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Abstract. We study the quantum mechanical Liouville model with attractive potential which is obtained by Hamiltonian symmetry reduction from the system of a free particle on $SL(2, \mathbb{R})$. The classical reduced system consists of a pair of Liouville subsystems which are 'glued together' in such a way that the singularity of the Hamiltonian flow is regularized. It is shown that the quantum theory of this reduced system is labelled by an angle parameter $\theta \in [0, 2\pi)$ characterizing the self-adjoint extensions of the Hamiltonian and hence the energy spectrum. There exists a probability flow between the two Liouville subsystems, demonstrating that the two subsystems are also 'connected' quantum mechanically, even though all the wave functions in the Hilbert space vanish at the junction.

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1. Introduction

The Liouville model has been under intense scrutiny in recent years, due to its relation to two dimensional quantum gravity, which is important in the theory of the world sheet of string theories. The model, which is solvable classically, has proved to be a rich source for developing techniques as well as for probing the universal features of quantum gravity in higher dimensions. However, the problem is that despite many fruitful achievements the model still resists a full understanding as a quantum theory. It is therefore heartening to observe that its toy model version obtained by ignoring the space dimension governed by the Hamiltonian,

$$H(\pi, x) = \pi^2 + \mu e^{2x}, \quad (1.1)$$

can be solved completely for $\mu > 0$ even quantum mechanically. The quantum mechanical Liouville model possesses a continuous energy spectrum and its eigenstates are given by modified Bessel functions [1]. A somewhat peculiar aspect of the quantum theory is that it has no vacuum state even though the energy is bounded from below, an aspect that stems from the repulsive exponential potential, which has no minimum. Now one might ask what happens if we replace the potential 'wall' with a 'well' by putting $\mu < 0$. This attractive exponential potential will undoubtedly give rise to difficulty in quantization because the energy would then be unbounded from below without any bound states, indicating the quantum instability of the system. This is a reflection of the classical instability whereby the particle sinks indefinitely fast into the well, as the classical solution develops a singularity and blows up at some finite time if $\mu < 0$. This is perhaps the reason why the quantum theory of the Liouville model with attractive potential has not been considered seriously so far.

Meanwhile, we have found in the study of \mathcal{W} -algebras and the generalized KdV systems that the Toda field theory, obtained by Hamiltonian symmetry reduction [2] from the Wess-Zumino-Novikov-Witten model based on a Lie group G , has a certain global structure [3]. More precisely, the reduced theory is not merely a Toda theory but consists of a multiple of Toda theories as subsystems having both repulsive and attractive potentials in general. In particular, for $G = SL(2, \mathbb{R})$ we have a pair of Liouville models as subsystems in the reduced system, both of which have an either repulsive or attractive potential depending on the reduction performed. The interesting observation made there [3, 4] (see also [5]) was that, in the toy model version where the Toda field theories become the Toda lattices, the singular classical solutions that arise in the Toda lattices are regularized automatically by the Hamiltonian reduction. An intuitive picture of the regularization may

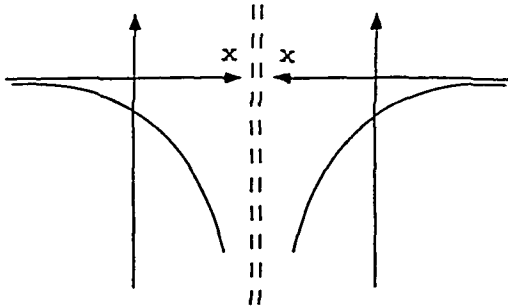


Figure 1. A schematic picture of the potential in the regularized Liouville system.

be gained by considering the simple case $G = SL(2, \mathbb{R})$ where we get two Liouville (toy) models which are ‘glued together’ by identifying the limits $x \rightarrow \infty$ of the two models (see Fig. 1). The singular solution in one Liouville model is regularized by continuing it in time to the solution in the other Liouville model, causing the particle to oscillate between the two subsystems. This observation motivated us to investigate more closely the Liouville model with attractive potential, now regularized in the above sense, to see if any sensible quantization is possible.

A first step in this direction was made in a paper by Fülöp [6], where, like the former repulsive potential case, the theory is solved completely at the quantum level yielding Bessel functions as energy eigenstates. The salient result of [6] is that the spectrum is discrete, which is perhaps a natural consequence of the regularized classical solutions being oscillations, and that there are *inequivalent quantizations* characterized by certain parameters specifying the self-adjoint quantum Hamiltonian and hence the spectrum. However, the argument in [6] appears to be unnecessarily complicated at a few crucial points, especially when the self-adjoint Hamiltonians are constructed over the entire reduced system. The aim of this paper is to present a quantization approach which is much simpler and more direct in these points, and to furnish a complete version of the (regularized) quantum mechanical Liouville model with attractive potential. We shall find that, as in [6], there arise inequivalent quantizations but they can be characterized just by an angle parameter $\theta \in [0, 2\pi)$, and that the discrete energy spectrum obtained turns out to be different from that of [6]. We also see more naturally a probability flow between the two Liouville subsystems, a fact demonstrating that these subsystems are also connected quantum mechanically.

The plan of the paper is as follows: To make the paper self-contained, in section 2 we provide a necessary background for the classical reduced system. Then in section 3 we present a quantum theory of the reduced system, which is a combined system of two Liouville models with attractive potential. The final section is devoted to discussion. We provide two appendices; Appendix A for a brief review of the general theory of self-adjoint extensions of symmetric operators, and Appendix B for a collection of formulae involving Bessel functions used in the text.

2. Hamiltonian reduction and the global structure

In order to set the scene for the system for which we discuss the quantization, we here recall the Hamiltonian symmetry reduction [2] which leads to the system of regularized Liouville models, together with the global description of the system developed recently [3].

The reduction is the special case $n = 2$ of the Hamiltonian reduction that yields a multiple (2^{n-1}) of open, finite Toda lattices from the free particle system on the group $G \simeq SL(n, \mathbb{R})$. In the reduced system these Toda lattices, which have in general both repulsive and attractive potentials, are ‘glued together’ in such a way that no singularity arises. In this sense the reduction provides a natural means to regularize the singularities which exist in those Toda lattices that have attractive potentials. (This regularization is an example of a more general idea put forwarded originally in [5].)

2.1. Classical Hamiltonian reduction

The free particle on a semisimple Lie group G is described by the Hamiltonian system (M, Ω, H) in the following way. The phase space M is the cotangent bundle of the group,

$$M = T^*G \simeq G \times \mathcal{G} = \{(g, J) \mid g \in G, J \in \mathcal{G}\}, \quad (2.1)$$

where \mathcal{G} is the Lie algebra of G (in our case $\mathcal{G} = \mathfrak{sl}(2, \mathbb{R})$), which is identified with its dual \mathcal{G}^* using the scalar product. The fundamental Poisson brackets are

$$\{g_{ij}, g_{kl}\} = 0, \quad \{g_{ij}, \text{tr}(T^a J)\} = (T^a g)_{ij}, \quad \{\text{tr}(T^a J), \text{tr}(T^b J)\} = \text{tr}([T^a, T^b]J), \quad (2.2)$$

where $\{T^a\}$ is a basis of \mathcal{G} . For $G = \mathfrak{sl}(2, \mathbb{R})$ we may take

$$T^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (2.3)$$

The Poisson brackets (2.2) derive from the symplectic form,

$$\Omega = d \operatorname{tr} (J dg g^{-1}). \quad (2.4)$$

The Hamiltonian is given by

$$H = \frac{1}{2} \operatorname{tr} (J^2), \quad (2.5)$$

which leads to the dynamics

$$\frac{dg}{dt} = \{g, H\} = Jg, \quad \frac{dJ}{dt} = \{J, H\} = 0, \quad (2.6)$$

and hence yields the geodesic equation $\frac{d}{dt}(\frac{dg}{dt} g^{-1}) = 0$ on the group manifold G . We note that J is the infinitesimal generator for the action of G on M defined by left translations, while the action of G defined by right translations is generated by $\tilde{J} : M \rightarrow \mathcal{G}$ where

$$\tilde{J}(g, J) := -g^{-1} Jg. \quad (2.7)$$

Let us now decompose the Lie algebra $\mathcal{G} = \mathfrak{sl}(2, \mathbb{R})$ into the subalgebras of strictly upper triangular, diagonal, and strictly lower triangular traceless matrices, that is, those subalgebras spanned by the basis (2.3),

$$\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_0 + \mathcal{G}_-. \quad (2.8)$$

Then we consider symmetry reduction based on the subgroup $N := N_+ \times N_- \subset G$, where $N_{\pm} = \exp(\mathcal{G}_{\pm})$, which acts on the phase space M according to

$$(n_+, n_-) : (g, J) \mapsto (n_+ g n_-^{-1}, n_+ J n_+^{-1}), \quad \forall (n_+, n_-) \in N, \quad (g, J) \in M. \quad (2.9)$$

The symmetry reduction is performed by decomposing the generators, $J = J_+ + J_0 + J_-$ and $\tilde{J} = \tilde{J}_+ + \tilde{J}_0 + \tilde{J}_-$, according to (2.8) and then fixing the value of the momentum map $\Phi(g, J) := (J_-, \tilde{J}_+)$ as

$$\Phi(g, J) = (I_-, -I_+), \quad (2.10)$$

where $I_- := \nu^- T_-$ and $I_+ := \nu^+ T_+$ are nonvanishing constant matrices ($\nu^{\pm} \neq 0$) belonging to \mathcal{G}_- and \mathcal{G}_+ , respectively. The reduced phase space is obtained as the factor space

$$M^{\text{red}}(I_-, I_+) = M^c(I_-, I_+)/N, \quad \text{where } M^c(I_-, I_+) := \Phi^{-1}(I_-, -I_+). \quad (2.11)$$

In Dirac's terminology, this Hamiltonian reduction amounts to imposing the first class constraints, $J_- = I_-$ and $\tilde{J}_+ = -I_+$, defining $M^c \subset M$, and getting the reduced phase space

by fixing the gauge associated with the symmetry group N generated by the constraints; hence (2.11).

Now the Bruhat (Gelfand-Naimark) decomposition for semisimple Lie groups [7] allows us to write $G = SL(2, \mathbb{R})$ as

$$G = G_e \cup G_{-e} \cup G_{\text{low}} \quad (\text{disjoint union}), \quad (2.12)$$

where

$$G_{\pm e} := \pm N_+ A N_- \quad \text{with } A := \exp(\mathcal{G}_0). \quad (2.13)$$

The two 'cells', G_e and G_{-e} , are open submanifolds in G containing e and $-e$ ($e \in G$ is the identity element), respectively (and their union is dense in G), while G_{low} is the union of 'borders', i.e., certain lower dimensional submanifolds of G . Correspondingly, any element $g \in G_e \cup G_{-e} \subset SL(2, \mathbb{R})$ admits the unique decomposition in the form,

$$g = \pm n_+ e^q n_- = \pm \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \quad (2.14)$$

where we put $q = xT_0$ and $a, c, x \in \mathbb{R}$. The two cells $G_{\pm e}$ are in fact the open submanifolds of determinant one matrices with $\operatorname{sign}(g_{22}) = \pm$, whereas G_{low} consists of those matrices with $g_{22} = 0$.

The Bruhat decomposition naturally induces the decomposition of the phase space $M = T^*G$ as $M = M_e \cup M_{-e} \cup M_{\text{low}}$. Since this decomposition is invariant under the action of the symmetry group N , we have the corresponding decomposition of $M^c = M_e^c \cup M_{-e}^c \cup M_{\text{low}}^c$ as well, which in turn induces the decomposition of the reduced phase space,

$$M^{\text{red}} = M_e^{\text{red}} \cup M_{-e}^{\text{red}} \cup M_{\text{low}}^{\text{red}} \quad (\text{disjoint union}). \quad (2.15)$$

We note that $M_{\pm e}^{\text{red}}$ are open submanifolds in M^{red} and $M_{\text{low}}^{\text{red}}$ is a union of lower dimensional submanifolds. In other words, the Bruhat decomposition introduces the cell-structure in the reduced phase space. We now show that each of the subsystems associated with the two cells $M_{\pm e}^{\text{red}}$ is a Liouville model. Indeed, since the submanifolds $M_{\pm e}^c$ are

$$M_{\pm e}^c = \left\{ (g, J) \mid g = \pm n_+ e^q n_-, \quad n_{\pm} \in N_{\pm}, \quad q \in \mathcal{G}_0, \quad J_- = I_-, \quad (g^{-1} Jg)_+ = I_+ \right\}, \quad (2.16)$$

we see that $M_{\pm e}^{\text{red}} = M_{\pm e}^c / (N_+ \times N_-)$ are given by the local gauge section,

$$M_{\pm e}^{\text{red}} = \left\{ (\pm e^q, J) \mid q \in \mathcal{G}_0, \quad J_- = I_-, \quad (e^{-q} J e^q)_+ = I_+ \right\}. \quad (2.17)$$

The condition in (2.17) can easily be solved for J :

$$J = J_{\pm e}(q, p) := I_- + p + e^q I_+ e^{-q}, \quad (2.18)$$

where $q, p \in \mathcal{G}_0$. Thus we may write

$$M_{\pm e}^{\text{red}} = \{ (\pm e^q, J_{\pm e}(q, p)) \mid (q, p) \in \mathcal{G}_0 \times \mathcal{G}_0 \}, \quad (2.19)$$

or simply $M_{\pm e}^{\text{red}} \simeq \mathcal{G}_0 \times \mathcal{G}_0 = \mathbb{R}^2$. By evaluating the symplectic form (2.4) on the reduced phase space (2.19), we find

$$\Omega_{\pm e}^{\text{red}} = d \operatorname{tr} (J_{\pm e} d(e^q) e^{-q}) = d \operatorname{tr} (pdq) = 2d\pi dx, \quad (2.20)$$

where we put $p = \pi T_0$ with $\pi \in \mathbb{R}$. Similarly, from (2.5) the reduced Hamiltonian turns out to be

$$H_{\pm e}^{\text{red}}(J_{\pm e}) = \frac{1}{2} \operatorname{tr} (J_{\pm e}^2) = \frac{1}{2} \operatorname{tr} (p^2) + \mu \operatorname{tr} (T_- e^q T_+ e^{-q}) = \pi^2 + \mu e^{2x}, \quad (2.21)$$

where $\mu = \nu^+ \nu^-$. Since the triples $(M_{\pm e}^{\text{red}}, \Omega_{\pm e}^{\text{red}}, H_{\pm e}^{\text{red}})$ are no other than that of the Liouville model, we conclude that the reduced system obtained by the Hamiltonian reduction possesses two, identical Liouville models as subsystems.

2.2. Global structure of the reduced system

The Hamiltonian flow on the manifold $M_{\pm e}^{\text{red}}$ is governed by the reduced Hamiltonian (2.21), which yields the equation of motion,

$$\frac{d^2 x}{dt^2} + \mu e^{2x} = 0. \quad (2.22)$$

The flow is incomplete (singular) if $\mu < 0$, as is intuitively clear from the rapidly decreasing potential well in which the particle sinks. For instance, the solution, $x(t) = -\ln(\cos t)$ which satisfies the initial condition $x(0) = 0$ and $\frac{dx}{dt}(0) = 0$ and corresponds to $\mu = -1$, blows up at $t = \frac{\pi}{2}$. But since there is no singularity in the full reduced system, the incompleteness of the Hamiltonian flow that arises in the reduced system when $\mu < 0$ is just a manifestation of the fact that the particle may leave the submanifold $M_{\pm e}^{\text{red}} \subset M^{\text{red}}$ at finite time. More concretely, the trajectory of the free particle on G determined by an initial value $(\pm e^q, J_{\pm e})$ at $t = 0$ as $g(t) = \pm e^{tJ_{\pm e}} e^q$ may leave the open submanifold $G_{\pm e}$, because the flow of the reduced system is obtained by projecting the original flow on $M^c \subset M$ to M^{red} . Thus the singularity occurs when g_{22} vanishes, which corresponds

to q (or x) reaching infinity. In this respect, one can say that the embedding of $M_{\pm e}^{\text{red}}$ into M^{red} provides a regularization of the Liouville model with $\mu < 0$ where the singular (blowing up) trajectories are glued together smoothly at x infinity. If, on the other hand, $\mu > 0$, then the Hamiltonian flow is complete, and the two Liouville models are completely disconnected from each other. Hereafter we confine ourselves to the case $\mu < 0$ where the two Liouville models are connected, and put $\mu = -1$ for simplicity since μ can be freely rescaled by shifting x in (2.21). We also choose $\nu^+ = -\nu^- = 1$ for definiteness.

So far, we have analyzed the structure of the reduced system only *locally*, using the gauge fixing to identify the reduced system $(M^{\text{red}}, \Omega^{\text{red}}, H^{\text{red}})$ as one containing a pair of Liouville models glued together along lower dimensional submanifolds. To furnish a tool to gain information on the *global* structure, we wish to have a global cross section (gauge fixing) of the gauge orbits in M^c . Such a cross section is furnished by the *Drinfeld-Sokolov gauge*, which is used in the context of generalized KdV equations [8] and \mathcal{W} -algebras (see, e.g., [2]). In our context we need to use it doubly for J and \bar{J} , and for this reason we call our global gauge fixing ‘double DS gauge’ here.

A double DS gauge is defined by requiring that the two generators J and \bar{J} of the symmetry be of the dual form,

$$J(u) = \begin{pmatrix} 0 & u_2 \\ 1 & 0 \end{pmatrix}, \quad \bar{J}(u) = \begin{pmatrix} 0 & 1 \\ u_2 & 0 \end{pmatrix}. \quad (2.23)$$

Note that by definition J and \bar{J} are not quite independent (see (2.7)). It is readily seen that the condition for \bar{J} in (2.23) is fulfilled if J is of the form in (2.23) and

$$g(u) = \begin{pmatrix} u_2 u_4 & -u_3 \\ u_3 & -u_4 \end{pmatrix}. \quad (2.24)$$

These parameters $(u_2, u_3, u_4) \in \mathbb{R}^3$ are subject to the condition,

$$F(u) := \det g(u) = u_3^2 - u_2 u_4^2 = 1, \quad (2.25)$$

which defines the hypersurface \mathcal{S} in \mathbb{R}^3 as a model of M^{red} . It is straightforward to check that $dF(u)|_{F(u)=1} \neq 0$, which implies that (2.25) gives a regular hypersurface diffeomorphic to M^{red} . Regarding the u_i as gauge invariant functions on the constrained manifold M^c , we find the Poisson brackets

$$\{u_4, u_3\} = \frac{u_4^2}{2}, \quad \{u_4, u_2\} = u_3, \quad \{u_3, u_2\} = u_4 u_2. \quad (2.26)$$

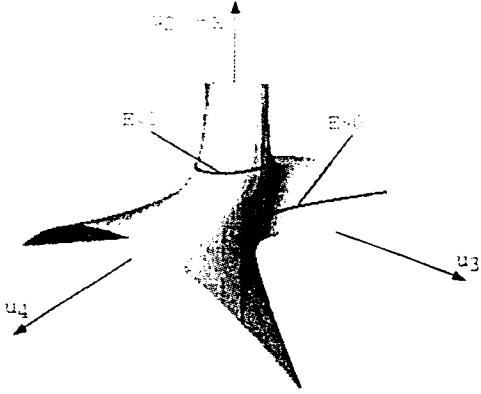


Figure 2. Two typical trajectories on the reduced phase space. The ellipse ($E < 0$) corresponds to a periodic motion whereas the hyperbola ($E > 0$) corresponds to a motion passing once for all from one cell to another.

The dynamics of the reduced system is determined by the Hamiltonian

$$H^{\text{red}}(u) = \frac{1}{2} \text{tr} J^2(u) = u_2. \quad (2.27)$$

The relationship to the local description of the reduced system given earlier is established by noting that, since $g_{22}(u) = -u_4$, the cells M_e^{red} and M_{-e}^{red} are represented by the domains $u_4 < 0$ and $u_4 > 0$, respectively.

Having obtained a global picture of the reduced system, we now see how the singularity of the Liouville model gets regularized when the two models are glued together. Consider the classical solution with a constant energy E , whose trajectory is the curve obtained by intersecting the hypersurface S in (2.25) with $u_2 = E$. The curve is a hyperbola or ellipse depending on the sign of the energy E (see Fig. 2),

$$u_3^2 - E u_4^2 = 1. \quad (2.28)$$

It is now clear that, for $E < 0$, the motion of the particle is periodic, which implies that the particle does pass the border $u_4 = 0$ along the curve, and when it does so it gives rise to a singularity in the solution when viewed as a local subsystem, although the solution is perfectly regular when viewed as a global system. It is worth noting that the hypersurface

S is not simply connected. In fact, the loop given by, say, the ellipse in (2.28) for some $E < 0$ cannot be contracted to a point on the surface S .

3. Quantization of the reduced system

We have learned that the classical reduced system consists of two Liouville subsystems glued together between which the point particle oscillates if the energy is negative. In this section we wish to define a quantum mechanics of the reduced system and examine if this global feature appears at the quantum level as well. In quantum mechanics observables are represented by self-adjoint operators. Here we consider the self-adjoint Hamiltonian of our reduced system, which is perhaps the most fundamental observable, as a crucial ingredient to set up the quantum theory. The standard procedure [9] for finding self-adjoint Hamiltonians is to choose a suitable domain where the quantum Hamiltonian, now takes some operator form, becomes a *symmetric* operator, and then find an extended domain where it becomes a *self-adjoint* operator. (For a brief review of the general procedure, see Appendix A.) After this procedure, we shall find that there exists a probability flow between the two subsystems even though all the wave functions vanish at the junction of the two Liouville subsystems, a fact that illustrates that the two subsystems are also connected at the quantum level.

3.1. Hamiltonian as a symmetric operator

The basic problem for quantizing our reduced system is that the reduced phase space M^{red} is not quite a cotangent bundle of some configuration space, but a system of two cotangent bundles nontrivially combined. To take this feature into account, we wish to formulate the quantum theory dealing with the two subsystems simultaneously, in such a way that the classical connectedness of the two subsystems will also be realized quantum mechanically.

In section 2 we have seen that there exists a global description of the reduced system using the hypersurface S as a model for the reduced phase space. Among the three parameters used we choose the variable

$$Q := g_{22}(u) = -u_4 \quad (3.1)$$

and regard it as ‘coordinate’ of the particle. The variable is chosen on the ground that the component g_{22} is gauge invariant under the symmetry action (2.9) and hence from (2.14) we have the direct relation $Q = \pm e^{-x}$ with the local Liouville coordinate x . The variable Q

is convenient for our purpose since $Q > 0$ corresponds to the subsystem $(M_e^{\text{red}}, \Omega_e^{\text{red}}, H_e^{\text{red}})$ and $Q < 0$ to $(M_{-e}^{\text{red}}, \Omega_{-e}^{\text{red}}, H_{-e}^{\text{red}})$, respectively. The canonical momentum conjugate to Q is then given by $P := -2u_3/u_4^2$ which satisfies

$$\{Q, P\} = 1, \quad (3.2)$$

under the reduced Poisson brackets (2.26). In terms of these variables the classical reduced Hamiltonian (2.27) reads

$$H = \frac{1}{4}Q^2 P^2 - \frac{1}{Q^2}. \quad (3.3)$$

Upon the identification $P = 2e^x\pi$ the Hamiltonian (3.3) agrees with the local expression (2.21). Now the trouble is that the Hamiltonian is ill-defined at $Q = 0$, i.e., at the junction between the two subsystems. We exclude this point from the domain where Q is defined: $Q \in \mathbb{R}^* := \mathbb{R} \setminus \{0\} = \mathbb{R}^+ \cup \mathbb{R}^-$. As we shall see soon, despite this exclusion and the apparent trivialization of the reduced system into two decoupled subsystems, it is possible to construct a quantum theory such that the two Liouville subsystems are connected nonetheless.

In quantization we elevate these canonical variables to linear operators on a Hilbert space with the Poisson bracket (3.2) replaced by the commutation relation,

$$[\hat{Q}, \hat{P}] = i. \quad (3.4)$$

Working with the coordinate representation, we define the Hilbert space by the space of square integrable functions,

$$\mathcal{H} := \{\phi \mid \|\phi\| < \infty\}, \quad (3.5)$$

where $\|\phi\| = \sqrt{\langle \phi, \phi \rangle}$ is the norm of the state ϕ . We furnish the innerproduct by

$$\langle \phi, \psi \rangle = \langle \phi, \psi \rangle_- + \langle \phi, \psi \rangle_+, \quad (3.6)$$

where

$$\langle \phi, \psi \rangle_- := \int_{-\infty}^{-0} \frac{dQ}{|Q|} \phi(Q)^* \psi(Q), \quad \langle \phi, \psi \rangle_+ := \int_{+0}^{+\infty} \frac{dQ}{|Q|} \phi(Q)^* \psi(Q). \quad (3.7)$$

The measure $dQ/|Q|$ used in the innerproduct (3.7) can be derived, for instance, by the path-integral reduction, where the original measure $\prod_t \Omega^3(t)$ for the free particle system on $SL(2, \mathbb{R})$ reduces to the form $\prod_t dQ/|Q|(t)$ after we integrate out the momentum variables

J taking into account the constraints and the gauge fixing conditions [10]. A more direct way to see this is to consider the phase space path-integral for the reduced system with the Hamiltonian (3.3) and then integrate on P to get the configuration space path-integral, which results precisely in the measure $dQ/|Q|$. Note that the measure $dQ/|Q|$ is just the standard Toda measure dx in terms of the local coordinate x , as expected. Note also that, because of the measure, all the wave functions $\phi(Q) \in \mathcal{H}$ must vanish at the junction of the two subsystems: $\phi(Q) \rightarrow 0$ as $Q \rightarrow \pm 0$.

In order to find a self-adjoint Hamiltonian operator, let us consider the differential operator \hat{H} of the form,

$$\hat{H} := -\frac{1}{4}Q \frac{d}{dQ} Q \frac{d}{dQ} - \frac{1}{Q^2}, \quad (3.8)$$

which is a naive choice for the operator which corresponds to the classical Hamiltonian (3.3). We can find a domain where the Hamiltonian operator is symmetric,

$$D(\hat{H}) := \{\psi \mid \psi \in \mathcal{H}, \hat{H}\psi \in \mathcal{H}, \lim_{Q \rightarrow \pm 0, \pm \infty} Q\psi(Q) = Q \frac{d}{dQ}\psi(Q) = 0\}. \quad (3.9)$$

In fact, it can be readily confirmed that on $D(\hat{H})$ we have

$$\langle \hat{H}\psi_1, \psi_2 \rangle = \langle \psi_1, \hat{H}\psi_2 \rangle \quad \forall \psi_1, \psi_2 \in D(\hat{H}). \quad (3.10)$$

It is also easy to see that the domain of the adjoint operator \hat{H}^* , which as a differential operator takes the same form as \hat{H} , is just $D(\hat{H}^*) := \{\psi \mid \psi \in \mathcal{H}, \hat{H}\psi \in \mathcal{H}\}$. This shows that the symmetric operator \hat{H} is not self-adjoint; $D(\hat{H}^*) \supset D(\hat{H})$.

Before discussing the self-adjoint extensions of the symmetric operator, let us consider the eigenvalue problem of the differential operator \hat{H} ,

$$\hat{H}\phi = E\phi. \quad (3.11)$$

Using the variable

$$z = \frac{2}{Q}, \quad (3.12)$$

we find that eq.(3.11) becomes

$$\left[z^2 \frac{d^2}{dz^2} + z \frac{d}{dz} + (z^2 - k^2) \right] \phi = 0, \quad (3.13)$$

where

$$k^2 = -4E. \quad (3.14)$$

The linearly independent eigenstates of eq.(3.11) are given by the Bessel functions $J_k(z)$ and $Y_k(z)$ with indices $k \in \mathbb{C}$. However, only those Bessel functions of the type $J_k(z)$ with $\text{Re } k > 0$ have finite norms and belong to the Hilbert space \mathcal{H} . In particular, for $E < 0$ there exists a unique k for which the corresponding eigenstate belongs to \mathcal{H} while there is no such eigenstate belonging to \mathcal{H} for $E \geq 0$, and for this reason we henceforth restrict ourselves to the negative energy states. Note however that *none* of these negative energy eigenstates belongs to the domain $D(\hat{H})$ where the Hamiltonian is symmetric, as can be seen from their asymptotic forms (see Appendix B),

$$\begin{aligned} QJ_k\left(\frac{2}{Q}\right) &\underset{Q \rightarrow 0}{\sim} \frac{1}{\sqrt{\pi}} Q^{\frac{3}{2}} \cos\left(\frac{2}{Q} - \frac{\pi}{2}\left(k + \frac{1}{2}\right)\right), \\ Q\frac{d}{dQ}J_k\left(\frac{2}{Q}\right) &\underset{Q \rightarrow 0}{\sim} -\frac{1}{\sqrt{\pi}} Q^{-\frac{1}{2}} \sin\left(\frac{2}{Q} - \frac{\pi}{2}\left(k + \frac{1}{2}\right)\right). \end{aligned} \quad (3.15)$$

Clearly, as Q tends to zero the first term $QJ_k(2/Q)$ goes to zero while the second term $Q\frac{d}{dQ}J_k(2/Q)$ blows up to infinity, showing that $J_k(z) \notin D(\hat{H})$ for $\text{Re } k > 0$.

3.2. Self-adjoint extensions

In order to find an extended domain where the symmetric operator \hat{H} becomes a self-adjoint operator \hat{H}_* for which $D(\hat{H}_*) = D(\hat{H}_*^*)$, we first examine, according to the general theory (see Appendix A), the deficiency indices (d_+, d_-) of the symmetric operator \hat{H} . The index d_+ (d_-) is given by the dimension of the eigenspace of the adjoint operator \hat{H}^* with eigenvalue i ($-i$),

$$d_+ := \dim \text{Ker}(\hat{H}^* - i), \quad d_- := \dim \text{Ker}(\hat{H}^* + i). \quad (3.16)$$

We recall that the eigenfunctions of the differential operator \hat{H} in (3.8) with eigenvalue i ($-i$) are given by the Bessel functions $J_{\pm k_0}$ ($J_{\pm k_0^*}$) with index $k_0 := 2 \exp(-\frac{\pi}{4}i)$. But since for these eigenfunctions to be in the Hilbert space \mathcal{H} the real part of the indices $\pm k_0$ ($\pm k_0^*$) must be positive, the only eigenstate allowed is J_{k_0} ($J_{k_0^*}$), which actually belongs to the domain of the adjoint operator $D(\hat{H}^*)$. We therefore observe that $(d_+, d_-) = (1, 1)$, a result ensuring that \hat{H} admits self-adjoint extensions.

The general theory of self-adjoint extension then asserts that the extended domain where the symmetric operator becomes self-adjoint consists of three spaces, i.e., the domain of the symmetric operator $D(\hat{H})$, the eigenspace of the adjoint operator \hat{H}^* with eigenvalue

i , and the eigenspace of \hat{H}^* with eigenvalue $-i$, with the latter two spaces being related unitarily. Concretely, it is given by

$$D(\hat{H}_*) = \{\psi \mid \psi = \phi + \alpha \xi_{k_0}, \phi \in D(\hat{H}), \alpha \in \mathbb{C}\}, \quad (3.17)$$

where

$$\xi_{k_0} := \frac{J_{k_0}}{\|J_{k_0}\|} + e^{i\theta} \frac{J_{k_0^*}}{\|J_{k_0^*}\|}. \quad (3.18)$$

The point to be noted is that the domain is parametrized by the angle,

$$\theta \in [0, 2\pi). \quad (3.19)$$

The angle parameter θ therefore characterizes the quantum theory we construct, whose existence is just a reflection of the ambiguity in quantizing a classical system.

Having achieved the self-adjoint extensions of the Hamiltonian, we now investigate which eigenstates J_k of the adjoint operator belong to $D(\hat{H}_*)$ in (3.17) for a specific θ . A necessary and sufficient condition for $J_k \in D(\hat{H}_*)$ is [11]

$$\langle \hat{H}^* J_k, \psi \rangle = \langle J_k, \hat{H}_* \psi \rangle \quad \forall \psi \in D(\hat{H}_*). \quad (3.20)$$

which takes the form

$$Q W(J_k^*, \psi) \Big|_{-\infty}^{-0} - Q W(J_k^*, \psi) \Big|_{+0}^{+\infty} = 0, \quad (3.21)$$

where $W(\phi_1, \phi_2) = \frac{d\phi_1}{dQ} \phi_2 - \phi_1 \frac{d\phi_2}{dQ}$ is the Wronskian. An equivalent condition is obtained if we replace ψ with ξ_{k_0} in (3.21), and by using the asymptotic forms (3.15) we arrive at the relation

$$e^{2\pi i(k - \text{Re } k_0)} = \frac{\cos \frac{\theta}{2} \cosh(\pi \text{Im } k_0) - i \sin \frac{\theta}{2} \sinh(\pi \text{Im } k_0)}{\cos \frac{\theta}{2} \cosh(\pi \text{Im } k_0) + i \sin \frac{\theta}{2} \sinh(\pi \text{Im } k_0)}. \quad (3.22)$$

Solving this relation in favour of k , we get

$$k = k(n, \theta) := \text{Re } k_0 - \frac{1}{\pi} \tan^{-1} \left[\tan \frac{\theta}{2} \tanh(\pi \text{Im } k_0) \right] + n, \quad (3.23)$$

where n are integers for which $k > 0$. We therefore see that the eigenstates allowed by the self-adjoint Hamiltonian labelled by θ are characterized by the indices $k(n, \theta)$ whose energy eigenvalues are discrete, and that from (3.14) the intervals of adjacent discrete energy levels depend on the angle parameter. Note that as we vary the parameter from $\theta = 0$ the spectrum changes accordingly and returns to the original spectrum only when

θ approaches to 2π . Since any two Bessel functions whose indices differ by an integer are orthogonal to each other with respect to the innerproduct on \mathbb{R}^* , so are any of the two eigenstates in the domain $D(\hat{H}_*)$, as required.

3.3. Probability flow between the two Liouville subsystems

We have seen in section 2 that the reduced classical system admits solutions $Q(t)$ oscillating between the two subsystems given by $Q > 0$ and $Q < 0$. We now analyze how this global aspect manifests itself in the quantum theory. More specifically, we are interested in the question if there exists a probability flow between the two subsystems. We shall find that the answer is positive, signaling the fact that the reduced system is a connected system also quantum mechanically.

For this, consider the state $\psi(t)$ given by a linear combination of two energy eigenstates

$$\psi(t) = c_1 J_{k_1}\left(\frac{2}{Q}\right) e^{-iE_{k_1}t} + c_2 J_{k_2}\left(\frac{2}{Q}\right) e^{-iE_{k_2}t}, \quad (3.24)$$

where $J_{k_1}, J_{k_2} \in D(\hat{H}_*)$ for some fixed θ and $E_{k_i} = -k_i^2/4$ for $i = 1, 2$. We take the two indices k_1 and k_2 which are different by some odd integer,

$$k_1 - k_2 = 2n + 1, \quad n \in \mathbb{Z}, \quad (3.25)$$

and choose the constants $c_1, c_2 \in \mathbb{C}$ so that the state be normalized $\|\psi\| = 1$ at $t = 0$. Then the orthogonality condition $\langle J_{k_1}, J_{k_2} \rangle = 0$ implies that the norm, i.e., the total probability on the full line \mathbb{R}^* , remains constant,

$$\frac{d}{dt} \langle \psi, \psi \rangle = \frac{d}{dt} \langle \psi, \psi \rangle_+ + \frac{d}{dt} \langle \psi, \psi \rangle_- = 0. \quad (3.26)$$

However, the probability on a half line, say \mathbb{R}^+ , does not remain constant. Indeed, a similar computation for the half line reveals that

$$\frac{d}{dt} \langle \psi, \psi \rangle_+ = ic_1^* c_2 \langle J_{k_1}, J_{k_2} \rangle_+ (E_{k_1} - E_{k_2}) e^{-i(E_{k_2} - E_{k_1})t} + \text{c.c.} \neq 0 \quad (3.27)$$

because the two Bessel functions with (3.25) are not orthogonal to each other $\langle J_{k_1}, J_{k_2} \rangle_+ \neq 0$ on the half line \mathbb{R}^+ . This shows that there exists a probability flow between the two subsystems where $Q \in \mathbb{R}^+$ and $Q \in \mathbb{R}^-$, even though the wavefunctions vanish at the junction. The reason why such a flow can exist is that the Hamiltonian operator is *not* self-adjoint with respect to the innerproduct on the half line \mathbb{R}^+ (or \mathbb{R}^-), although it

becomes so if we cut the domain ‘in half’ so that it consists only of those eigenstates with indices given by even (or odd) integers n in (3.23).

4. Discussion

We have seen in this paper that the quantum mechanical Liouville model with attractive potential obtained by the Hamiltonian reduction (which regularizes the Liouville model classically) can be solved completely. Although the energy spectrum is unbounded from below, the fact that only discrete levels are allowed suggests that the system is ‘quasi-stable’ at the quantum level. The connectedness of the two subsystems can be observed by a probability flow, which we have shown to exist. The quantum theory is labelled by the angle parameter θ which arises in constructing self-adjoint Hamiltonian operators. In this respect, it is worth noting that the reduced phase space is topologically $\mathbb{R}^2 \setminus \{0\}$, that is, the two dimensional Euclidean plane with a hole, which is homeomorphic to $T^*S^1 = S^1 \times \mathbb{R}$. The appearance of the angle parameter may perhaps be understood as a common phenomenon observed in quantizing on a configuration space with a hole, as in the case of the quantum theory on S^1 or of the Yang-Mills theory, whose quantization yields the θ -vacua by an analogous mechanism.

Our procedure of quantization is similar to that of Fülöp [6] but differs in some important points. First, unlike our measure $dQ/|Q|$ for the innerproduct (3.7), the measure used for the innerproduct in [6] is $|Q|dQ$, which is obtained from the Haar measure of the group $SL(2, \mathbb{R})$ by eliminating the degrees of freedom that correspond to the symmetry. However, in Hamiltonian reduction one has to use the phase space volume element on $T^*SL(2, \mathbb{R})$ to derive the correct reduced measure [10], which is the measure $dQ/|Q|$ we used. This causes a certain alteration in the energy spectrum. Second, and more importantly, the self-adjoint extensions of the Hamiltonian operator are achieved in [6] by considering a domain such that the eigenstates on the full line \mathbb{R}^* are formed out of linear combinations of two eigenstates, each defined on the half lines \mathbb{R}^+ and \mathbb{R}^- , respectively. Since one can take distinct angle parameters to specify the self-adjoint extensions on the two half lines, one needs two angle parameters in general to specify the self-adjoint extensions on the full line \mathbb{R}^* (plus an extra parameter to render the eigenstates mutually orthogonal). This we find an unnecessary complication, given that the self-adjoint extensions can be achieved on the full line without referring to those on the half lines. Our simpler quantization yields the energy spectrum given by a single class of discrete levels with indices (3.23) specified

by the angle θ , a result which we feel is natural to associate with the classical periodic motions on the smooth phase space. In contrast, the spectrum in [6] consists of two classes of discrete levels similar to (3.23) but with even integers $n \in 2\mathbb{Z}$.

Finally, we wish to stress that the quantization discussed in this paper is not the unique one available to the Liouville system. Indeed, from the way the classical system is defined, it is perhaps more natural to quantize first the system of a free particle on G and then carry out quantum Hamiltonian reduction. This will provide a way to confirm what we have learned in the quantum mechanical Liouville model given in this paper.

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Appendix A. Symmetric operator and self-adjoint extension

Here we briefly summarize the general theory of self-adjoint extensions of symmetric operators [9]. Let \hat{A} be a linear operator on a dense domain $D(\hat{A})$ in a Hilbert space \mathcal{H} . Consider $\psi \in \mathcal{H}$ for which

$$\langle \psi, \hat{A}\phi \rangle = \langle \psi', \phi \rangle \quad \forall \phi \in D(\hat{A}). \quad (\text{A.1})$$

is satisfied for some $\psi' \in \mathcal{H}$. Define the domain $D(\hat{A}^*)$ by the set consisting of those ψ . It is then easily confirmed that $\psi \mapsto \psi'$ gives a linear map. The *adjoint* operator \hat{A}^* is defined by this linear map $\hat{A}^*: \psi' = \hat{A}^*\psi$. If the operator \hat{A} fulfills the condition

$$D(\hat{A}^*) \supset D(\hat{A}), \quad \hat{A}^* = \hat{A} \quad \text{on} \quad D(\hat{A}), \quad (\text{A.2})$$

then \hat{A} is a *symmetric* operator. A symmetric operator is called a *self-adjoint* operator when the two domains coincide, $D(\hat{A}^*) = D(\hat{A})$. If \hat{A}, \hat{B} are two operators such that

$$D(\hat{B}) \supset D(\hat{A}), \quad \hat{B} = \hat{A} \quad \text{on} \quad D(\hat{A}), \quad (\text{A.3})$$

then the operator \hat{B} is an *extension* of the operator \hat{A} . A symmetric operator \hat{A} can be extended to be a self-adjoint operator \hat{A}_* if there exists a domain $D(\hat{A}_*)$ such that

$$D(\hat{A}^*) \supset D(\hat{A}_*) = D(\hat{A}_*) \supset D(\hat{A}). \quad (\text{A.4})$$

When a certain condition is fulfilled (which we discuss shortly), such *self-adjoint extensions* of a symmetric operator \hat{A} are possible in the following way. We begin by decomposing the domain $D(\hat{A}^*)$ of the adjoint operator \hat{A}^* as

$$D(\hat{A}^*) = D(\hat{A}) + K_-(\hat{A}^*) + K_+(\hat{A}^*), \quad \text{where} \quad K_{\pm}(\hat{A}^*) := \text{Ker}(\hat{A}^* \pm i). \quad (\text{A.5})$$

To see that this decomposition is possible, we first decompose any state $\psi \in D(\hat{A}^*)$ as $\psi = \zeta + \alpha$ where $\zeta \in D(\hat{A})$ and $\alpha \notin D(\hat{A})$. Applying $(\hat{A}^* - i)$ to ψ we find

$$(\hat{A}^* - i)\psi = (\hat{A} - i)\zeta + (\hat{A}^* - i)\alpha. \quad (\text{A.6})$$

Note that on account of the property $\|(\hat{A} \pm i)\phi\|^2 = \|\hat{A}\phi\|^2 + \|\phi\|^2$ for $\forall \phi \in D(\hat{A})$ the spaces $(\hat{A} \pm i)D(\hat{A})$ are closed subspaces in \mathcal{H} , and that they are orthogonal to $K_{\mp}(\hat{A}^*)$, respectively. Thus we can write $(\hat{A}^* - i)\alpha = (\hat{A} - i)\beta - 2i\xi$, where $\beta \in D(\hat{A})$ and $\xi \in K_+(\hat{A}^*)$. But since $-2i\xi = (\hat{A}^* - i)\xi$, we have

$$(\hat{A}^* - i)(\psi - \phi - \xi) = 0, \quad (\text{A.7})$$

where $\phi := \zeta + \beta \in D(\hat{A})$. It then follows that $\eta := \psi - \phi - \xi \in K_-(\hat{A}^*)$, that is, any state $\psi \in D(\hat{A}^*)$ can be decomposed as

$$\psi = \phi + \xi + \eta, \quad \phi \in D(\hat{A}), \quad \xi \in K_-(\hat{A}^*), \quad \eta \in K_+(\hat{A}^*), \quad (\text{A.8})$$

which proves our claim (A.5).

It can be shown that a necessary and sufficient condition for satisfying the relation

$$\langle \psi, \hat{A}^*\psi \rangle = \langle \hat{A}^*\psi, \psi \rangle \quad (\text{A.9})$$

for $\psi \in D(\hat{A}^*)$ in the form (A.8) is $\|\xi\| = \|\eta\|$, i.e., the latter two components must be related by a unitary transformation $\eta = \hat{U}\xi$. Such a unitary transformation exists if and only if the two *deficiency indices* (d_+, d_-) defined by

$$d_+ = \dim K_-(\hat{A}^*), \quad d_- = \dim K_+(\hat{A}^*), \quad (\text{A.10})$$

are equal $d_+ = d_-$. Hence, if this is the case, the symmetric operator \hat{A} can be extended to a self-adjoint operator \hat{A}_* with the domain

$$D(\hat{A}_*) = \{\psi \mid \psi = \phi + \xi + \hat{U}\xi, \quad \phi \in D(\hat{A}), \quad \xi \in K_-(\hat{A}^*), \quad \hat{U}\xi \in K_+(\hat{A}^*)\}. \quad (\text{A.11})$$

Appendix B. Orthogonality and the asymptotic forms of Bessel functions

Bessel functions $J_k(z)$ are defined by the series,

$$J_k(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(k+m+1)} \left(\frac{z}{2}\right)^{k+2m}, \quad \arg |z| < \pi, \quad k \in \mathbb{C}. \quad (\text{B.1})$$

The other type of Bessel functions are $Y_k(z) := (\cos(k\pi)J_k(z) - J_{-k}(z))/\sin(k\pi)$. The Bessel functions (B.1) have the following asymptotic forms for z approaching infinity and zero:

$$\begin{aligned} J_k(z) &\underset{z \rightarrow +\infty}{\sim} \sqrt{\frac{2}{\pi z}} \cos\left[z - \frac{\pi}{2}(k+1)\right], \\ J_k(z) &\underset{z \rightarrow 0}{\sim} \frac{1}{\Gamma(k+1)} \left(\frac{z}{2}\right)^k. \end{aligned} \quad (\text{B.2})$$

We define the Bessel functions on \mathbb{R}^- by analytically continuing them from the upper half plane $\text{Im}z > 0$, that is, we give the values on \mathbb{R}^- by rotating the functions on \mathbb{R}^+ by π ,

$$J_k(z) = J_k(e^{\pi i}(-z)) = e^{k\pi i} J_k(-z), \quad \arg z = \pi. \quad (\text{B.3})$$

Note that $J_k^*(z) = J_{k^*}(z)$ for $z \in \mathbb{R}^+$ whereas $J_k^*(z) = e^{-2ik^*\pi} J_{k^*}(z)$ for $z \in \mathbb{R}^-$.

Due to the formula (B.3), integrals of the Bessel functions on the full line $\mathbb{R} \setminus \{0\}$ read

$$\langle J_k, J_l \rangle = (1 + e^{-i(k^*-l)\pi}) \langle J_k, J_l \rangle_+. \quad (\text{B.4})$$

The integrals $\langle J_k, J_l \rangle_+$ on the half line \mathbb{R}^+ can be evaluated by means of Lommel's integral [12],

$$\int_a^b \frac{dz}{z} J_k(z) J_l(z) = \frac{1}{k^2 - l^2} \left[z \left(J_k(z) \frac{d}{dz} J_l(z) - J_l(z) \frac{d}{dz} J_k(z) \right) \right]_a^b. \quad (\text{B.5})$$

If $\text{Re}(k+l) > 0$ then the above integral converges, and from $Q = 2/z$ we obtain

$$\langle J_k, J_l \rangle_+ = \lim_{\substack{a \rightarrow +0 \\ b \rightarrow +\infty}} \int_a^b \frac{dQ}{|Q|} J_k\left(\frac{2}{Q}\right) J_l\left(\frac{2}{Q}\right) = \frac{2\sqrt{2}}{\pi(k^2 - l^2)} \sin\left[\frac{\pi}{2}(k^* - l)\right], \quad (\text{B.6})$$

where we used the asymptotic forms (B.2). If $k, l \in \mathbb{R}$ and $k-l = n \in \mathbb{Z}$ then the integral on the full line vanishes, $\langle J_k, J_l \rangle = 0$. This occurs for two different reasons depending on whether n is even or odd; for even n , the integral on the half line vanishes $\langle J_k, J_l \rangle_+ = 0$, whereas for odd n the integral is not zero but the factor $(1 + e^{-i(k^*-l)\pi})$ vanishes.

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