Quantum Chaotic Dynamics and Random Polynomials

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Abstract

We investigate the distribution of roots of polynomials of high degree with random coefficients which, among others, appear naturally in the context of "quantum chaotic dynamics". It is shown that under quite general conditions their roots tend to concentrate near the unit circle in the complex plane. In order to further increase this tendency, we study in detail the particular case of self-inversive random polynomials and show that for them a finite portion of all roots lies exactly on the unit circle. Correlation functions of these roots are also computed analytically, and compared to the correlations of eigenvalues of random matrices. The problem of ergodicity of chaotic wave-functions is also considered. For that purpose we introduce a family of random polynomials whose roots spread uniformly over phase space. While these results are consistent with random matrix theory predictions, they provide a new and different insight into the problem of quantum ergodicity. Special attention is devoted all over the paper to the role of symmetries in the distribution of roots of random polynomials.

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1 Introduction

Semiclassical approximations for multidimensional quantum systems and the manifestations of chaotic behaviour in quantum mechanics have attracted wide attention during the last years (see e.g. [1], [2] and references therein). In these approximations one quite often needs to locate the roots of polynomials of high degree whose coefficients are rapidly-varying erratic functions of the energy. As a consequence, these coefficients may be considered as random variables, even in a small energy interval. Though the distribution of roots of polynomials with random coefficients have been studied in the past (see e.g. [3]-[5]), its relevance with respect to the specific problems which naturally arise in the context of quantum chaotic dynamics (and in other domains of physics as well) has been underestimated, and a number of elementary and basic questions have not yet been solved

The main purpose of this paper is to study properties of random polynomials with emphasis on the above mentioned connection. A short version containing some of our results was already published in [6]. Except for some general symmetry considerations, the coefficients of the polynomials will be considered independent random variables. This assumption is in some cases justified with physical arguments, and in others just made because of mathematical simplicity. Our investigations could have direct application in other fields. In fact, zeros in the complex plane of polynomials with random coefficients occur in a variety of problems of science and engineering (zeros of the partition function in statistical mechanics, theory of noise, etc). The present investigation may also be of interest if one views zeros of polynomials as interacting particles in two dimensions, as, for instance, eigenvalues of random asymmetric matrices can be physically interpreted as a two-dimensional electron gas confined in a disk [7, 8].

The use of a statistical approach in the description of complex systems is an old idea. In particular the random matrix theory (RMT), originally formulated in the context of nuclear physics, has had a great success and impact in the study of quantum chaotic dynamics[1] and disordered systems [1, 9]. As it will become clearer in the following sections, the statistical analysis of these systems by random polynomials is, in some sense, complementary to the RMT.

In Section 2 we consider general random polynomials of degree N whose coefficients are independent random variables having zero mean. We show that under quite general conditions their roots tend to concentrate in an annulus near the unit circle of the complex plane, and that the width of this annulus goes to zero as $N \to \infty$. In [3] this result was proved by a different method for the particular case when all second moments of the coefficients are equal. Our method, based on the existence of a saddle point configuration, seems to be more general and physically transparent.

[1] we will use this expression to avoid the more proper but lengthy term 'quantum mechanics of a classically chaotic system'.

Aside from section 2, which constitutes a general introduction, the paper can be divided into two main parts (sections 3 and 4) which are to a large extent independent. In Section 3 we investigate the important case of self-inversive polynomials, namely polynomials whose coefficients satisfy Eq.(3.6) below and whose roots are hence distributed symmetrically with respect to the unit circle. This type of polynomials appear when considering either the semiclassical quantization via the transfer-operator method [10] or quantum maps [11, 12]. We prove that for them a finite fraction of the roots lies exactly on the unit circle. Moreover, we compute analytically the two-point correlation function of these roots. We find that there is linear repulsion between them at short distances, and compare to the Gaussian Orthogonal Ensemble of RMT [13, 14]. The possibility of locating in a statistical sense all the roots on the unit circle is also discussed.

In Section 4 we consider the properties of eigenfunctions of chaotic systems and their relation with the classical concept of ergodicity, in the spirit of Refs [15, 16]. Using a phase-space representation, we write down the eigenfunctions of spin systems in a polynomial form and consider the coefficients to be random. We then analyse how the roots of these polynomials are distributed in phase space. The second moments of the coefficients now grow with N so fast that the resulting distribution of roots turns out to be uniform over the phase space, a sort of quantum ergodicity. We moreover compare this result with the predictions concerning the coefficients of the eigenfunctions of a random matrix ensemble.

It turns out that the existence of a symmetry of the roots with respect to a line increases considerably the probability of finding roots exactly on that line. In subsection 4.2 we consider the influence of symmetries on the distribution of the roots of chaotic eigenstates. For that purpose we analyze the eigenfunctions of a certain quantum map having two antiunitary symmetries and show numerically that their roots tend to concentrate over the associated phase-space symmetry lines. We also investigate how this phenomenon disappears as the symmetries are broken.

In the appendices we explain in detail our computations.

2 Some General Properties of Random Polynomials

Given a distribution function

$$\int \mathcal{D}(a_0, a_1, \ldots, a_N) d^2a_0 d^2a_1 \ldots d^2a_N \tag{2.1}$$

for some complex coefficients $\{a_k\}$, we are interested in the distribution in the complex plane of the roots $\{z_k\}$, $k = 1, \ldots, N$ of the polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_N z^N. \tag{2.2}$$

For clarity, we consider the particular case where the real and imaginary part of the coefficients $\{a_k\}$ are independent normally distributed real-valued random variables with zero mean and standard deviation σ_k (henceforth denoted as a GRI distribution)

$$\mathcal{D}(a_0, \ldots, a_N) = \prod_{k=0}^{N} \frac{1}{2\pi\sigma_k^2} \exp\left(-\frac{|a_k|^2}{2\sigma_k^2}\right) . \tag{2.3}$$

The joint probability density for the zeros is obtained by changing from the variables $\{a_0, a_1, a_2, \ldots, a_N\}$ to the variables $\{a_N, z_1, z_2, \ldots, z_N\}$ using the standard formulae

$$\begin{aligned} a_{N-k} &= (-1)^k a_N u_k(z), \qquad k = 1, \cdots, N \\ u_k(z) &= \sum_{1=i_1<i_2<\ldots<i_k=N} z_{i_1} z_{i_2} \cdots z_{i_k}, \end{aligned}$$

and integrating over a_N, with the result

$$\mathcal{D}(z_1, \ldots, z_N) = C_N \frac{D_N(z)}{[G_N(z)]^{N+1}}, \tag{2.4}$$

where

$$\begin{aligned} C_N &= \tfrac{N!}{\pi^N} \prod_{k=0}^{N} \eta_k, \qquad \eta_k = \left(\tfrac{\sigma_N}{\sigma_{N-k}}\right)^2, \\ G_N(z) &= 1 + \sum_{k=1}^{N} \eta_k \, |u_k(z)|^2, \\ D_N(z) &= \prod_{j<k} |z_j - z_k|^2. \end{aligned} \tag{2.5}$$

The factor $D_N(z)$ comes from the Jacobian of the transformation.

Some properties of this distribution are:

(i) if for all k $\sigma_k = \sigma$, then $\mathcal{D}(z_1, \ldots, z_N)$ is independent of σ. This is a consequence of the fact that the roots of $P(z)$ are unchanged by multiplying $P(z)$ by a constant.

(ii) $\mathcal{D}(z_1, \ldots, z_N)$ is invariant under a rotation of the coordinates in the complex plane, i.e.

$$\mathcal{D}(z_1, \ldots, z_N) = \mathcal{D}(z_1 e^{i\varphi}, \ldots, z_N e^{i\varphi}) .$$

(iii) if $\sigma_{N-k} = \sigma_k$, then $\mathcal{D}(z_1, \ldots, z_N)$ is invariant under inversion with respect to the unit circle C in the complex plane,

$$\mathcal{D}(z_1, \ldots, z_N) = \mathcal{D}(1/\bar{z}_1, \ldots, 1/\bar{z}_N),$$

where the bar means complex conjugate.

In order to find the most probable distribution of roots, we have to locate the maximum of $\mathcal{D}$. The equation for the extrema $d\log\mathcal{D}/d\bar{z}_p = 0$ can be transformed into the following form:

$$\sum_{k=1}^{N} \eta_k u_k \left[\overline{\left(\frac{du_k}{dz_p}\right)} - g_p \bar{u}_k \right] = g_p, \qquad p = 1,\ldots,N \tag{2.6}$$

where

$$g_p = \frac{1}{(N+1)} \sum_{j\neq p} \frac{1}{\bar{z}_p - \bar{z}_j} .$$

One can easily check that the following configuration is a solution of these equations:

$$z_k = r_N \exp\left[\mathrm{i}\frac{2\pi}{N} k + \mathrm{i}\phi \right], \quad k = 1,\ldots,N , \tag{2.7}$$

where

$$r_N = \left(\frac{N-1}{N+3}\right)^{1/2N} \left(\frac{\sigma_0}{\sigma_N}\right)^{1/N} \overset{N\to\infty}{\simeq} 1 + \frac{1}{N} \ln\left(\frac{\sigma_0}{\sigma_N}\right) .$$

We will refer to this configuration as to the crystal solution. To prove that it obeys Eq.(2.6) we remark that the z_k given by Eq.(2.7) are solutions of the equation $z^N = const$, and therefore for this solution all u_k with $k = 1, 2, \ldots, N-1$ vanish and Eq.(2.6) reduces to

$$\eta_N u_N \left[\overline{\left(\frac{du_N}{dz_p}\right)} - g_p \bar{u}_N \right] = g_p \qquad p = 1,\ldots,N. \tag{2.8}$$

But $du_N/dz_p = u_N/z_p$ and $g_p = C/\bar{z}_p$ where

$$C = \frac{1}{N+1} \sum_{j=1}^{N-1} \frac{1}{1 - \exp(\mathrm{i}2\pi j/N)} = \frac{1}{2}\frac{(N-1)}{(N+1)}.$$

Then Eq.(2.8) becomes

$$\eta_N |u_N|^2 (1 - C) = C ;$$

using the definitions of η_N, u_N and C, the reader can verify that this equation is satisfied. Therefore all Eqs.(2.6) for $p = 1, \ldots, N$ will be fulfilled by the crystal solution (2.7).

It follows that (except in the case of an exponential (or faster) dependence of σ_0/σ_N on N) in the limit $N \to \infty$ the radius r_N tends always to one. Moreover, the phase ϕ in Eq.(2.7) is arbitrary because of the property (ii). In the particular case $\sigma_0 = \sigma_N$, r_N is equal to one, as it must be according to property (iii).

The existence of the crystal solution implies that the distribution of roots $\mathcal{D}(z_1, \ldots, z_N)$ will always have a maximum on the line $|z| = r_N$. But the actual distribution of roots in the complex plane depends, however, on the second derivatives of $\log\mathcal{D}$. There is a direct connection between their magnitude and the dominance of the crystal solution. For example, putting $\sigma_0 = \sigma_N$ and choosing all the other σ's such that $N\eta_k \to 0$, $(k = 1,\ldots,N-1)$ as $N \to \infty$, one concludes that the crystal solution will not dominate the distribution since the intermediate terms in the denominator of (2.4) automatically cancel, independently of the crystal solution. An extreme case of this behaviour is what we call, for reasons which will become clear later, the $SU(2)$ polynomials, for which $\sigma_k/\sigma_N = \sqrt{C_N^k}$. As we shall see in section 4, for these polynomials the roots spread all over the complex plane.

On the contrary, the case $N\eta_k \to \infty$ enforces the crystal solution since the factors u_k in the denominator of (2.4) are multiplied by an increasing function of N. We shall consider in more detail several cases of this type in the next sections.

These simple considerations show that under quite general conditions the roots of random polynomials of high degree for which the mean value of the coefficients is equal to zero will tend to concentrate near the circle $|z| = 1$. In order to illustrate this with a numerical example, in Fig.1a are plotted, in the complex plane, the roots of 200 trials of a polynomial of degree $N = 48$ whose coefficients are GRI-distributed all having the same second moment. The observed distribution clearly satisfies the properties (ii) and (iii).

On the other hand, it can be easily proved that if the mean value of the coefficients are non-zero, the roots tend to concentrate around the roots of the mean polynomial [3].

3 The Self-Inverse Symmetry

As shown in the previous section and as illustrated in Fig.1a, in the large-N limit and under certain circumstances the roots of random polynomials tend to concentrate around the unit circle but not, in general, on it. We now want to study some simple conditions to be imposed over the coefficients of a random polynomial in order to locate as much zeros as possible *exactly* on the unit circle. Among other areas of physics and mathematics, this problem is of interest in the context of semiclassical approximations in quantum mechanics because of the following reasons.

Some recent methods to solve quantum problems incorporate the physical information into a transfer operator T which is an $N \times N$ unitary matrix [11, 10]. The operator T has N complex eigenvalues $\{z_k\}$, $k = 1,\ldots,N$ lying on the unit circle $\mathcal{C}$ of the complex plane, which are determined by the roots of the characteristic polynomial

$$P(z) = \det(z - T) = \sum_{k=0}^{N} a_k\, z^k \ . \tag{3.1}$$

The eigenvalues are functions of the energy of the system, $z_k = z_k(E)$, since T is a function of E. They move on $\mathcal{C}$ as E is varied, and the quantization condition, which takes the form of a Fredholm determinant, states that whenever one of the roots crosses the point $z = 1$ then the corresponding energy is an eigenvalue of the system

$$\det(1 - T(E)) = 0 \; . \tag{3.2}$$

A similar equation but with no energy dependence is obtained when considering the quantization of maps, where T is now the one step unitary evolution operator of the map [11]. Since T is a finite matrix, it has N nontrivial invariants which can be chosen either as the coefficients $\{a_k\}$ of the characteristic polynomial, or as the traces of the first N powers of T

$$M_L = \mathrm{Tr}\; T^L(E) = \sum_{k=1}^{N} z_k^L \; , \qquad L = 0, \ldots, N \; . \tag{3.3}$$

One can express one set of invariants in terms of the other set by means of the following recurrent formulas

$$a_{N-k} = -\frac{1}{k} \sum_{j=1}^{k} a_{N-k-j}\, M_j \; , \qquad k = 1, \ldots, N \; , \tag{3.4}$$

with $a_N = 1$ by definition of $P(z)$. These relations are of interest because in the semiclassical limit $N \to \infty$ one can express the traces of the powers of T as an amplitude sum over the classical periodic orbits

$$M_L \simeq \sum_{\gamma(L)} A_\gamma(E) e^{\mathrm{i}S_\gamma(E)/\hbar - \mathrm{i}\pi\mu_\gamma/2} \; . \tag{3.5}$$

Here, $\gamma(L)$ are all the periodic orbits corresponding to L iterations of the initial map at energy E (including repetitions), $S_\gamma(E)$ is the action of the periodic orbit, μ_γ is the Maslov index and $A_\gamma(E)$ is a real function depending on the stability of the orbit. Inserting (3.5) into (3.4) we obtain a semiclassical approximation for each coefficient a_k, now written in terms of sums over all the periodic orbits up to period k (the special combinations between them given by Eqs.(3.4) are called pseudo-orbits). To compute all the coefficients, all the periodic orbits up to period N are needed.

Since the action $S_\gamma(E)$ is a function of the energy, in the semiclassical limit the moments are rapidly varying functions of the energy and the coefficients a_k are sums of products of these rapidly varying functions. It thus seems natural to adopt a statistical approach and to consider these coefficients as random variables. This way of proceeding differs from the usual statistical approach to complex systems in which, instead of the coefficients of its characteristic polynomial, the matrix elements of a relevant operator are assumed to be random.

There is, however, an intrinsic difficulty when approximating the coefficients of the characteristic polynomial by Eqs.(3.4) and (3.5), or more generally when considering them as independent variables. We are ignoring the correlations existing among them that guarantee the unitarity of T. (Without correlations and as was pointed out in section 2, the roots will lie close to $\mathcal{C}$ but not on it.) A consequence of this is that the eigenvalues of T are no more located on C, and Eq.(3.2) fails to determine the full spectrum (typically, some of the eigenvalues are missed).

It may therefore be useful to find some simple conditions to be imposed on the coefficients $\{a_k\}$ in order to restore – at least partially – the unitarity of T. A necessary but not sufficient condition is the self-inversive (SI) property

$$a_{N-k} = \exp(\mathrm{i}\Theta)\,\bar{a}_k \tag{3.6}$$

which can easily be obtained from Eq.(3.1) factorizing the polynomial and making the substitution $z_k = \exp(\mathrm{i}\theta_k)$. Θ is a real function of the energy, $\Theta = \pi N + \sum_{k=1}^{N} \theta_k(E)$. In the semiclassical limit [10]

$$\Theta \simeq \pi \overline{N(E)},$$

where $\overline{N(E)}$ is the mean number of levels with energy less than E. Polynomials obeying Eq.(3.6) satisfy a functional equation

$$\overline{P(1/\bar{z})} = \mathrm{e}^{-\mathrm{i}\Theta} P(z)/z^N \; . \tag{3.7}$$

It follows from this equation that the symmetry (3.6) of the coefficients is reflected into a symmetry of the zeros: if z_k is a root, then $1/\bar{z}_k$ is also a root, i.e. the roots either lie on C or are symmetrically located under inversion with respect to it.

From the semiclassical point of view, the advantages of imposing the self-inversive symmetry are twofold: firstly because it implements, in a simple manner, part of the unitarity of T; and secondly because it lowers the number of periodic orbits to be computed. In fact, we only need to know the periodic orbits up to the period $N/2$ instead of N, since we only need to compute half of the coefficients (the others being determined by symmetry) (see e.g. [17, 12]).

3.1 The fraction of roots lying on C

Being a necessary but not sufficient condition we don't know, however, how many zeros are located on the unit circle by the SI symmetry. In order to answer this question, we compute the fraction of zeros lying on C for SI polynomials of the form (remember that $a_N = 1$)

$$P(z) = 1 + \sum_{k=1}^{N-1} a_k z^k + z^N \; , \qquad a_{N-k} = \bar{a}_k \; , \tag{3.8}$$

and consider the coefficients $\{a_k\}, k = 1, \ldots, (N-1)/2$ as GRI distributed complex variables with arbitrary variances σ_k^2. For simplicity we consider the particular case $\Theta = 0$ and consider N to be an odd integer, without loss of generality. By substitution $z = \exp(\mathrm{i}\theta)$ in (3.8), SI polynomials transform into real trigonometric polynomials:

$$f(\theta) = \frac{1}{2}\mathrm{e}^{-\mathrm{i}N\theta/2} P\left(\mathrm{e}^{\mathrm{i}\theta}\right) = \cos\left(\frac{N}{2}\theta\right) + \sum_{k=1}^{M} \left\{ c_k \cos\left[\left(\frac{N}{2} - k\right)\theta\right] + d_k \sin\left[\left(\frac{N}{2} - k\right)\theta\right] \right\} \; , \tag{3.9}$$

where $M = (N-1)/2$, $c_k = \Re e(a_k)$ and $d_k = \Im m(a_k)$. The zeros of $P(z)$ lying on $\mathcal{C}$ correspond now to the real zeros of the real function $f(\theta)$.

The average fraction of zeros lying on $\mathcal{C}$ is defined as

$$< \nu > = \frac{1}{N} \int_0^{2\pi} < \rho(\theta) > d\theta \tag{3.10}$$

where $< \rho(\theta) >$ is the average density of zeros on $\mathcal{C}$

$$\rho(\theta) = \sum_k \delta(\theta - \theta_k) = \delta[f(\theta)] |f'(\theta)| \ . \tag{3.11}$$

(Primes indicate derivative with respect to θ.) To compute $< \rho(\theta) >$, we use the method of Kac [5] which exploits the following representations of $\delta[f]$ and $|f'|$

$$\delta[f] = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{i\xi f} \ , \quad |f'| = \int_{-\infty}^{\infty} \frac{d\eta}{\pi\eta^2} \left(1 - e^{i\eta f'}\right) \ . \tag{3.12}$$

The advantage of such representations is that, when performing the ensemble average over the coefficients

$$< \rho(\theta) > = \int \mathcal{D}(a_0, a_1, \ldots, a_N)\, \rho(\theta)\, d^2a_0 \ldots d^2a_N \tag{3.13}$$

the exponentiation of f and f' (who are linear function of the $\{a_k\}$) allows an easy computation of the integrals.

The computation of $< \rho(\theta) >$ and $< \nu >$ for arbitrary N and arbitrary second moments $\{\sigma_k\}$ are straightforward but lengthy, and we include them in the appendix A. Here we give the result for the particular case of constant second moments $\sigma_k = \sigma \, \forall k$ and in the limit $N \to \infty$ (see Eqs.(3.20),(3.28) below for the exact answer for arbitrary N and arbitrary variances in the case of a SI polynomial of the form (3.19)). We find that, to leading order in $1/N$, the average density of roots depends on σ through the scaled parameter $\varepsilon = \sigma\sqrt{N}$, with the result

$$< \rho(\theta) > \simeq \frac{N}{2} \exp\left[-\cos^2\left(\frac{N}{2}\theta\right)/\varepsilon^2\right] \left\{ \frac{1}{\sqrt{\pi}\varepsilon} \left|\sin\left(\frac{N}{2}\theta\right)\right| + \frac{1}{\pi\sqrt{3}} \int_0^1 dx \exp\left[-3\sin^2\left(\frac{N}{2}\theta\right)/(\varepsilon x)^2\right] \right\} . \tag{3.14}$$

Integration over θ gives

$$< \nu(\varepsilon) > \simeq \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{2}/\varepsilon}^{\sqrt{2}/\varepsilon} dy\, e^{-y^2/2} + \frac{1}{\pi\sqrt{3}} \int_0^1 dx \int_0^\pi d\varphi \exp\left[-\frac{1}{\varepsilon^2}\left(\cos^2\varphi + \frac{3\sin^2\varphi}{x^2}\right)\right] . \tag{3.15}$$

To understand these results, consider first the limit $\varepsilon \to 0$. In this case, the coefficients $\{a_k\}$ tend to have a very narrow distribution centered around zero, and the polynomial (3.8) is well approximated by

$$P(z) \stackrel{\varepsilon \to 0}{\simeq} 1 + z^N \ ;$$

the roots of this polynomial are given by Eq.(2.7), with $r_N = 1$ and $\phi = \pi/N$. This is indeed the behaviour recovered from Eqs.(3.14) and (3.15). In (3.15), when $\varepsilon \to 0$ the first term in the r.h.s. tends to one while the second tends to zero, implying $<\nu>= 1$. Moreover, in (3.14) $\exp[-\cos^2(\frac{N}{2}\theta)/\varepsilon^2] \to 0$ for any θ except at $\theta = \frac{2\pi}{N}(k+\frac{1}{2})$, $k = 0,\ldots,N-1$ where we get a delta peak, in agreement with the crystal distribution.

On the other extreme, when $\varepsilon \to \infty$, we can neglect in (3.8) the term $1 + z^N$ and

$$P(z) \stackrel{\varepsilon\to\infty}{\simeq} \sum_{k=1}^{N-1} a_k\, z^k \;, \qquad a_{N-k} = \bar{a}_k \;. \tag{3.16}$$

In this limit, the first term in the r.h.s. of Eq.(3.15) tends to zero, while the double integral of the second term tends to π, and therefore

$$<\nu(\varepsilon)> \stackrel{\varepsilon\to\infty}{\simeq} 1/\sqrt{3}, \tag{3.17}$$

while from (3.14) we recover a uniform density

$$<\rho(\theta)> \stackrel{\varepsilon\to\infty}{\simeq} \frac{N}{2\pi\sqrt{3}}. \tag{3.18}$$

Eq.(3.17) answers the question of how efficient is the SI-symmetry to locate roots of a random polynomial on $\mathcal{C}$. We observe that indeed it has a strong effect in the distribution of roots, since it manages to locate a fraction $1/\sqrt{3} \simeq 57\%$ of the roots exactly on the unit circle. (For a different proof of this result see Ref. [18].)

Fig.1b shows the superposition of the roots of 200 iterations of a $N = 48$ self-inversive polynomial with all the second moments equal. The total number of zeros is the same as in Fig.1a; the strong concentration of roots on $\mathcal{C}$ is stressed in that figure by the reduction of the black intensity outside $\mathcal{C}$.

Fig.2 displays the fraction of points lying on $\mathcal{C}$ as a function of ε, Eq.(3.15). For small ϵ we observe the existence of a plateau which can be interpreted in the following way. At $\epsilon = 0$, as we said before the roots coincide with the crystal lattice and $<\nu>= 1$. When ϵ increases, and since the zeros are analytic functions of that parameter, they cannot immediately move outside $\mathcal{C}$ because this would violate the self-inversive symmetry (zeros come by symmetric pairs with respect to $\mathcal{C}$). The only way they can get out from $\mathcal{C}$ is to first move along $\mathcal{C}$ until two roots become degenerate, and then split in the radial direction, one zero moving in the positive radial direction, the other towards the origin. The size of the plateau can be estimated as the typical perturbation needed to produce a coalescence of two roots starting from the crystal solution.

It is instructive to compare our results to an analogous result due to M. Kac [5]. He considers the case of polynomials with *real* coefficients $\{a_k\}$ having a GRI distribution with all the second moments equal. These polynomials satisfy the functional equation

$$\overline{P(z)} = P(\bar{z})$$

(the roots lie either on the real axis or come by symmetric pairs under reflection with respect to it). He computes the fraction of real roots and finds a much smaller effect of the symmetry as compared to Eq.(3.17), since he shows that $<\nu>\sim \ln N/N$ as $N \to \infty$ (he also considers the distribution of the zeros on the real axis, see [5]). A numerical simulation of the distribution of roots of random polynomials with real coefficients is included in Fig.1c. The weak concentration of roots on the real axis can be appreciated in the figure from the fact that the density of points surrounding $\mathcal{C}$ is essentially unchanged as compared to Fig.1a. For completeness, we plot in Fig.1d the distribution of roots of 200 trials of $N = 48$ SI-polynomials with real coefficients. See also subsection 4.2 for an analogous result concerning SU(2) polynomials.

To conclude this subsection, let us point out that there is a simple way to put *all* the roots of a self-inversive polynomial on the unit circle $\mathcal{C}$ (in a statistical sense). In appendix A we prove that the general formula for the fraction of roots lying on $\mathcal{C}$ for SI polynomials of the form

$$P(z) = \sum_{k=0}^{N} a_k\, z^k \ , \qquad a_{N-k} = \bar{a}_k \tag{3.19}$$

where the $\{a_k\}$ are complex GRI distributed variables is given by

$$<\nu>= \frac{2}{N}\sqrt{\frac{g_2}{g_1}} \tag{3.20}$$

where

$$g_1 = \sum_{k=1}^{\frac{N-1}{2}} \sigma_k^2 \quad , \qquad g_2 = \sum_{k=1}^{\frac{N-1}{2}} \left(\frac{N}{2} - k\right)^2 \sigma_k^2 \ . \tag{3.21}$$

This is an exact formula valid for arbitrary N and arbitrary variances σ_k^2 of the coefficients. The particular case of equal variances is explicitly written in Eq.(3.28) below. This result was also obtained recently in [19], where the reader can also find a geometrical interpretation of it. Consider now the parametrization

$$\sigma_k = k^s \ ,$$

where s is an arbitrary real number. Using the asymptotic expansion

$$\sum_{k=1}^{L} k^\alpha \simeq \frac{L^{\alpha+1}}{\alpha+1} + O(L^\alpha), \qquad \alpha > -1 \ ,$$

from Eqs.(3.21) and (3.20) it follows, to leading order in N

$$<\nu>\simeq \frac{1}{\sqrt{(s+1)(2s+3)}}, \qquad s > -\frac{1}{2} \ . \tag{3.22}$$

For $s = 0$ (all the σ's equal), we recover the previous result $<\nu>= 1/\sqrt{3}$. For $s \to -1/2$, then $<\nu>\to 1$. For $s = -1/2$ we can estimate the rate of convergence towards $<\nu>= 1$ as $N \to \infty$ from

$$\sum_{k=1}^{L} k^{-1} \simeq \ln N + C + O(1/N) \tag{3.23}$$

(where C is Euler's constant). Then, from (3.20) and (3.21) we get

$$<\nu>\simeq 1 - \frac{3}{4}\frac{1}{\ln(N/2)} + O(1/N), \qquad s = -1/2\ . \tag{3.24}$$

The convergence is therefore quite slow (for example, to put 98% of the zeros on C we need $N \simeq 3.8 \times 10^{16}$).

3.2 Correlations between the roots

Having determined that the SI symmetry locates, in the large-N limit, $1/\sqrt{3}$ of the zeros on C, the next relevant question is what are the correlations existing between those roots. In particular, we would like to know if they repel each other or not and how their correlations compare to the random matrix theory (RMT). We have therefore computed, using the same techniques as before, the average two-point correlation function

$$R_2(\tau) = <\rho(\theta)\rho(\theta+\tau)>$$

for the set of roots lying on C for polynomials of the form (3.19). For this kind of polynomials the function $R_2(\tau)$ does not depend on θ since the distribution of roots is invariant under rotations.

The exact result valid for arbitrary N and arbitrary variances is presented in appendix B (cf Eq.(B.15)). In the particular case of equal variances and in the limit $N \to \infty$, $\tau \to 0$, $N\tau \to$ constant the two-point correlation function normalized to the square of the mean density (3.18) takes the form

$$\tilde{R}_2(\delta) = R_2(\delta)/(N/2\pi\sqrt{3})^2 = \frac{24}{\sqrt{C}}\left[B \arcsin\left(\frac{B}{A}\right) + \sqrt{A^2 - B^2}\right], \tag{3.25}$$

where $\delta = \tau N/2\pi$ and

$$\begin{aligned} A &= \frac{1}{8(\pi\delta)^2}\left\{\frac{(\pi\delta)^2}{3} - \left[\cos(\pi\delta) - \frac{\sin(\pi\delta)}{\pi\delta}\right]^2 \Big/ \left[1 - \frac{\sin^2(\pi\delta)}{(\pi\delta)^2}\right]\right\} \\ B &= \frac{1}{4(\pi\delta)^2}\left\{\cos(\pi\delta) + \left[\frac{(\pi\delta)^2}{2} - 1\right]\frac{\sin(\pi\delta)}{\pi\delta} - \frac{1}{2}\frac{\sin(\pi\delta)}{\pi\delta}\left[\cos(\pi\delta) - \frac{\sin(\pi\delta)}{\pi\delta}\right]^2 \Big/ \left[1 - \frac{\sin^2(\pi\delta)}{(\pi\delta)^2}\right]\right\} \\ C &= 1 - \frac{\sin^2(\pi\delta)}{(\pi\delta)^2}\ . \end{aligned} \tag{3.26}$$

Fig.3 is a plot of the function $\tilde{R}_2(\delta)$. For short distances there is a repulsion between the zeros. More precisely, from (3.25) it follows that

$$\tilde{R}_2(\delta) \overset{\delta\to 0}{\simeq} \frac{\pi^2}{10\sqrt{3}}\delta\ .$$

This behaviour is reminiscent of the linear repulsion between eigenvalues obtained for the orthogonal ensemble (GOE) of the random matrix theory, but with a different slope. However our computations correspond to systems without time reversal invariance, since the coefficients are complex. We therefore could have expected a quadratic repulsion, like in the GUE case. This particular point deserves further investigation. The long-range behavior shows on the other hand pronounced oscillations.

We have also computed numerically the nearest-neighbour spacing distribution for the zeros lying on C, shown in Fig.4. This distribution was investigated in Ref. [20], where a recursive procedure for $p(\delta)$ was developed. Our numerical results are in agreement with those obtained in that reference.

From the general results of Appendix B, it is also possible to compute the two-point correlation function for the case $\sigma_k = k^{-1/2}$, when statistically 100% of the zeros lie on C. We now find that the correlations between zeros tend to be much closer to a crystal-like solution (i.e., strong oscillations that survive for large values of δ) than in the case of constant second moments. Fig.5 displays the function $\tilde{R}_2(\delta)$ computed analytically from Eq.(B.15) for a self-inversive polynomial with $\sigma_k = k^{-1/2}$ and $N = 3601$; for such a value of N the fraction of roots lying on C is, from Eq.(3.24), $<\nu>= 0.9$. In the light of the results of section 2, the emergence of a crystal-like behaviour when $\sigma_k = k^{-1/2}$ is to be expected since in this case $N\eta_k \to 0$ as $N \to \infty$.

3.3 The variance of the number of zeros lying on C

In section 3.1 we have established that SI polynomials of the form (3.19) have asymptotically on the average a fraction $<\nu>= 1/\sqrt{3}$ of their roots lying on C if the complex coefficients $\{a_k\}$ are GRI-distributed with the same second moment. Our purpose now is to compute the asymptotic behaviour of the variance of that average fraction, defined by

$$\sigma_\nu^2 =<\nu^2> - <\nu>^2 . \tag{3.27}$$

From Eqs.(3.20) and (B.16) the exact form of the average number of roots lying on C for arbitrary N and equal variances is

$$<\nu>=\sqrt{\frac{1/3+1/N+2/(3N^2)}{1+1/N}} \simeq \frac{1}{\sqrt{3}}\left(1+\frac{1}{N}\right)+O(1/N^2) \tag{3.28}$$

and therefore

$$<\nu>^2 \simeq \frac{1}{3}+\frac{2}{3N}. \tag{3.29}$$

On the other hand,

$$\begin{aligned} <\nu^2> &= \tfrac{1}{N^2}\int_0^{2\pi}\int_0^{2\pi} <\rho(\theta)\rho(\theta')> d\theta\, d\theta' \\ &= \tfrac{1}{N^2}\left[\int_0^{2\pi} <\rho(\theta)> d\theta + \int_0^{2\pi}\int_0^{2\pi} R_2(\theta,\theta')d\theta\, d\theta'\right] \\ &\simeq \tfrac{1}{N^2}\left[\tfrac{N}{\sqrt{3}} + 2\int_0^{2\pi}(2\pi-\tau)\, R_2(\tau)d\tau\right] + O(1/N^2) \end{aligned} \tag{3.30}$$

where we have replaced the average number of roots on C by its asymptotic value $N/\sqrt{3}$ and we have also exploited the fact that $R_2(\theta,\theta')$ depends only on the difference τ between θ and θ' to express the double integral as a simple one. Because $R_2(\tau)$ is symmetric with respect to $\tau=\pi$, Eq.(3.30) can be rewritten as

$$< \nu^2 > \simeq \frac{1}{\sqrt{3}N} + \frac{4\pi}{N^2}\int_0^\pi R_2(\tau)\,d\tau$$

or, again normalizing $R_2(\tau)$ to the square of the asymptotic mean density $\tilde{R}_2(\tau) = R_2(\tau)/(N/2\pi\sqrt{3})^2$

$$< \nu^2 > \simeq \frac{1}{\sqrt{3}N} + \frac{1}{3\pi}\int_0^\pi \tilde{R}_2(\tau)\,d\tau. \tag{3.31}$$

As shown in Fig.3, in the large-N limit $\tilde{R}_2(\tau)$ has some oscillations on a scale $\tau \sim O(1/N)$ and tends to one for larger values of τ. In order to separate the contribution to the integral in Eq.(3.31) from the oscillatory part of $\tilde{R}_2(\tau)$, we rewrite it in the form

$$\int_0^\pi \tilde{R}_2(\tau)\,d\tau = \frac{2\pi}{N}\int_0^q \tilde{R}_2(\delta,N)\,d\delta + \int_{2\pi q/N}^\pi \tilde{R}_2(\tau,N)\,d\tau\ , \tag{3.32}$$

where $\delta = \tau N/2\pi$ and q is a parameter which is large compare to one but much smaller than N. In Eq.(3.32) we have explicitly indicated the N-dependence of R_2. Asymptotically (cf appendix B)

$$\begin{aligned}\tilde{R}_2(\delta,N) &\rightarrow \tilde{R}_2(\delta) + O(1/N)\\ \tilde{R}_2(\tau,N) &\rightarrow 1 + 2/N + O(1/N^2)\end{aligned}$$

where $\tilde{R}_2(\delta)$ is given by Eq.(3.25). Therefore, keeping terms up to order $1/N$

$$\int_0^\pi \tilde{R}_2(\tau)\,d\tau \simeq \frac{2\pi}{N}\int_0^q \tilde{R}_2(\delta)\,d\delta + \pi - \frac{2\pi q}{N} + \frac{2\pi}{N} = \pi + \frac{2\pi}{N}\left(1 - \int_0^q \left[1 - \tilde{R}_2(\delta)\right]\,d\delta\right). \tag{3.33}$$

Because $\tilde{R}_2(\delta)$ tends to one as $\delta \rightarrow \infty$, we are allowed to take the limit $q \rightarrow \infty$ in the integral.

Collecting Eqs.(3.33), (3.31), (3.29) and (3.27) we finally get

$$\sigma_\nu^2 \simeq \left(\frac{1}{\sqrt{3}} - \frac{2}{3}\Delta\right)\frac{1}{N}, \tag{3.34}$$

with $\Delta = \int_0^\infty \left[1 - \tilde{R}_2(\delta)\right]\,d\delta$. We were not able to compute analytically this integral, and made instead a numerical calculation. We obtained $\Delta \simeq 0.44733$.

4 $SU(2)$ Polynomials

4.1 A theorem concerning the ergodicity of wave-functions

In the previous section we were mainly concerned with the problem of locating on C as many as possible of the roots of a random polynomial and studying their correlations. This problem concerns in particular the spectral statistics of asymptotic approximations of chaotic systems. In this section we explore a somewhat different and in some sense opposite problem: polynomials whose roots distribute uniformly (at least as $N \to \infty$) on certain surfaces. Our motivation is related to the asymptotic $\hbar \to 0$ structure of quantum eigenstates of classically chaotic systems.

Consider a system characterized by its total angular momentum $\vec{J} = (J_x, J_y, J_z)$ whose modulus J is conserved by the dynamics. The motion of the arrow $\vec{J}$ in the three-dimensional space can be represented by a point moving on the surface of a two-dimensional sphere, a Riemann sphere denoted $\mathcal{S}$, which is in fact the phase space of the system.

The equations of motion for $\vec{J}$ are such that the representative point is assumed to move on $\mathcal{S}$ in a chaotic way. This is possible if the angular momentum $\vec{J}$ is coupled to some external time-dependent field, typically a magnetic field. The simplest case is a periodic time-dependence, which we henceforth assume. Integrating the equations of motion of $\vec{J}$ over one period of the field, the classical dynamics for the point moving on the surface of the sphere reduces to a discrete map M acting on $\mathcal{S}$

$$\vec{J}^{(n+1)} = M\left(\vec{J}^{(n)}\right) . \tag{4.1}$$

These equations determine the position of the arrow at time $t = n + 1$ knowing its position at time $t = n$.

The quantization of such a map introduces a one-period evolution operator U, the analog of the classical map M

$$|\psi^{(n+1)}> = U|\psi^{(n)}> \ , \tag{4.2}$$

where $|\psi^{(n)}>$ defines the quantum state of the system at time $t = n$. Because the modulus of $\vec{J}$ is conserved, then $[U, \vec{J}^2] = 0$ and the Hilbert space is finite and $(2J+1)$-dimensional. We can choose as a basis of that space the eigenstates of J_z, $J_z|m> = \hbar m|m>$, $m = -J, -J+1, \ldots, J$. In particular, the eigenstates of the unitary operator U

$$U|\psi_\alpha> = e^{i\omega_\alpha}|\psi_\alpha>, \qquad \alpha = 1, \ldots, 2J+1 \tag{4.3}$$

can be written as

$$|\psi_\alpha> = \sum_{m=-J}^{J} a_m^{(\alpha)}|m> \ .$$

The classical limit of such models corresponds to $N = 2J \to \infty$. For convenience we normalize the radius of the sphere, given by $\hbar\sqrt{J(J+1)}$, to one.

In order to have a unified semiclassical framework for both classical and quantum mechanics, it is convenient to introduce a phase-space representation of the Hilbert space. For that purpose, we project the eigenstates $|\psi_\alpha>$ into $SU(2)$ coherent-states $|z>$ [21, 22], $\psi_\alpha(z) = < z|\psi_\alpha>$ with the result (we drop from now on the subscript α)

$$\psi(z) = \sum_{k=0}^{N} \sqrt{C_N^k}\, a_k\, z^k, \tag{4.4}$$

where $N = 2J$, C_N^k are the binomial coefficients and where we have shifted to the new label $k = m + J$.

The complex variable z labeling the coherent states and appearing in the polynomial (4.4) is connected to the variables (θ, ϕ) spanning the Riemann sphere by a stereographic projection of the plane onto the sphere, $z = \cot(\theta/2)\, e^{i\phi}$. The function $\psi(z)$ is therefore an analytic function defined on the two-dimensional sphere. Because it is a polynomial of degree N, it has N *zeros* in that space which completely determine (up to a global normalizing factor) the quantum state.

Our purpose here is to analyze the structure of the eigenstates of U in a regime where the classical dynamics is dominated by chaotic trajectories; this means that classically the iteration of a typical initial point covers in time the entire two-dimensional sphere $\mathcal{S}$ in a more or less uniform way. According to the correspondence principle, in the semiclassical limit the quantum states of such a system must tend, at least in a weak sense, to the microcanonical density [23]. The simplest asymptotic realization would be a function $\psi(z)$ whose modulus is uniform over $\mathcal{S}$. This, however, is not an allowed solution since $\psi(z)$ has to have N zeros in $\mathcal{S}$, and therefore cannot be uniform. Moreover, the number of zeros proliferates in the semiclassical limit. The closest approximation to a uniform density would then be a function $\psi(z)$ whose zeros spread all over $\mathcal{S}$. This behaviour was in fact already observed in Ref.[15] for eigenstates of chaotic systems. In the following, we make a precise statement concerning the ergodicity of the distribution of zeros for such systems. Assuming that the coefficients $\{a_k\}$ in Eq.(4.4) are GRI-distributed, we prove that the zeros of $\psi(z)$ are indeed spread all over $\mathcal{S}$ and that their distribution is moreover *uniform*.

Our assumption concerning the coefficients $\{a_k\}$ is motivated by the random matrix theory. As is well known, the statistical properties of the spectrum of classically chaotic systems are well described, in the universal regime, by the results of the RMT [24, 13]. Much less explored are the eigenstates of such systems, *i.e.* the statistical properties of the coefficients $\{a_k\}$ for chaotic systems and how they compare to the RMT (some results concerning this problem can be found in [25]). The invariance under unitary transformations of the GUE ensemble of random matrices implies that the joint distribution function for the amplitudes must be

$$\mathcal{D}_{RMT}(\vec{a}) = \frac{1}{|S_{2(N+1)}|}\delta\left[1 - \sum_{k=0}^{N} |a_k|^2\right]$$

where $|S_n| = 2\pi^{n/2}/\Gamma(n/2)$ is the surface of a $(n-1)$-dimensional sphere of unit radius. When computing average properties of the zeros of $\psi(z)$, this distribution is strictly equivalent to a GRI distribution (see ref. [5], p.6)

$$\mathcal{D}(a_k) = \frac{1}{(2\pi)^N} \exp\left\{-\frac{1}{2}\sum_{k=0}^{N} |a_k|^2\right\} . \tag{4.5}$$

Thus, the amplitudes $\{a_k\}$ in RMT turn out to be gaussian uncorrelated random variables.

Assuming this distribution function, our purpose now is to compute the associated average density of zeros on $\mathcal{S}$, $< \rho(\theta, \varphi) >$, and compare it to the "ergodic" distribution conjectured above. Expressed in terms of the complex variable z, the density of zeros can be written (see appendix C)

$$\rho(z) = \delta[\psi(z)] \left|\frac{d\psi}{dz}\right|^2 . \tag{4.6}$$

The next step in the computation would then be to exponentiate both terms in the r.h.s. of Eq.(4.6) and compute the ensemble average over the coefficients. We find, however, that it is not necessary to exponentiate the Jacobian $|d\psi/dz|^2$. From Eqs.(4.4) and (4.6), using the exponential expression of the delta function and computing the ensemble average we find, for *arbitrary* N, the result

$$< \rho(z) > d^2 z = \frac{N}{\pi} \frac{d^2 z}{(1+|z|^2)^2} = \frac{N}{4\pi} \sin\theta \, d\theta d\phi \tag{4.7}$$

i.e., a uniform distribution of the zeros on the Riemann sphere. The proof of Eq.(4.7) is given in appendix C.

The result (4.7) constitutes a precise statement concerning the ergodicity of the (zeros of) eigenstates of chaotic systems, and makes a connection between the concept of ergodicity and the RMT. Since the zeros are genuine wavefunction parameters, their equidistribution constitutes a stronger statement than just the limiting ergodicity of the Husimi function (which is a smooth quantity and bilinear in ψ).

As already mentioned, Eq.(4.7) holds for arbitrary N. In particular in the extreme case $N = 1$ ($J = 1/2$) it means that the single root $z_1 = -a_0/a_1$ of the monomial $\psi(z) = a_0 + a_1 z$ is uniformly distributed on the Riemann sphere if a_1 and a_0 are complex variables having a GRI distribution and the same second moment. This result can be directly checked from Eq.(2.3),

$$\mathcal{D}(a_0, a_1) d^2a_0 d^2a_1 = \frac{1}{(2\pi)^2\sigma^4} \exp\left\{-\frac{1}{2\sigma^2}\left[|a_0|^2 + |a_1|^2\right]\right\} d^2a_0 d^2a_1,$$

$$\Rightarrow \quad \mathcal{D}(z_1, a_1) d^2z_1 d^2a_1 = \frac{1}{(2\pi)^2\sigma^4} \exp\left\{-\frac{|a_1|^2}{2\sigma^2}\left[1 + |z_1|^2\right]\right\} |a_1|^2 d^2z_1 d^2a_1 .$$

Integrating over a_1 we find

$$\mathcal{D}(z_1) = < \rho(z_1) > = \frac{1}{\pi} \frac{1}{(1+|z_1|^2)^2}$$

in agreement with (4.7). A polynomial like (4.4) generalizes this result to arbitrary N. The validity of the equidistribution of roots for arbitrary N is however related to the gaussian nature of the distribution of the coefficients. We have numerically tested other distributions. For example, we find that for a uniform distribution of the coefficients in the interval [-1,1], the distribution of roots tends to be uniform only for large values of N.

To conclude this subsection, let us mention that correlations between roots of random SU(2) polynomials were recently studied in [26] and compare to results obtained from quantum chaotic systems. Moreover, the general k-point correlation functions were also recently computed analytically by J. Hannay [27].

4.2 A physical example

In order to illustrate the theorem (4.7) we consider a kicked-spin model, classically defined by the Hamiltonian

$$H = \frac{p}{2} J_z^2 + \mu J_x \sum_{n=-\infty}^{\infty} \delta(t-n)$$

where μ and p are constant parameters. Integrating over a period $\Delta t = 1$ the equations of motion take the form of a discrete map

$$\vec{J}^{(n+1)} = R_x(\mu)\, R_z\left(pJ_z^{(n)}\right) \vec{J}^{(n)}. \tag{4.8}$$

$R_i(\lambda)$ represents a rotation around the i-th axis by an angle λ, and $\vec{J} = (J_x, J_y, J_z)$. The quantum mechanical analog of Eq.(4.8) is the one-step unitary operator

$$U = \mathrm{e}^{-\frac{i}{\hbar}\mu J_x} \mathrm{e}^{-\frac{i}{\hbar}\frac{p}{2}J_z^2} \tag{4.9}$$

acting on a $(2J+1)$-dimensional Hilbert space. The stationary equation (4.3) determines the eigenphases ω_α and eigenstates $|\psi_\alpha >$ of U. According to the result Eq.(4.7), for parameters (μ, p) for which the classical map (4.8) is dominated by chaotic trajectories we expect the zeros of the polynomials (4.4) associated with the eigenstates of U to be uniformly distributed over the Riemann sphere. Before checking this, let us briefly mentioned some properties of U.

The operator U is not generic since it has two symmetries: it commutes with two antiunitary operators [28]

$$T_1 = e^{i\pi J_z} \mathrm{e}^{i\mu J_x} K, \qquad T_2 = \mathrm{e}^{-i\mu J_x} \mathrm{e}^{i\pi J_y} K,$$

where K is the usual antiunitary complex conjugation operator. These operators satisfy $T_1^2 = T_2^2 = 1$ and the time reversal property $T_1 U T_1 = T_2 U T_2 = U^{-1}$. These two symmetries are nongeneric in the sense that (a) they are not just the conjugation operator usually connected to time-reversal invariance and (b) they depend on the parameter μ controlling, together with p, the dynamics of the system.

Because $T_i^2 = 1$, the existence of an operator T_i commuting with U implies that the eigenstates of U can always be chosen T-invariant, $T|\psi_\alpha> = |\psi_\alpha>$. In the coherent-state representation, this latter equation imposes a functional equation on each polynomial $\psi_\alpha(z)$

$$\overline{\psi_\alpha(\tau_i(z))} = \psi_\alpha(z) \qquad i = 1,2 \; ; \qquad \alpha = 1, \ldots, 2J+1 \; , \tag{4.10}$$

where τ_i, $i = 1,2$ are the classical versions of the quantum operators T_i which associates to each point of the classical phase space z an image $\tau_i(z)$. Each of these transformations has a symmetry line, defined by the set of points z satisfying $\tau_i(z) = z$. In terms of the canonical conjugate variables $(\phi, \cos\theta)$ spanning the sphere, the two symmetry lines are given by

$$\cos\theta = \pm \frac{\sin\phi}{\sqrt{\sin^2\phi + \frac{1\mp\cos\mu}{1\pm\cos\mu}}} \; , \tag{4.11}$$

where the upper and lower sign holds for the T_1 and T_2 symmetry, respectively.

Because of (4.10), if z_k is a root of $\psi(z)$, then $\tau_i(z_k)$ will also be a root. According to the results of section 3, if the roots are symmetric with respect to a line and if the coefficients of the polynomial are random, we expect a concentration of roots over that line. In the present case, we have two symmetry lines. Figure 6a shows the superposition of the 60 roots of the 61 eigenstates obtained by a numerical diagonalization of (4.9) for $J = 30$, $\mu = 1$ and $p = 4\pi$, which classically looks fully chaotic [16]. We observe the expected concentration of roots over the two symmetry lines, a free-of-roots region close to them, and a tendency to cover in a more or less uniform way the remaining phase space (see ref.[26] for more details). Although we haven't computed analytically, we suspect that the concentration of roots over those lines is not macroscopic (i.e., asymptotically tends to zero), like the number of real roots of random polynomials having real coefficients. In this context, let us mention that it has been shown recently [19] that the asymptotic fraction of real roots of an SU(2) polynomial having real coefficients is $1/\sqrt{N}$. This should be compare to the $\log N/N$ fraction valid for the original problem proposed by Kac.

In order to break both antiunitary symmetries T_1 and T_2 we add an extra term in the propagator

$$U = \mathrm{e}^{-\frac{it}{2\hbar}J_x^2}\mathrm{e}^{-\frac{i}{\hbar}\mu J_x}\mathrm{e}^{-\frac{i}{2\hbar}pJ_z^2} . \tag{4.12}$$

Figure 6b and 6c show the superposition of the roots of the 61 eigenstates for the same p and μ as in Fig.6a for $t = 1$ and $t = 6$, respectively. After a transition regime where the symmetry lines are still observed (even though the symmetry has been broken), for $t = 6$ the roots spread in a more or less uniform way over the whole phase space, as predicted in Eq.(4.7).

Notice that although in order to have better statistics we have superimposed in the figures the roots of all the eigenstates, the ergodic theorem (4.7) holds for *individual* eigenstates. However, a direct numerical test based on a single eigenstate would need much higher values of J.

For other systems having different phase spaces, it is natural to expect – guided by semiclassical intuition – a result equivalent to Eq.(4.7), at least in the semiclassical limit. And indeed a related result was recently found [29] for the usual Bargmann representation [30] of quantum mechanics. This is a representation of Hilbert space in terms of entire functions

$$\psi(z) = \sum_{k=0}^{\infty} \frac{a_k}{\sqrt{n!}} z^k. \tag{4.13}$$

The associated phase space is the two-dimensional plane labelled by the canonical variables (q,p), and $z = (q - ip)/\sqrt{2}$. It was shown [29] that if the coefficients $\{a_k\}$ in Eq.(4.13) are GRI-distributed with all the second moments equal, the zeros of $\psi(z)$ are uniformly distributed over the whole plane. Moreover, if the sum in Eq.(4.13) is truncated at a finite value N, then the density of roots is uniform inside a circle of radius $\sqrt{N}$, and tends to zero outside.

5 Concluding remarks

We have studied the distribution and correlation of roots of random polynomials under several conditions. In section 3, we have explored, motivated by the problem of the semiclassical spectral properties of chaotic systems, different ways to increase the number of roots of a random polynomial located on $\mathcal{C}$, and studied correlations between the roots. If all the coefficients of the characteristic polynomial have the same standard deviation, then asymptotically the self-inversive symmetry locates a fraction $1/\sqrt{3}$ of the roots on $\mathcal{C}$ with a variance inversely proportional to the degree of the polynomial (section 3.3). The two-point correlation function of these zeros behaves linearly for short distances (Fig.3). However, if the standard deviations are not equal for all the coefficients but instead are given by $\sigma_k = k^{-1/2}$, then the fraction of roots lying on $\mathcal{C}$ tends to one as $N \to \infty$. But the convergence is slow (logarithmic), and the two-point correlation function much more crystalline-like (Fig.5). Surprisingly, and unlike the case of standard polynomials with real coefficients (the problem of Kac), we were able to compute explicitly the *exact* fraction of roots lying on C and their two-point correlation function for arbitrary N and arbitrary variances.

To improve by less artificial means the number of roots lying on $\mathcal{C}$ we need to incorporate additional correlations between the coefficients. However no simple procedure exists, and the exact conditions for all the roots of the characteristic polynomial to lie on $\mathcal{C}$ are certain complicated determinantal inequalities for the coefficients (see e.g. [4]).

The self-inversive property arises naturally in other problems of physics, as in the case of certain Ising models in statistical mechanics where the partition function takes a polynomial form when written in terms of the fugacity [31]. There, the SI-symmetry is connected to a spin-up spin-down (or particle-hole) symmetry of the system. As shown in [31], under certain assumptions all the necessary correlations implying unitarity are present and as a consequence all the roots of the partition function lie on $\mathcal{C}$. In Ref. [32] the reader can find additional examples and references concerning the distribution of roots of partition functions in connection with the theory of phase transitions.

Another famous example of concentration of the roots of a certain function on some simple curve in the complex plane is provided by the Riemann zeta-function $\zeta(z)$. This function, which has no explicit random parameters entering its definition, satisfies the functional equation $\xi(1-z)=\xi(z)$, where $\xi(z)=\pi^{-z/2}\Gamma(z/2)\zeta(z)$. Accordingly, its roots are symmetric with respect to the critical line $\Re e(z)=1/2$. The Riemann hypothesis asserts that all the nontrivial roots of $\zeta(z)$ lie on that line.

In section 4 we have shown that if we assume for the coefficients $\{a_k\}$ a GRI distribution with second moments equal to $\sqrt{C_N^k}$ then the roots of the eigenstates of a classically chaotic spin system, Eq.(4.4), are uniformly distributed over the phase space (in this case, the two-dimensional sphere).

Appendix A

In this appendix we compute the average density and average fraction of roots lying on $\mathcal{C}$ for a self-inversive polynomial of the form

$$P(z) = 1 + \sum_{k=1}^{N-1} a_k z^k + z^N \ , \qquad a_{N-k} = \bar{a}_k \tag{A.1}$$

for arbitrary N. The coefficients a_k are assumed to be complex independent variables having a Gaussian distribution

$$\mathcal{D}(a_1, \ldots, a_M) = \frac{1}{(2\pi)^M \prod \sigma_k^2} \exp\left(-\frac{1}{2}\sum_{k=1}^{M} |a_k|^2/\sigma_k^2\right) \ , \tag{A.2}$$

where $M = (N-1)/2$ (N is an odd integer). By the substitution $z = \exp(\mathrm{i}\theta)$, the problem reduces to the computation of the average density and average number of roots of a real function in the interval $0 \leq \theta < 2\pi$ (cf. Eq.(3.9))

$$f(\theta) = \frac{1}{2}\mathrm{e}^{-\mathrm{i}N\theta/2} P(\exp(\mathrm{i}\theta)) = \cos\left(\frac{N}{2}\theta\right) + \sum_{k=1}^{M}\left\{c_k \cos\left[\left(\frac{N}{2}-k\right)\theta\right] + d_k \sin\left[\left(\frac{N}{2}-k\right)\theta\right]\right\} \tag{A.3}$$

where $c_k = \Re e(a_k)$ $d_k = \Im m(a_k)$. The derivative of this function with respect to θ is

$$f'(\theta) = -\frac{N}{2}\sin(\frac{N}{2}\theta) - \sum_{k=1}^{M}\left\{\left(\frac{N}{2}-k\right)c_k \sin\left[\left(\frac{N}{2}-k\right)\theta\right] - \left(\frac{N}{2}-k\right)d_k \cos\left[\left(\frac{N}{2}-k\right)\theta\right]\right\} \ . \tag{A.4}$$

The density of roots of $f(\theta)$ is defined by

$$\rho(\theta) = \delta[f(\theta)]\,|f'(\theta)|$$

and using the Kac's representation (3.12) for $\delta[f]$ and $|f'|$ we get

$$\rho(\theta) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} d\xi\, \mathrm{e}^{\mathrm{i}\xi f(\theta)} \int_{-\infty}^{\infty} d\eta \frac{1-\mathrm{e}^{\mathrm{i}\eta f'(\theta)}}{\eta^2} \ . \tag{A.5}$$

In order to compute the average of $\rho(\theta)$ over the ensemble (A.2), we proceed in the following way. We first replace $f(\theta)$ and $f'(\theta)$ in (A.5) by its definition Eqs.(A.3) and (A.4). Then we average $\rho(\theta)$ over the coefficients c_k and d_k (it can be shown that the order of integration can be interchanged; see [kac]),

$$<\rho(\theta)>= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} d^2a_1 \ldots d^2a_M \rho(\theta)\mathcal{D}(a_1, \ldots, a_M) \ . \tag{A.6}$$

The final step is to integrate over η and ξ.

1. We will need the average of $e^{\mathrm{i}\xi f} e^{\mathrm{i}\eta f'}$. From (A.3) and (A.4) we have

$$\xi f + \eta f' = \xi \cos\left(\frac{N}{2}\theta\right) - \frac{N}{2}\eta\sin\left(\frac{N}{2}\theta\right) + \sum_{k=1}^{M}(s_k c_k + t_k d_k) \ , \tag{A.7}$$

where

$$s_k = \xi \cos\left[\left(\tfrac{N}{2} - k\right)\theta\right] - (\tfrac{N}{2} - k)\eta \sin\left[\left(\tfrac{N}{2} - k\right)\theta\right]$$
$$t_k = \xi \sin\left[\left(\tfrac{N}{2} - k\right)\theta\right] + (\tfrac{N}{2} - k)\eta \cos\left[\left(\tfrac{N}{2} - k\right)\theta\right] . \tag{A.8}$$

From these equations and the fact that

$$\frac{1}{\sqrt{2\pi\sigma_k^2}} \int_{-\infty}^{+\infty} e^{-a_k^2/(2\sigma_k^2)+b_k a_k} \, da_k = e^{b_k^2\sigma_k^2/2} \tag{A.9}$$

we get from (A.6)

$$< e^{i\xi f} e^{i\eta f'} >= \exp\left\{i\left[\xi \cos\left(\frac{N}{2}\theta\right) - \frac{N}{2}\eta \sin\left(\frac{N}{2}\theta\right)\right]\right\} \exp\left\{-\frac{1}{2}\sum_{k=1}^{M}(s_k^2 + t_k^2)\sigma_k^2\right\} .$$

But $s_k^2 + t_k^2 = \xi^2 + (N/2 - k)^2 \eta^2$. Then defining

$$g_1 = \sum_{k=1}^{M} \sigma_k^2$$
$$g_2 = \sum_{k=1}^{M} \left(\frac{N}{2} - k\right)^2 \sigma_k^2 \tag{A.10}$$

we can write

$$< e^{i\xi f} e^{i\eta f'} >= \exp\left[-\frac{1}{2}(g_1\xi^2 + g_2\eta^2)\right] \exp\left\{i[\cos\left(\frac{N}{2}\theta\right)\xi - \frac{N}{2}\sin\left(\frac{N}{2}\theta\right)\eta]\right\} . \tag{A.11}$$

Note that the functions g_1 and g_2 contain all the information concerning the variances of the random coefficients. For $\eta = 0$, the result is

$$< e^{i\xi f} >= \exp\left(-\frac{1}{2}g_1\xi^2\right) \exp\left[i\cos(\frac{N}{2}\theta)\,\xi\right] . \tag{A.12}$$

Then from (A.11), (A.12) and (A.5)

$$< \rho(\theta) >= \frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\xi \, e^{-\frac{1}{2}g_1\xi^2 + i\cos(\frac{N}{2}\theta)\xi} \int_{-\infty}^{\infty} \frac{d\eta}{\eta^2}\left(1 - e^{-\frac{1}{2}g_2\eta^2 - i\frac{N}{2}\sin(\frac{N}{2}\theta)\eta}\right) . \tag{A.13}$$

2. The next step is to evaluate the integrals in (A.13). The integral over η can be written

$$I(\beta) = \int_{-\infty}^{\infty} \frac{d\eta}{\eta^2}\left(1 - e^{-\beta\eta^2} e^{-i\frac{N}{2}\sin(\frac{N}{2}\theta)\eta}\right)$$

where $\beta = g_2/2$. Since

$$\frac{\partial I}{\partial \beta} = \int d\eta \, e^{-\beta\eta^2} e^{-i\frac{N}{2}\sin(\frac{N}{2}\theta)\eta} = \sqrt{\frac{\pi}{\beta}}\, e^{-\frac{N^2}{16}\sin^2(\frac{N}{2}\theta)/\beta}$$

and, from (3.12)

$$I(\beta = 0) = \int_{-\infty}^{\infty} \frac{d\eta}{\eta^2}\left(1 - e^{-i\frac{N}{2}\sin(\frac{N}{2}\theta)\,\eta}\right) = \frac{\pi N}{2}\left|\sin\left(\frac{N}{2}\theta\right)\right| ,$$

then

$$I(\beta)=\frac{\pi N}{2}\left|\sin\left(\frac{N}{2}\theta\right)\right|+\sqrt{\pi}\int_0^{\beta}\frac{dy}{\sqrt{y}}e^{-\frac{N}{16}\sin^2(\frac{N}{2}\theta)/y} . \tag{A.14}$$

Moreover, the integral over ξ in (A.13) is

$$\int_{-\infty}^{\infty} d\xi e^{-\frac{1}{2}g_1\xi^2+i\cos(\frac{N}{2}\theta)\xi}=\sqrt{\frac{2\pi}{g_1}}e^{-\cos^2(\frac{N}{2}\theta)/2g_1} . \tag{A.15}$$

Doing the change of variable $x=\sqrt{y/\beta}$ in the integral of Eq.(A.14), the final result for the average density of zeros on $\mathcal{C}$ is

$$<\rho(\theta)>=\frac{e^{-\cos^2(\frac{N}{2}\theta)/2g_1}}{\sqrt{2\pi g_1}}\left\{\frac{N}{2}\left|\sin\left(\frac{N}{2}\theta\right)\right|+2\sqrt{\frac{g_2}{2\pi}}\int_0^1 dx\, e^{-N^2\sin^2(\frac{N}{2}\theta)/(8g_2x^2)}\right\} . \tag{A.16}$$

3. The average fraction of zeros lying on $\mathcal{C}$ is defined as

$$<\nu>=\frac{1}{N}\int_0^{2\pi}<\rho(\theta)>d\theta . \tag{A.17}$$

The integral involving the first term between curly brackets in (A.16) can be rewritten (by an obvious change of variables)

$$\int_0^{2\pi} e^{-\cos^2(\frac{N}{2}\theta)/(2g_1)}\left|\sin\left(\frac{N}{2}\theta\right)\right|d\theta=\frac{2}{N}\int_0^{N\pi}d\varphi\, e^{-\cos^2(\varphi)/(2g_1)}|\sin\varphi| .$$

The function to be integrated is periodic of period π, and since $\sin\varphi$ is a positive function in that interval the previous equation takes the form

$$2\int_0^{\pi}d\varphi\, e^{-\frac{1}{2g_1}\cos^2\varphi}\sin\varphi=2\sqrt{g_1}\int_{-1/\sqrt{g_1}}^{1/\sqrt{g_1}}dye^{-y^2/2} . \tag{A.18}$$

The integral over the second term between curly brackets in (A.16) can be written, putting $\varphi=N\theta/2$ and by the same argument as before

$$\int_0^1 dx\int_0^{2\pi}d\theta\, e^{-\left[\cos^2(\frac{N}{2}\theta)/g_1+N^2\sin^2(\frac{N}{2}\theta)/(4g_2x^2)\right]/2}=2\int_0^1 dx\int_0^{\pi}d\varphi\exp\left\{-\left[\frac{cos^2\varphi}{g_1}+\frac{N^2}{4g_2}\frac{\sin^2\varphi}{x^2}\right]/2\right\} . \tag{A.19}$$

Putting together Eqs.(A.16)-(A.19), the fraction of zeros lying on $\mathcal{C}$ is given by

$$<\nu>=\frac{1}{\sqrt{2\pi}}\int_{-1/\sqrt{g_1}}^{1/\sqrt{g_1}}dy\, e^{-y^2/2}+\frac{2}{\pi N}\sqrt{\frac{g_2}{g_1}}\int_0^1 dx\int_0^{\pi}d\varphi\, e^{-\left(\frac{1}{2g_1}\cos^2\varphi+\frac{N}{8g_2}\frac{\sin^2\varphi}{x^2}\right)} . \tag{A.20}$$

Both results (A.16) and (A.20) are valid for arbitrary N and arbitrary variances $\sigma_k^2, k=1,\ldots,M$. Moreover, both expressions are the sum of two terms: the first one in each expression is related to the term $1 + z^N$ in (A.1), while the second comes from the random part of the polynomial. In the case where all the variances are equal, $\sigma_k^2 = \sigma^2 \; \forall k$, then the coefficients g_1 and g_2 reduce to

$$\begin{aligned} g_1 &= \sigma^2 \sum_{k=1}^{M} 1 = \sigma^2 (N-1)/2 \\ g_2 &= \sigma^2 \sum_{k=1}^{M} (\tfrac{N}{2} - k)^2 = \sigma^2 \left(\tfrac{N^3}{24} - \tfrac{N^2}{8} + \tfrac{N}{12} \right) . \end{aligned} \tag{A.21}$$

In the limit $N \to \infty$, $g_1 \simeq \sigma^2 N/2 = \epsilon^2/2$, $g_2 \simeq \sigma^2 N^3/24 = \epsilon^2 N^2/24$, where we have introduced the scaled parameter $\epsilon = \sigma\sqrt{N}$. Replacing these asymptotic expressions for g_1 and g_2 in (A.16) and (A.20), we find that $< \rho(\theta) >$ and $< \nu >$ depend only on the rescaled parameter ϵ, and are given by Eqs.(3.14) and (3.15), respectively.

On the other hand, when $|\sigma_k| >> 1 \; \forall k$, then g_1 and $g_2 \to \infty$ and we can ignore the first term in (A.20). In this case (the pure random limit, which corresponds to neglect the term $z^N + 1$ in $P(z)$), the fraction of roots lying on $\mathcal{C}$, Eq.(A.20), reduces to Eqs.(3.20) which in turn simplifies to Eq.(3.17) if all the σ's are equal and if we keep only the leading term in $1/N$. Moreover, the average density of zeros tends to

$$< \rho(\theta) > \to \frac{1}{\pi} \sqrt{\frac{g_2}{g_1}} \, . \tag{A.22}$$

Appendix B

We compute here the average two-point correlation function

$$R_2(\tau) = < \rho(\theta) \rho(\theta + \tau) > \tag{B.1}$$

for the roots lying on $\mathcal{C}$ of a self-inverse random polynomial of the form

$$P(z) = \sum_{k=0}^{N} a_k \, z^k \, , \qquad a_{N-k} = \bar{a}_k \tag{B.2}$$

where the coefficients a_k , $k = 0, \ldots , \frac{N-1}{2} = M$ are complex independent variables having a Gaussian distribution (A.2) (we again assume for simplicity that N is odd; for even N, we must take $M = N/2$). Substituting $z = \exp(i\theta)$ in (B.2) and ignoring prefactors we end up with the real function

$$f(\theta) = \sum_{k=0}^{M} \left\{ c_k \cos \left[\left(\frac{N}{2} - k \right) \theta \right] + d_k \sin \left[\left(\frac{N}{2} - k \right) \theta \right] \right\} \tag{B.3a}$$

where $c_k = \Re e(a_k)$, $d_k = \Im m(a_k)$, and whose derivative is

$$f'(\theta) = - \sum_{k=0}^{M} \left\{ (\frac{N}{2} - k) c_k \sin \left[\left(\frac{N}{2} - k \right) \theta \right] - \left(\frac{N}{2} - k \right) d_k \cos \left[\left(\frac{N}{2} - k \right) \theta \right] \right\} . \tag{B.3b}$$

The function $f(\theta)$ has, as shown in appendix A, an average number $N/\sqrt{3}$ of zeros in the interval $0 \leq \theta < 2\pi$. Moreover, since $< \rho(\theta) >$ is in this case independent of θ (cf. Eqs.(A.22) and (3.18)), R_2 depends on τ but not on θ. Using the definition (3.11) of $\rho(\theta)$ and Eqs.(3.12), we can write the two-point correlation function as

$$\rho(\theta)\rho(\theta+\tau) = \frac{1}{4\pi^2} \iiint\int_{-\infty}^{\infty} d\xi_1 d\xi_2 \frac{d\eta_1}{\eta_1^2} \frac{d\eta_2}{\eta_2^2} e^{i\xi_1 f(\theta)} e^{i\xi_2 f(\theta+\tau)} \left(1 - e^{i\eta_1 f'(\theta)}\right) \left(1 - e^{i\eta_2 f'(\theta+\tau)}\right) \ . \quad \text{(B.4)}$$

We proceed as in appendix A. We first replace Eqs.(B.3) in (B.4) and compute the average over the coefficients (c_k, d_k) according to the definition (A.6). Thenceforth we evaluate the integrals over ξ_i and η_i.

1. We will need $< e^{i[\xi_1 f(\theta)+\xi_2 f(\theta+\tau)+\eta_1 f'(\theta)+\eta_2 f'(\theta+\tau)]} >$. From Eqs.(B.3)

$$\xi_1 f(\theta) + \xi_2 f(\theta+\tau) + \eta_1 f'(\theta) + \eta_2 f'(\theta+\tau) = \sum_{k=0}^{M} [(s_k + u_k)c_k + (t_k + v_k)d_k]$$

where

$$\begin{aligned} s_k &= \xi_1 \cos[(N/2-k)\,\theta] - (N/2-k)\,\eta_1 \sin[(N/2-k)\,\theta] \\ t_k &= \xi_1 \sin[(N/2-k)\,\theta] + (N/2-k)\eta_1 \cos[(N/2-k)\,\theta] \\ u_k &= \xi_2 \cos[(N/2-k)\,(\theta+\tau)] - (N/2-k)\,\eta_2 \sin[(N/2-k)\,(\theta+\tau)] \\ v_k &= \xi_2 \sin[(N/2-k)\,(\theta+\tau)] + (N/2-k)\,\eta_2 \cos[(N/2-k)\,(\theta+\tau)] \ . \end{aligned} \quad \text{(B.5)}$$

Averaging over the Gaussian ensemble gives

$$< e^{i[\xi_1 f(\theta)+\xi_2 f(\theta+\tau)+\eta_1 f'(\theta)+\eta_2 f'(\theta+\tau)]} >= \exp\left\{-\frac{1}{2}\sum_{k=0}^{M}[(s_k+u_k)^2 + (t_k+v_k)^2]\sigma_k^2\right\} \ .$$

But

$$\begin{aligned} (s_k+u_k)^2 + (t_k+v_k)^2 = \ & \xi_1^2 + \xi_2^2 + (N/2-k)^2(\eta_1^2+\eta_2^2) \\ & +2\,[\xi_1\xi_2 + (N/2-k)^2\eta_1\eta_2] \cos[(N/2-k)\,\tau] \\ & +2(N/2-k)(\xi_2\eta_1 - \xi_1\eta_2) \sin[(N/2-k)\,\tau] \ , \end{aligned}$$

which is independent of θ, as it should be. Then defining

$$\left\{ \begin{aligned} g_1 &= \sum_{k=0}^{M} \sigma_k^2 \\ g_2 &= \sum_{k=0}^{M} (N/2-k)^2 \sigma_k^2 \\ g_3 &= \sum_{k=0}^{M} \cos[(N/2-k)\tau]\sigma_k^2 \\ g_4 &= -\partial g_3/\partial\tau = \sum_{k=0}^{M} (N/2-k)\sin[(N/2-k)\tau]\sigma_k^2 \\ g_5 &= -\partial^2 g_3/\partial\tau^2 = \sum_{k=0}^{M} (N/2-k)^2 \cos[(N/2-k)\tau]\sigma_k^2 \end{aligned} \right. \quad \text{(B.6)}$$

we obtain the result

$$
\begin{aligned}
&< \exp\{\mathrm{i}[\xi_1 f(\theta) + \xi_2 f(\theta+\tau) + \eta_1 f'(\theta) + \eta_2 f'(\theta+\tau)]\} >= \\
&\quad \exp\{-[g_1(\xi_1^2+\xi_2^2) + g_2(\eta_1^2+\eta_2^2) + 2(g_3\xi_1\xi_2 + g_5\eta_1\eta_2) + 2g_4(\xi_2\eta_1 - \xi_1\eta_2)]/2\} \ .
\end{aligned}
\tag{B.7}
$$

From this expression the average of the different terms appearing in (B.4) can be evaluated, with the result

$$
\begin{aligned}
R_2(\tau) = \ & \tfrac{1}{4\pi^4} \int\!\!\int\!\!\int\!\!\int_{-\infty}^{\infty} d\xi_1 d\xi_2 \frac{d\eta_1}{\eta_1^2}\frac{d\eta_2}{\eta_2^2} e^{-[g_1(\xi_1^2+\xi_2^2)+2g_3\xi_1\xi_2]/2} \times \\
& \left\{1 - e^{-(g_2\eta_1^2+2g_4\xi_2\eta_1)/2} - e^{-(g_2\eta_2^2-2g_4\xi_1\eta_2)/2} + e^{-[g_2(\eta_1^2+\eta_2^2)+2g_5\eta_1\eta_2+2g_4(\xi_2\eta_1-\xi_1\eta_2)]/2}\right\} .
\end{aligned}
\tag{B.8}
$$

2. The next step is to compute the integrals in (B.8). The integrals over ξ_1 and ξ_2 are straightforward, since they involve exponentials of quadratic forms:

$$
R_2(\tau) = \frac{1}{2\pi^3\sqrt{g_1^2-g_3^2}} \int\!\!\int_{-\infty}^{\infty} \frac{d\eta_1}{\eta_1^2}\frac{d\eta_2}{\eta_2^2}\left[1 - \mathrm{e}^{-\frac{\alpha}{2}\eta_1^2} - \mathrm{e}^{-\frac{\alpha}{2}\eta_2^2} + \mathrm{e}^{-\frac{\alpha}{2}(\eta_1^2+\eta_2^2)}\mathrm{e}^{-\beta\eta_1\eta_2}\right] , \tag{B.9}
$$

where we have introduced

$$
\alpha = g_2 - \frac{g_1 g_4^2}{(g_1^2-g_3^2)} , \qquad \beta = g_5 - \frac{g_3 g_4^2}{(g_1^2-g_3^2)} . \tag{B.10}
$$

Consider now the integral

$$
I(\beta) = \int\!\!\int_{-\infty}^{\infty} \frac{d\eta_1}{\eta_1^2}\frac{d\eta_2}{\eta_2^2}\left[1 - \mathrm{e}^{-\frac{\alpha}{2}\eta_1^2} - \mathrm{e}^{-\frac{\alpha}{2}\eta_2^2} + \mathrm{e}^{-\frac{\alpha}{2}(\eta_1^2+\eta_2^2)}\mathrm{e}^{-\beta\eta_1\eta_2}\right] \tag{B.11}
$$

and write it in the form

$$
I(\beta) = I(\beta)|_{\beta=0} + \left.\frac{\partial I}{\partial\beta}\right|_{\beta=0}\beta + \int\!\!\int_0^{\beta} dy I(y) . \tag{B.12}
$$

From Eq.(A.15) evaluated at $\theta = 0$ we get the first term of the previous equation

$$
I(\beta)|_{\beta=0} = \int\!\!\int_{-\infty}^{\infty} \frac{d\eta_1}{\eta_1^2}\frac{d\eta_2}{\eta_2^2}\left(1 - e^{-\frac{\alpha}{2}\eta_1^2}\right)\left(1 - e^{-\frac{\alpha}{2}\eta_2^2}\right) = 2\pi\alpha . \tag{B.13}
$$

Moreover

$$
\frac{\partial I}{\partial\beta} = -\int\!\!\int_{-\infty}^{\infty} \frac{d\eta_1}{\eta_1}\frac{d\eta_2}{\eta_2}\mathrm{e}^{-\frac{\alpha}{2}(\eta_1^2+\eta_2^2)}\mathrm{e}^{-\beta\eta_1\eta_2} ,
$$

implying $\partial I/\partial\beta|_{\beta=0} = 0$ by antisymmetry. Furthermore,

$$
\frac{\partial^2 I}{\partial\beta^2} = \int\!\!\int_{-\infty}^{\infty} d\eta_1 d\eta_2 e^{-\frac{\alpha}{2}(\eta_1^2+\eta_2^2)}e^{-\beta\eta_1\eta_2} = \frac{2\pi}{\sqrt{\alpha^2-\beta^2}} ,
$$

and integrating twice

$$
\int_0^{\beta} dx \int_0^{x} dy \frac{2\pi}{\sqrt{\alpha^2-y^2}} = 2\pi\left[\beta\arcsin(\beta/\alpha) + \sqrt{\alpha^2-\beta^2} - \alpha\right] . \tag{B.14}
$$

We thus obtain, substituting in (B.12) the different terms

$$I(\beta) = 2\pi \left[\beta \arcsin\left(\frac{\beta}{\alpha}\right) + \sqrt{\alpha^2 - \beta^2}\right] .$$

Then the exact average two-point correlation function, valid for arbitrary N and arbitrary variances, is

$$R_2(\tau) = \frac{1}{\pi^2 \sqrt{g_1^2 - g_3^2}} \left[\beta \arcsin(\beta/\alpha) + \sqrt{\alpha^2 - \beta^2}\right] . \tag{B.15}$$

The sums (B.6) – needed in the computation of the coefficients α and β – can be evaluated explicitly if the variances are equal. We find

$$\begin{cases} g_1 = \sigma^2 (N+1)/2 \\ g_2 = \sigma^2 \left(N^3/24 + N^2/8 + N/12\right) \\ g_3 = \sigma^2 \sin\left[\tau(N+1)/2\right] / \left[2\sin(\tau/2)\right] \\ g_4 = -\frac{\sigma^2}{4\sin(\tau/2)} \left\{N \cos\left[\tau(N+1)/2\right] - \frac{\sin(\tau N/2)}{\sin(\tau/2)}\right\} \\ g_5 = \frac{\sigma^2}{4} \left\{\frac{\sin(\tau N/2)\cos(\tau/2)}{\sin(\tau/2)} \left[\frac{N^2}{2} - \frac{1}{\sin^2(\tau/2)}\right] + N\cos(\tau N/2)\left[\frac{N}{2} + \frac{1}{\sin^2(\tau/2)}\right]\right\} \end{cases} \tag{B.16}$$

In this latter case, taking the limit $N \to \infty$, $\tau \to 0$, $N\tau \to$ constant, and normalizing R_2 to the square of the asymptotic mean density (3.18), Eqs.(B.15), (B.16) and (B.10) can be rewritten in the form (3.25)-(3.26).

Appendix C

We prove here the theorem Eq.(4.7), stating that the average density of roots of a polynomial of the form

$$\psi(z) = \sum_{k=0}^{N} \sqrt{C_N^k}\, a_k\, z^k \tag{C.1}$$

is *uniform* over the Riemann sphere for arbitrary N. In Eq.(C.1) the coefficients a_k, $k = 0, \ldots, N$ are assumed to be complex independent variables having a gaussian distribution with the same standard deviation $\sigma_k = \sigma\ \forall k$, and the C_N^k are the binomial coefficients $C_N^k = N!/k!(N-k)!$.

We want to compute the density of zeros of a complex function in the complex plane. By definition

$$\rho(z) = \delta[\Re e\,(\psi(z))]\,\delta[\Im m\,(\psi(z))] \begin{vmatrix} \frac{\partial \Re e(\psi)}{\partial \Re e(z)} & \frac{\partial \Re e(\psi)}{\partial \Im m(z)} \\ \frac{\partial \Im m(\psi)}{\partial \Re e(z)} & \frac{\partial \Im m(\psi)}{\partial \Im m(z)} \end{vmatrix} . \tag{C.2}$$

By the Lemma 7.1, page 150 of Ref.[3] concerning the Jacobian of complex analytic functions, (C.2) can be rewritten

$$\rho(z) = \delta[\Re e\,(\psi(z))]\,\delta[\Im m\,(\psi(z))] \left|\frac{d\psi}{dz}\right|^2 . \tag{C.3}$$

We will use, for convenience, polar coordinates in the z-plane

$$z = r\,\mathrm{e}^{\mathrm{i}\varphi}$$

and write $a_k = c_k + \mathrm{i}d_k$. Then, from (C.1)

$$\begin{aligned} f(r,\varphi) &= \psi(z = r\,\mathrm{e}^{\mathrm{i}\varphi}) \\ &= \sum_{k=0}^{N}\left\{\sqrt{C_N^k}r^k[c_k\cos(k\varphi) - d_k\sin(k\varphi)] + \mathrm{i}\sqrt{C_N^k}r^k[d_k\cos(k\varphi) + c_k\sin(k\varphi)]\right\}. \end{aligned} \tag{C.4}$$

Moreover

$$\frac{d\psi}{dz} = \sum_{k=0}^{N}\sqrt{C_N^k}\,k\,a_k\,z^{k-1}$$

and then, in polar coordinates

$$\left|\frac{d\psi}{dz}\right|^2 = \sum_{k,\ell=0}^{N}\sqrt{C_N^k C_N^\ell}(c_k + id_k)(c_\ell - id_\ell)r^{k+\ell-2}\mathrm{e}^{\mathrm{i}(k-\ell)\varphi}k\,\ell\,. \tag{C.5}$$

To compute the average over the Gaussian ensemble we will only exponentiate the delta functions in (C.3), but not the Jacobian. Using the expression (3.12) for the delta functions, and (C.4)-(C.5) for f and the Jacobian, respectively, the density (C.3) can be expressed as

$$\begin{aligned} \rho(r,\varphi) = &\frac{1}{(2\pi)^2}\int\!\!\int_{-\infty}^{\infty} d\xi_1 d\xi_2 \left\{\sum_{k=0}^{N} C_N^k k^2(c_k^2 + d_k^2)r^{2(k-1)}\right. \\ &\left. + \sum_{k\neq\ell=0}^{N}\sqrt{C_N^k C_\ell^N}k\ell\,[c_k c_\ell + d_k d_\ell + \mathrm{i}(d_k c_\ell - c_k d_\ell)]\,r^{k+\ell-2}\mathrm{e}^{\mathrm{i}(k-\ell)\varphi}\right\}\exp\left\{\sum_{n=0}^{N}(\alpha_n c_n + \beta_n d_n)\right\} \end{aligned} \tag{C.6}$$

where

$$\begin{aligned} \alpha_n &= \mathrm{i}\sqrt{C_N^n}r^n[\cos(n\varphi)\xi_1 + \sin(n\varphi)\xi_2] \\ \beta_n &= \mathrm{i}\sqrt{C_N^n}r^n[\cos(n\varphi)\xi_2 - \sin(n\varphi)\xi_1]. \end{aligned} \tag{C.7}$$

1. The average over the coefficients c_k and d_k in (C.6) involves expressions of the type

$$< c_k^{j_1} d_\ell^{j_2} \exp\left\{\sum_{n=0}^{N}(\alpha_n c_n + \beta_n d_n)\right\} >$$

where the symbol $< . >$ represents the average over the ensemble (2.3) taking all variances σ^2 equal; the parameters j_i in the latter expression can take the values 1 or 2. The computation is straightforward; for example, for $j_1 = 2$ and $j_2 = 0$

$$< c_k^2 \exp\left\{\sum_{n=0}^{N}(\alpha_n c_n + \beta_n d_n)\right\} >= \sigma^2(1 + \alpha_k^2\sigma^2)\exp\left\{\frac{\sigma^2}{2}\sum_{n=0}^{N}(\alpha_n^2 + \beta_n^2)\right\}\,.$$

From (C.7) we can write

$$\alpha_n^2 + \beta_n^2 = -C_N^n r^{2n}(\xi_1^2 + \xi_2^2)$$

and hence

$$< c_k^2 \exp\left\{\sum_{n=0}^{N}(\alpha_n c_n + \beta_n d_n)\right\} >= \sigma^2(1+\alpha_k^2\sigma^2)\exp\left\{-\frac{\sigma^2}{2}(1+r^2)^N(\xi_1^2+\xi_1^2)\right\} .$$

The other averages are computed analogously. The result of averaging (C.6) is

$$\begin{aligned} < \rho(r,\varphi) >= &\ \tfrac{\sigma^2}{(2\pi)^2}\int\int_{-\infty}^{\infty} d\xi_1 d\xi_2 \exp\left[-\tfrac{\sigma^2}{2}(1+r^2)^N(\xi_1^2+\xi_2^2)\right]\left\{\sum_{k=0}^{N} C_N^k k^2\left[2-\sigma^2 C_k^N r^{2k}(\xi_1^2+\xi_2^2)\right]\right. \\ &\left. +\sigma^2\sum_{k\neq \ell=0}^{N}\sqrt{C_N^k C_N^\ell}\, k\ell\, [\alpha_k\alpha_\ell+\beta_k\beta_\ell+\mathrm{i}(\beta_k\alpha_\ell-\alpha_k\beta_\ell)]\, r^{k+\ell-2}e^{\mathrm{i}(k-\ell)\varphi}\right\} . \end{aligned} \tag{C.8}$$

2. The integrals involving the first term between curly brackets in (C.8) give

$$\int\int_{-\infty}^{\infty} d\xi_1 d\xi_2 \left[2-\sigma^2 C_N^k r^{2k}(\xi_1^2+\xi_2^2)\right]\exp\left\{-\sigma^2(1+r^2)^N(\xi_1^2+\xi_2^2)/2\right\} = \frac{4\pi}{\sigma^2(1+r^2)^N}\left[1-\frac{C_N^k r^{2k}}{(1+r^2)^N}\right] . \tag{C.9}$$

Moreover, from Eqs.(C.7) it follows that

$$\alpha_k\alpha_e+\beta_k\beta_\ell+\mathrm{i}(\beta_k\alpha_\ell-\alpha_k\beta_\ell) = -\sqrt{C_N^k C_N^\ell}\, r^{k+\ell} e^{-\mathrm{i}(k-\ell)\varphi}(\xi_1^2+\xi_2^2) .$$

Using this result, we evaluate the integrals involving the second term between curly brackets

$$\int\int_{-\infty}^{\infty} d\xi_1 d\xi_2 e^{-\sigma^2(1+r^2)^N(\xi_1^2+\xi_2^2)/2}\,[\alpha_k\alpha_\ell+\beta_k\beta_\ell+i(\beta_k\alpha_\ell-\alpha_k\beta_\ell)] = -\frac{4\pi}{\sigma^4(1+r^2)^N}\sqrt{C_N^k C_N^\ell}\, r^{k+\ell}e^{-\mathrm{i}(k-\ell)\varphi} . \tag{C.10}$$

Using Eqs.(C.9) and (C.10), (C.8) can be expressed as

$$< \rho(r,\varphi) >= \frac{1}{\pi(1+r^2)^N}\left[\sum_{k=0}^{N} C_N^k k^2 r^{2(k-1)} - \frac{1}{(1+r^2)^N}\left(\sum_{k=0}^{N} C_N^k k\, r^{2k-1}\right)^2\right] . \tag{C.11}$$

In order to evaluate the sums, we consider the identity

$$\sum_{k=0}^{N} C_k^N r^{2k} = (1+r^2)^N . \tag{C.12}$$

By differentiating Eq.(C.12) once and twice with respect to r^2 we get

$$\sum_{k=0}^{N} C_N^k k\, r^{2k-1} = r\frac{\partial(1+r^2)^N}{\partial(r^2)} = rN(1+r^2)^{N-1} \tag{C.13}$$

and

$$\begin{aligned} \sum_{k=0}^{N} C_N^k k^2\, r^{2(k-1)} &= r^2\frac{\partial^2(1+r^2)^N}{\partial(r^2)^2}+\frac{\partial(1+r^2)^N}{\partial(r^2)} \\ &= r^2N(N-1)(1+r^2)^{N-2}+N(1+r^2)^{N-1} , \end{aligned} \tag{C.14}$$

respectively. Substitution of (C.13) and (C.14) into (C.11) gives, finally

$$< \rho(r,\varphi) > r dr d\varphi = \frac{N}{\pi}\frac{r}{(1+r^2)^2} dr d\varphi = \frac{N}{\pi}\frac{d^2 z}{(1+|z|^2)^2} \ . \tag{C.15}$$

A simpler form for the density of zeros is obtained projecting (by a stereographic projection from the north pole) the complex plane into the two-dimensional Riemann sphere (having unit radius), spanned by the spherical variables (θ, φ)

$$z = \cot(\theta/2)\, e^{i\varphi} \ .$$

This transformation explicitly shows that, in fact, the density (C.15) is *uniform* on that surface

$$< \rho(\theta,\varphi) > d\theta d\varphi = \frac{N}{4\pi} \sin\theta \, d\theta d\varphi \ . \tag{C.16}$$

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FIGURES

FIG. 1: Distribution in the complex plane of the roots of random polynomials. In all the ((a) to (d)) cases we have superimposed the roots of 200 different trials of a polynomial of degree $N = 48$ whose coefficients obey a GRI distribution all having the same second moment σ. (a) Standard random polynomial with all its coefficients complex and independent. (b) SI random polynomial with complex coefficients; only half of the coefficients are random, the other half being determined by complex conjugation. (c) standard random polynomial with real coefficients. (d) SI random polynomial with real coefficients.

FIG. 2: The asymptotic average number of roots lying on the unit circle $\mathcal{C}$ for a SI random polynomial of the form (3.8) as a function of the parameter $\epsilon = \sigma\sqrt{N}$.

FIG. 3: The asymptotic two-point correlation function $\tilde{R}_2$ (Eq.(3.25)) for the roots lying on $\mathcal{C}$ of SI polynomials with complex GRI-distributed coefficients.

FIG. 4: Nearest-neighbour spacing distribution for the case of Fig.3.

FIG. 5: The two-point correlation function $\tilde{R}_2$ for the roots lying on $\mathcal{C}$ of $N = 3601$ SI polynomials with complex GRI-distributed coefficients and second moments $\sigma_k = k^{-1/2}$.

FIG. 6: Phase-space distribution of the zeros of eigenstates of a chaotic system. In the three parts of the figure we have superimpose the 60 roots of the $N + 1 = 61$ eigenstates obtained by numerical diagonalization of the kicked-top map (4.12) for $\mu = 1$ and $p = 4\pi$. (a) t=0, where two antiunitary symmetries exist. We observe a concentration of roots over the two associated phase-space symmetry lines given by Eq.(4.11). (b) t=1, both symmetries are now broken but there is still a concentration of roots on the symmetry lines. (c) t=6, all vestiges of the symmetries have disappeared, and we recover a uniform-like distribution, in agreement with the prediction (4.7).

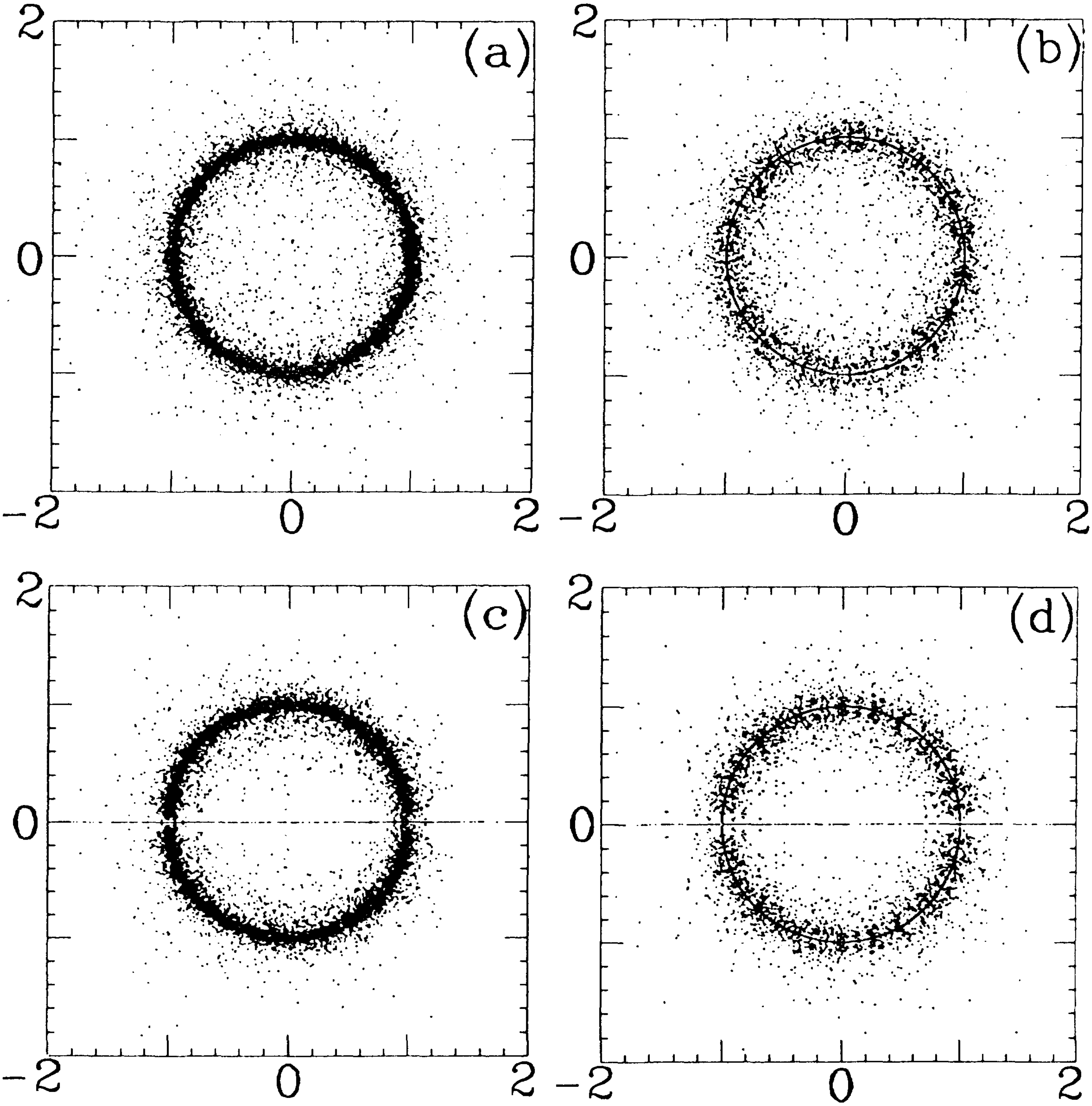

Fig. 1

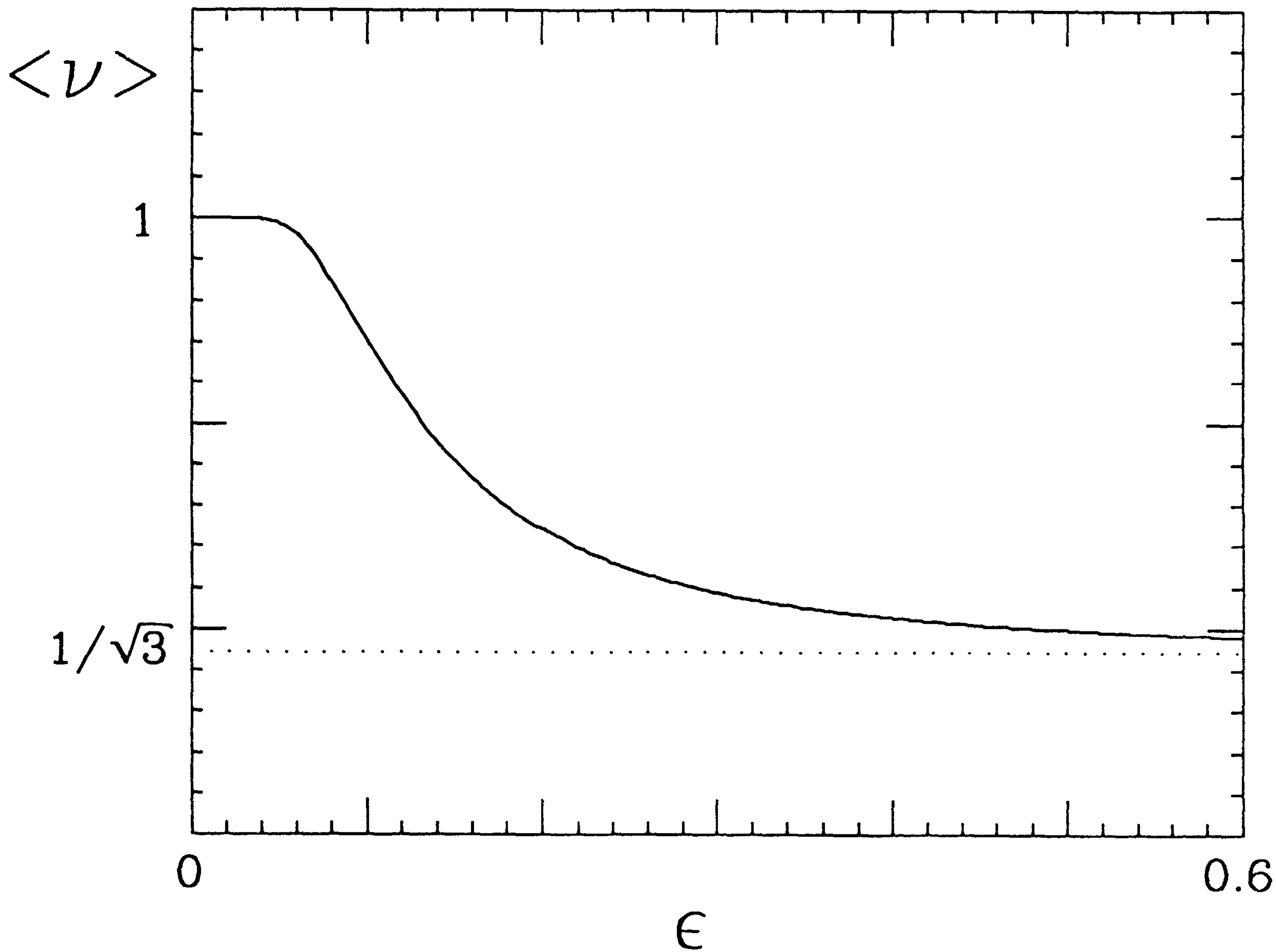

Fig. 2

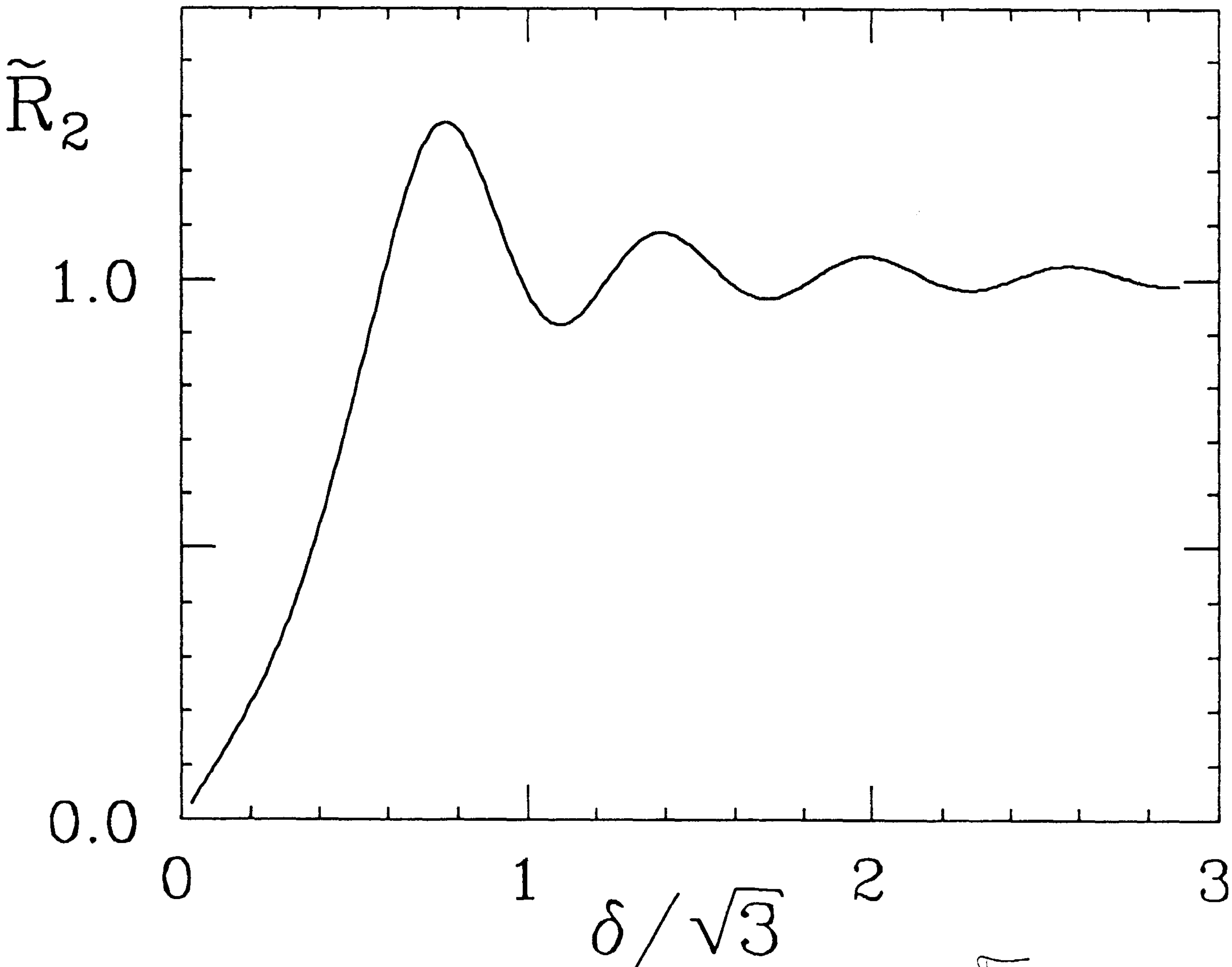

Fig. 3

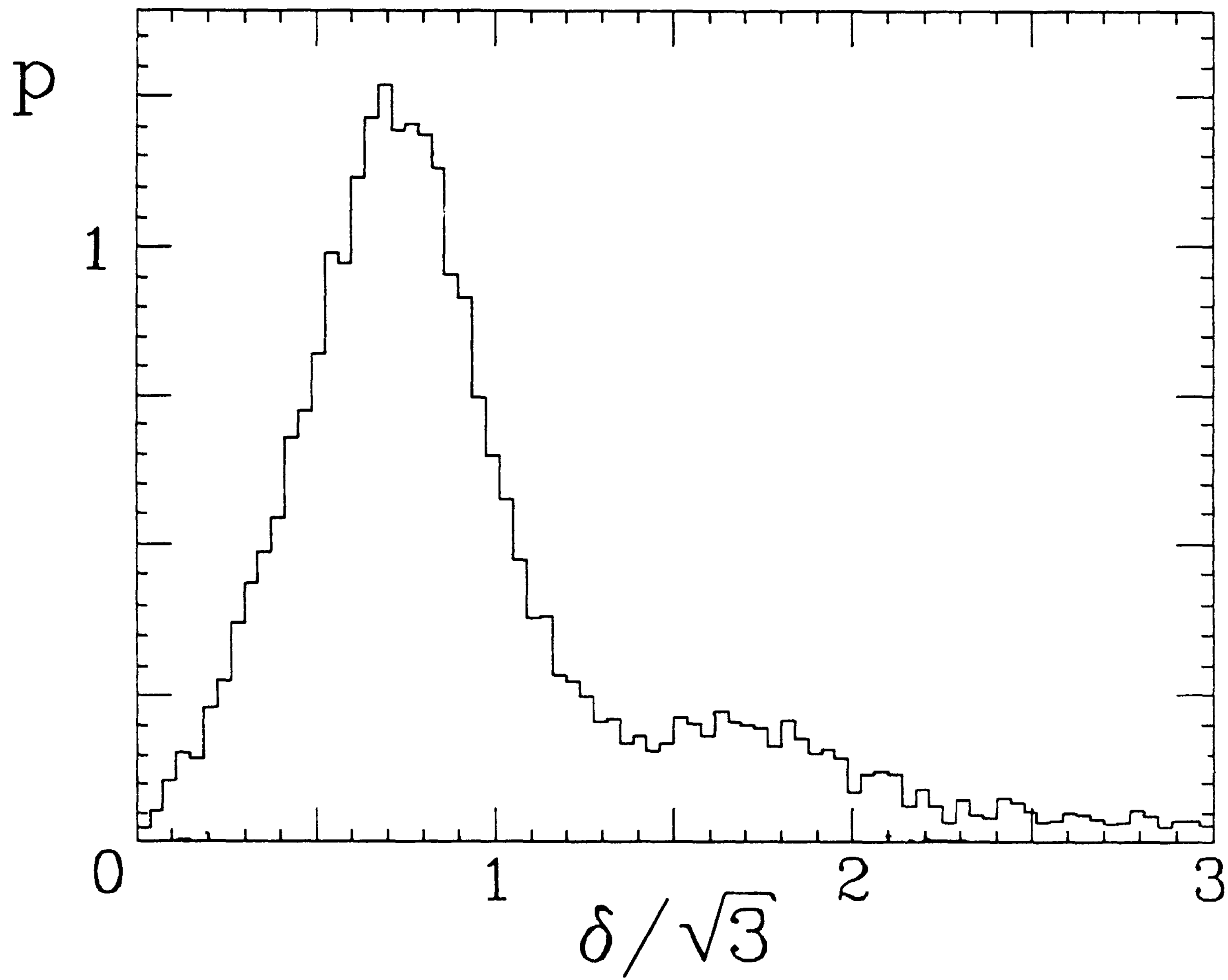

Figure 4

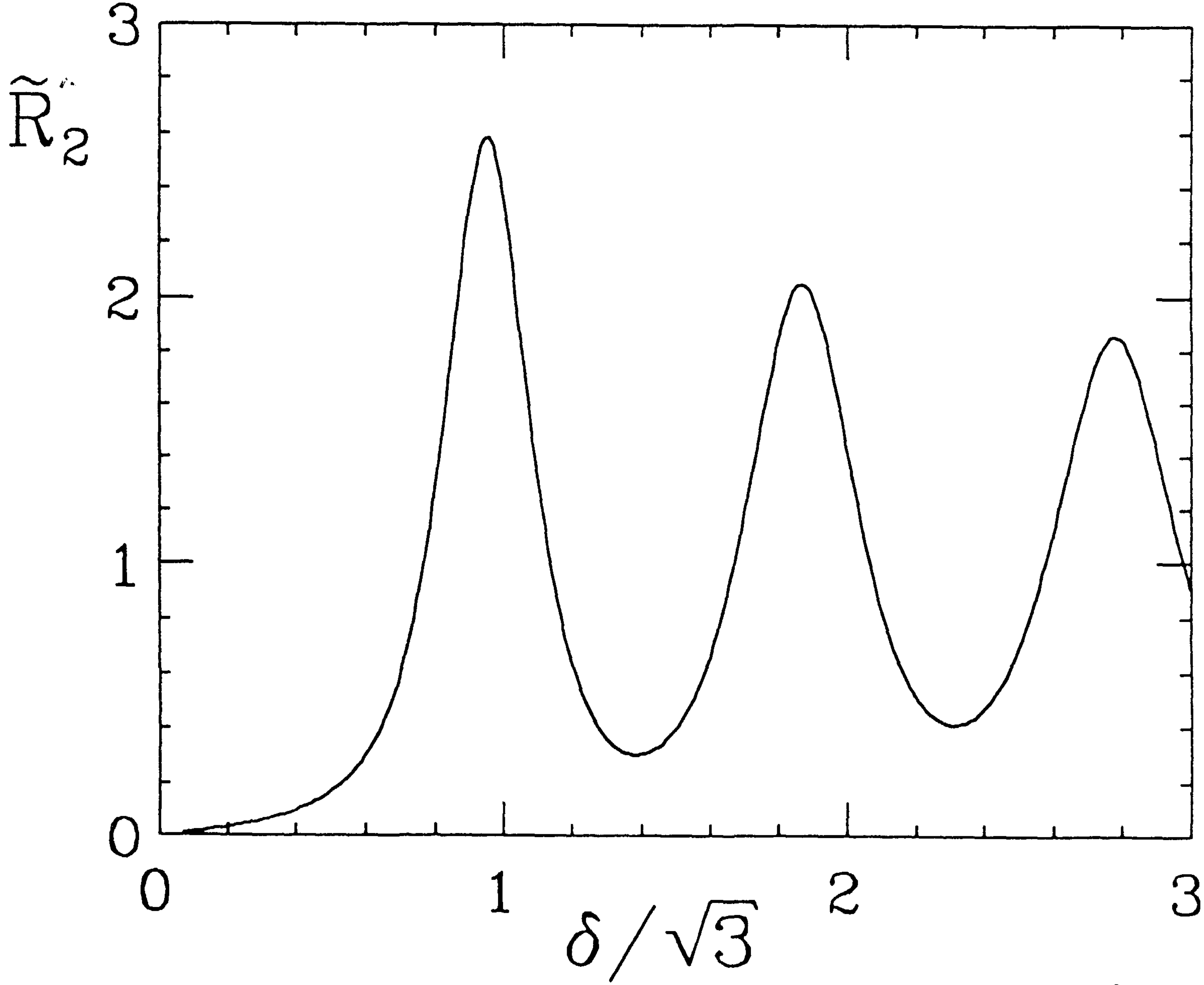

Figure 5

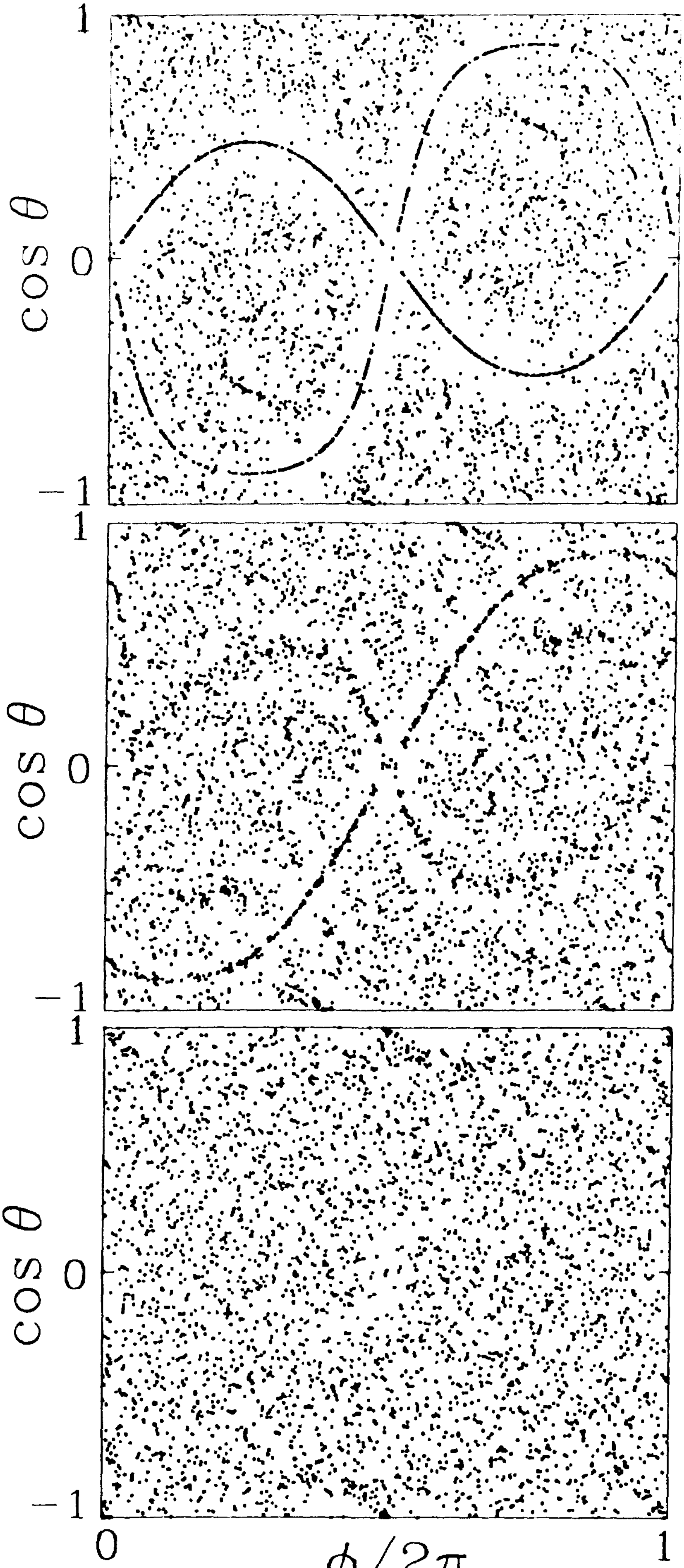

Figure 6