully stochastic situation until later.

For the purpose of this Letter, we consider the standard single-band tight binding equation:

$$(E - \epsilon_n)c_n = V(c_{n-1} + c_{n+1}). \tag{1}$$

The system under consideration is a sample consisting of N lattice points (lattice constant unity). The quantum system is a *closed* one when it is isolated from the rest of the world. The *open quantum system*, on the other hand, is coupled to the external world (two reservoirs at a very slightly different electrochemical potentials) with two identical semi-infinite perfect leads on either side. Here, E is the fermionic energy, V is the constant nearest neighbor hopping term which is the same both in the leads and in the sample, ϵ_n is the site-energy, and c_n is the site amplitude at the nth site. Without any loss of generality, we choose $\epsilon_n = 0$ in the leads and $V = 1$ to set the energy scale. Inside the sample, we choose $\epsilon_n = i\eta$ where i is the square root of negative unity and the imaginary part η is a fixed real number which may be either positive or negative. It is clear that for an isolated scattering potential with a positive (negative) imaginary part, the wave-vector (k) has a positive (negative) imaginary part which means that the wave ($\sim e^{ikx}$) decays (grows) exponentially with x. Thus we call

a medium with all the scatterers having positive η an absorbing medium and a medium with all the scatterers having negative η an amplifying medium. The physical reason for such a description lies in the fact that the scattering in any real medium is never perfectly elastic and that in many cases the deviation from perfectly elastic scattering may be described by an absorption (or, even amplification) of incident particles or waves. Physical situations with these behavior for light or electronic waves have already been described in the literature[2-5].

First we look at the spectrum of the closed system (i.e., in the absence of any leads coupling it to the external world). The Hamiltonian (a tridiagonal matrix) is non-Hermitian with complex entries in the diagonal terms only and may or may not admit of real eigenvalues. Since the imaginary part here is fixed, one may simply shift the energy along the imaginary axis to reduce the problem to the regular tight binding case. As usual with a periodic boundary condition (for a finite chain length) this gives Bloch wave solutions with $c_n = c_0 e^{ikn}$ and a complex spectrum: $E = 2V\cos k + i\eta$ which is dense (band-like) for an infinite chain within the energy domain $[-2V + i\eta, +2V + i\eta]$. On the other hand one may look for solutions with real energy eigenvalues but complex wave-vectors. Clearly, in that case, the wave-function may be exponentially growing or decaying. Normalizability condition for infinite-sized closed systems would then rule out the growing solution in preference to the decaying solution. This vanishing boundary condition at the edges of the infinite sample and not the periodic (non-decaying) one discussed above seems to be relevant in realistic situations as demonstrated in the sequel. Let us consider a wave function which is peaked at the site n_0 and whose site-amplitudes far away from the peak have the form: $c_n \sim e^{\pm ik_1|n-n_0|}$ where $k_1 = k_s + i\gamma$ is a complex wave-vector. Then we find that

$$E = (e^{\gamma} + e^{-\gamma})\cos k_s, \tag{2}$$

and

$$\eta = (e^{\gamma} - e^{-\gamma})\sin k_s. \tag{3}$$

The two truncated tight binding equations at the two ends of the sample are not of the form of Eq.(1) because of the absence of one of the two neighbors and hence does not

support the solution given by Eqs.(2) and (3). But as the system size tends to infinity and the boundaries move infinitely far away from the peak of the decaying solution, these two truncated equations are trivially satisfied with vanishing amplitudes at the two ends and hence the solutions given above remain intact. It may be noted again that we say nothing about the form of the wave function near its peak, and claim that the general (asymptotic) form of the amplitudes

$$c_n \sim c_r e^{+ik_1|n-n_0|} + c_l e^{-ik_1|n-n_0|} \tag{4}$$

should suffice to obtain the spectrum. One can solve for the oscillatory (k_s) and the decay properties (γ) of the wave-function from the Eqs.(2) and (3). In particular when $\eta > 0$, one finds that $\gamma > 0$ in the region $0 < k_s < \pi$ since in this case $\sin k_s > 0$ in the Eq.(3) above. Thus one must have $c_l = 0$ in Eq.(4) above to ensure the normalizability for all the states in the spectral region $E_L < E < E_U$, where $E_U = (e^{\gamma} + e^{-\gamma})$, and $E_L = -E_U$. By the same argument, when $\eta > 0$, $\gamma < 0$ in the domain $-\pi < k_s < 0$ and hence one must choose $c_r = 0$ in Eq.(4). On the other hand when $\eta < 0$, $\gamma < 0$ for $0 < k_s < \pi$ so that $c_r = 0$ and $\gamma > 0$ for $-\pi < k_s < 0$ so that $c_l = 0$. Two interesting points to note regarding the effect of an imaginary part in the site potential (any $|\eta| > 0$) are that when the spectrum is real and dense (i) all the extended Bloch states transform into localized states in the normalizable cases, and (ii) comparing the spectral region to that in the regular Bloch case, the hopping term in the sample is seen to increase to an effective value $V_s(E) = (e^{\gamma} + e^{-\gamma})/2 > 1$, since $\gamma = \gamma(E) \neq 0$. Indeed near $k_s = 0$ or π, $\gamma \to \infty$ and this fact implies using Eq.(2) that *the spectrum gets infinitely broadened* compared to the Bloch case. More interestingly, as we will see below, this inequality between the sample and the lead hopping terms is at the heart of an oscillatory pattern (superposed on the decaying function) in the mesoscopic regime ($L < L_m \simeq 1/\gamma$ at zero temperature) in the transmittance (as well as reflectance) as reported[8] recently by us.

Next we consider the open quantum system. We study first the transmittance (or, the two-probe conductance, g_2, in units of e^2/h) and its evolution with length ($L = N - 1$) by

the numerical transfer matrix method[8]. It may be noted that the complex transfer matrix at each site is still unimodular (i.e., has a determinant of unity), and hence the transfer matrix for the whole system is still unimodular. So, this is the only property we have made sure to keep intact in our numerical procedure as the system evolves in length. Also since the electronic energy is real in the perfect leads it must also be real inside the sample. It may be noted here that the numerical transfer matrix method does not use any of the dispersion relations [8][i.e., neither the oscillatory Bloch-like case nor the combination of Eqs.(2) and (3)] obtained above for the isolated quantum system and hence there is no pre-determined preference to one or the other boundary condition.

Figure 1 shows both the reflectance and the transmittance as a function of length for the absorptive case with $E = 0.1$ and $\eta = 0.1$. The transmittance in this case decays exponentially (with very minute oscillations invisible in Fig.1) to zero. But more interestingly the reflectance (or, the two-probe resistance) evolves in a non-monotonic, oscillatory fashion and saturates to a value of about 6.27×10^{-4} even though there is no backscattering due to disorder (impurity). Consistent with these the absorption grows from zero to a saturation value of about 0.999373. In Fig.2, we choose $E = 0.1$, but $\eta = -0.1$ characterizing an *active medium* with coherent amplification. In this case, the interesting point to note for the open quantum system is that the transmittance first grows from quite small values to very large (compared to unity) values through large oscillations, and that eventually it decays to zero as L tends to infinity. Further, the reflectance also evolves non-monotonically with length through large oscillations. It reaches a peak value of 2.368×10^3 and finally saturates to a very large constant value of about 1.594×10^3 as $L \rightarrow \infty$. For a classical (light) wave, one would obtain expressions similar to Eqs.(2) and (3), and this would mean a very large coherent amplification in the reflected wave due to the lasing medium, but without any assistance (or, synergy) from any backscattering mechanism due to disorder, interaction etc (cf. as seen in Fig.2 for an electron wave).

To be able to ascertain which dispersion relation gives rise to this behavior, we follow a

method used in our recent work[7] and assume the solutions in different regions of the open system to be of the form:

$$c_n = \begin{cases} Ae^{ikn} + Be^{-ikn}, & -\infty < n \leq 0 \\ Ce^{ik_1 n} + De^{-ik_1 n}, & 1 \leq n \leq N \\ Fe^{ikn} + Ge^{-ikn}, & N+1 \leq n < \infty \end{cases} \tag{5}$$

where k is given by the dispersion relation $E = 2cosk$. The electrons continue with the same real energy inside the sample and for the wave vector k_1 inside the sample we choose to use the Eqs.(2) and (3) instead of the Bloch-like solution. Further to mimic an experimental setup, we take $B = rA$ (r=complex reflection amplitude), $F = tA$ (t=complex transmission amplitude), and $G = 0$. From the set of four tight binding equations at $n = 0, 1, N$ and $N + 1$, one gets the complex reflection amplitude r as

$$r = \frac{ae^{ik_1 L} - be^{-ik_1 L}}{-ce^{ik_1 L} + de^{-ik_1 L}}, \tag{6}$$

where the constant complex co-efficients a, b, c and d are given by

$$a = (e^{ik_1 - ik} - 1)(e^{ik_1 + ik} - 1), \tag{7}$$

$$b = (e^{-ik_1 - ik} - 1)(e^{-ik_1 + ik} - 1), \tag{8}$$

$$c = (e^{ik_1 - ik} - 1)^2, \tag{9}$$

and

$$d = (e^{-ik_1 - ik} - 1)^2. \tag{10}$$

The complex transmission amplitude may be similarly calculated from the above to obtain

$$t = \frac{(e^{ik_1} - e^{-ik_1})(e^{ik} - e^{-ik})e^{-ik(L+2)}}{de^{-ik_s L}e^{\gamma L} - ce^{ik_s L}e^{-\gamma L}}. \tag{11}$$

We make note of the fact that in contrast to the Eq.(4) where $\gamma < 0$ is not allowed, both positive and negative γ must be allowed in Eq.(5) and hence for the reflectance and the

transmittance since we donot have any normalizability constraint for the open quantum system. Using the above equations, we obtain the reflectance $R = |r|^2$ as an oscillatory function whose asymptotic value saturates for large L to some constant dependent on E and η. In the absorbing case ($\gamma > 0$), one obtains this constant value $R_\infty = |b|^2/|d|^2$ and in the amplifying case ($\gamma < 0$) $R_\infty = |a|^2/|c|^2$. The saturation values obtained from the above expressions match those obtained from the numerical transfer matrix method (for example, for the cases of Figs.1 and 2) exactly. It may be noted how a non-zero $|\eta|$ not only gives rise to absorption or amplification but also to backscattering or reflection (as found in previous numerical[3,4,6] works) even in the absence of disorder. Next one observes from the Eq.(11) that the transmittance $T = |t|^2$ does decay to zero at very large lengths whether γ is a positive or a negative real number. That the above analytic results are correct has been checked from the fact that the values of R and T as a function of L obtained from the Eq.(11) match those shown in Figs.1 and 2 exactly. Further, since the expressions for R and T are obtained using the form of the wave-amplitudes and the dispersion relations [Eqs.(2) and (3)] derived for the closed quantum system, the results for the closed system are also hereby verified to be correct. Naively thought $\gamma < 0$ for a forward-scattered wave should make it grow exponentially and thus this decaying transmittance for the open quantum system seems counter-intuitive. But indeed the middle line of Eq.(5) indicates that there is an interference between the incident and the reflected waves inside the sample. Thus the physical reason for the decay of T even in the absence of any disorder lies in the fact that the scatterer with the imaginary (in general, complex) site-potential plays the dual role of both an amplifier/ absorber and a backscatterer and that the latter role dominates the former from the beginning.

In conclusion then, we have looked at a periodic 1D chain with coherent absorbing or amplifying characteristic at each site to model either a *lossy* or an *active* (i.e., lasing in the context of light waves and a corresponding Helmholtz wave equation) medium. The spectrum gets infinitely broadened in either case. The reflectance in the form of backscattered wave

(even in the absence of any disorder) saturates to a value greater (lesser) than unity for an amplifying (absorbing) medium while the transmittance in the form of forward-scattered wave decays to zero even in both the cases. Thus the main interesting point is that even in the case of an amplifying medium, one does not need any disorder-induced backscattering to confine the forward-scattering part. Further work, specially in the presence of disorder, is under progress.

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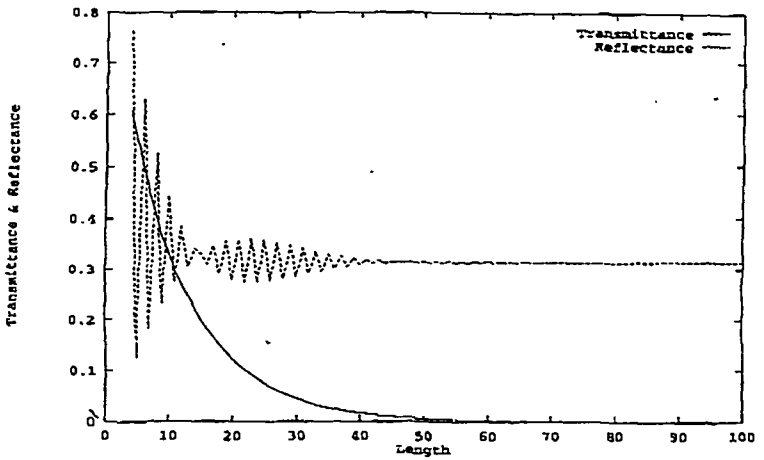
Figure Captions:

Fig.1 Reflectance (R) and transmittance (T) in units of e^2/h as a function of length L ($L = N-1$ where N is the number of sites in the sample). The Fermi energy of the electrons is $E = 0.1$, the hopping energy is $V = 1.0$ and the imaginary site-energy is $\eta = 0.1$. The absorption in this case is A such that $R + T + A = 1$. Further R is so small for all the lengths considered here that we have actually plotted $500R$ against L to make its structure visible. The transmittance T also has a similar oscillatory structure which is not visible to the eye at this scale of the graph.

Fig.2 The same as in Fig.1 for an amplifying medium with $\eta = -0.1$. In this case the amplification A provided by the medium is $A = R + T - 1$. The graph of R has been shifted upwards by 3000 units (but no multiplication as in Fig.1) to make it clearly visible. Both R and T evolve non-monotonically with L and their peak values are 2.754×10^3 and 2.368×10^3 respectively. But whereas T decays to zero, R flattens out to a constant value of about 1.594×10^3 as the length L becomes very large.


Transmittance
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Transmittance & Reflectance
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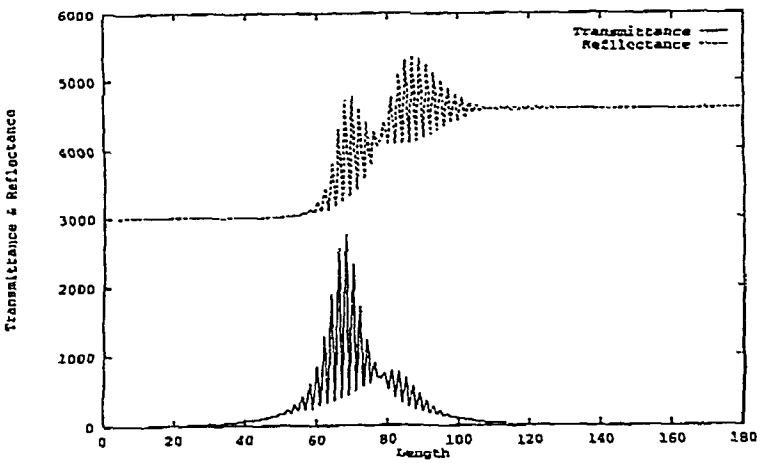
Fig.1


Transmittance
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Transmittance & Reflectance
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Fig.2