Canonical and magnetic coordinates applied to relativistic guiding centre drifts

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Abstract. The formulation of the guiding centre drift orbits is extended to relativistic particles in canonical coordinates with arbitrary time dependent electric and magnetic fields. The transformation to these coordinates from general flux coordinates is cumbersome for practical applications. This transformation is straightforward for Boozer magnetic coordinates which are canonical if the perturbed magnetic field is constrained to the form $\delta \boldsymbol{B} = \boldsymbol{\nabla}\times(\Upsilon \boldsymbol{B})$.

A Hamiltonian formalism in canonical coordinates constitutes the most transparent and compact approach to treat the guiding centre drift orbit problem. In this paper, we extend to relativistic particles the formulation in a canonical coordinate system that is valid for arbitrary three dimensional (3D) time dependent electric and magnetic fields [1,2]. For a 3D equilibrium calculated in a different coordinate system, we demonstrate that the transformation to the canonical coordinates is too complicated for useful applications. For Boozer magnetic coordinates [3], the transformation is simple. However, the perturbed magnetic field structure cannot be arbitrary for this system to retain canonical properties.

To define canonical coordinates in a torus with arbitrary time dependent electric and magnetic fields requires that these coordinates (r, θ, ζ) satisfy the property that the vector potential and the magnetic field be expressed as [1]

$$\boldsymbol{A} = \Phi(r,\theta,\zeta,t)\boldsymbol{\nabla}\theta - \psi(r,\theta,\zeta,t)\boldsymbol{\nabla}\zeta \ , \tag{1}$$

$$\boldsymbol{B} = B_\theta(r,\theta,\zeta,t)\boldsymbol{\nabla}\theta + B_\zeta(r,\theta,\zeta,t)\boldsymbol{\nabla}\zeta \ . \tag{2}$$

where r is the radial variable θ is the poloidal angle and ζ is the toroidal angle. In other words, the coordinates are canonical when the radial components of $\boldsymbol{A}$ and $\boldsymbol{B}$ in the covariant representation vanish. As $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$, the magnetic field in the contravariant representation is $\boldsymbol{B} = \boldsymbol{\nabla}\zeta\times\boldsymbol{\nabla}\psi + \boldsymbol{\nabla}\Phi\times\boldsymbol{\nabla}\theta$. The canonical momenta in the drift approximation are given by $\boldsymbol{P} = p_{\parallel}\boldsymbol{B}/B + e\boldsymbol{A}$. The corresponding relativistic Hamiltonian is

$$H = H(\boldsymbol{p},\boldsymbol{x},t) = \sqrt{p_{\parallel}^2c^2 + 2\mu B m_0 c^2 + m_0^2c^4} + e\chi(\boldsymbol{x},t) = \gamma m_0 c^2 + e\chi(\boldsymbol{x},t)\ , \qquad (3)$$

where $\rho_{\parallel} = p_{\parallel}/eB$ is the parallel gyroradius, χ is the electrostatic potential, μ is the magnetic moment, e is the particle charge, c is the speed of light, m_0 is the particle rest mass and γ is the relativistic gamma factor. Applying the Hamiltonian formalism, we derive the equations of motion in the drift approximation [2]

$$\dot{r} = \frac{\partial r}{\partial t} - \frac{B_\zeta}{D}\left[\frac{\partial\chi}{\partial\theta}\Big|_{r,\zeta,t} + \frac{1}{\gamma}\left(\frac{\mu}{e} + \frac{eB}{m_0}\rho_{\parallel}^2\right)\frac{\partial B}{\partial\theta}\Big|_{r,\zeta,t}\right] + \frac{B_\theta}{D}\left[\frac{\partial\chi}{\partial\zeta}\Big|_{r,\theta,t} + \frac{1}{\gamma}\left(\frac{\mu}{e} + \frac{eB}{m_0}\rho_{\parallel}^2\right)\frac{\partial B}{\partial\zeta}\Big|_{r,\theta,t}\right]$$
$$- \frac{eB^2\rho_{\parallel}}{\gamma m_0 D}\left(\frac{\partial\psi}{\partial\theta}\Big|_{r,\zeta,t} - \rho_{\parallel}\frac{\partial B_\zeta}{\partial\theta}\Big|_{r,\zeta,t} + \frac{\partial\Phi}{\partial\zeta}\Big|_{r,\theta,t} + \rho_{\parallel}\frac{\partial B_\theta}{\partial\zeta}\Big|_{r,\theta,t}\right), \qquad (4)$$

$$\dot{\theta} = \frac{eB^2\rho_{\parallel}}{\gamma m_0 D}\left(\frac{\partial\psi}{\partial r}\Big|_{\theta,\zeta,t} - \rho_{\parallel}\frac{\partial B_\zeta}{\partial r}\Big|_{\theta,\zeta,t}\right) + \frac{B_\zeta}{D}\left[\frac{\partial\chi}{\partial r}\Big|_{\theta,\zeta,t} + \frac{1}{\gamma}\left(\frac{\mu}{e} + \frac{eB}{m_0}\rho_{\parallel}^2\right)\frac{\partial B}{\partial r}\Big|_{\theta,\zeta,t}\right], \qquad (5)$$

$$\dot{\zeta} = \frac{eB^2\rho_{\parallel}}{\gamma m_0 D}\left(\frac{\partial\Phi}{\partial r}\Big|_{\theta,\zeta,t} + \rho_{\parallel}\frac{\partial B_\theta}{\partial r}\Big|_{\theta,\zeta,t}\right) - \frac{B_\theta}{D}\left[\frac{\partial\chi}{\partial r}\Big|_{\theta,\zeta,t} + \frac{1}{\gamma}\left(\frac{\mu}{e} + \frac{eB}{m_0}\rho_{\parallel}^2\right)\frac{\partial B}{\partial r}\Big|_{\theta,\zeta,t}\right]. \qquad (6)$$

$$\dot{\rho}_{\parallel} = \frac{\partial\rho_{\parallel}}{\partial t} - \frac{1}{D}\left(\frac{\partial\psi}{\partial r}\Big|_{\theta,\zeta,t} - \rho_{\parallel}\frac{\partial B_\zeta}{\partial r}\Big|_{\theta,\zeta,t}\right)\left[\frac{\partial\chi}{\partial\theta}\Big|_{r,\zeta,t} + \frac{1}{\gamma}\left(\frac{\mu}{e} + \frac{eB}{m_0}\rho_{\parallel}^2\right)\frac{\partial B}{\partial\theta}\Big|_{r,\zeta,t}\right]$$
$$- \frac{1}{D}\left(\frac{\partial\Phi}{\partial r}\Big|_{\theta,\zeta,t} + \rho_{\parallel}\frac{\partial B_\theta}{\partial r}\Big|_{\theta,\zeta,t}\right)\left[\frac{\partial\chi}{\partial\zeta}\Big|_{r,\theta,t} + \frac{1}{\gamma}\left(\frac{\mu}{e} + \frac{eB}{m_0}\rho_{\parallel}^2\right)\frac{\partial B}{\partial\zeta}\Big|_{r,\theta,t}\right]$$
$$+ \frac{1}{D}\left(\frac{\partial\psi}{\partial\theta}\Big|_{r,\zeta,t} - \rho_{\parallel}\frac{\partial B_\zeta}{\partial\theta}\Big|_{r,\zeta,t}\right)\left[\frac{\partial\chi}{\partial r}\Big|_{\theta,\zeta,t} + \frac{1}{\gamma}\left(\frac{\mu}{e} + \frac{eB}{m_0}\rho_{\parallel}^2\right)\frac{\partial B}{\partial r}\Big|_{\theta,\zeta,t}\right]$$
$$+ \frac{1}{D}\left(\frac{\partial\Phi}{\partial\zeta}\Big|_{r,\theta,t} + \rho_{\parallel}\frac{\partial B_\theta}{\partial\zeta}\Big|_{r,\theta,t}\right)\left[\frac{\partial\chi}{\partial r}\Big|_{\theta,\zeta,t} + \frac{1}{\gamma}\left(\frac{\mu}{e} + \frac{eB}{m_0}\rho_{\parallel}^2\right)\frac{\partial B}{\partial r}\Big|_{\theta,\zeta,t}\right], \qquad (7)$$

where $D = B_\zeta\partial\Phi/\partial r + B_\theta\partial\psi/\partial r + \rho_{\parallel}\Big(B_\zeta\partial B_\theta/\partial r - B_\theta\partial B_\zeta/\partial r\Big)$.

Now we consider a 3D static equilibrium with nested magnetic flux surfaces. The magnetic field in the contravariant representation is $\boldsymbol{B} = \boldsymbol{\nabla}\alpha\times\boldsymbol{\nabla}\psi$ and in the covariant representation is $\boldsymbol{B} = \boldsymbol{\nabla}\eta + \beta\boldsymbol{\nabla}s$ where $\alpha = \zeta - q(s)[\theta + \lambda(s,\theta,\zeta)]$, $\eta = \mu_0[J(s)\theta - I(s)\zeta + Q(s,\theta,\zeta)]$ and $\beta = \mu_0[I'(s)\zeta - J'(s)\theta - \nu(s,\theta,\zeta)]$, where $q(s)$ is the inverse rotational transform, $2\pi I(s)$ and $2\pi J(s)$ are the poloidal and toroidal current fluxes, respectively. The periodic function λ is determined from the condition $\boldsymbol{j}\cdot\boldsymbol{\nabla}s = 0$. The periodic functions Q and ν are still unspecified. It is usually the case that a 3D equilibrium is known in a different set of coordinates (s,u,v) (calculated for example with the VMEC code) and a transformation procedure to the desired canonical coordinates

(s,θ,ζ) must be prescribed. To do this, we identify $\theta = u + h(s,u,v)$ and $\zeta = v + k(s,u,v)$ where h and k are periodic functions of the known poloidal and toroidal angles u and v, respectively. For straight field lines, $\lambda(s,\theta,\zeta) = 0$. The functions α, η and β, being scalars, are invariant with respect to coordinate transformations. From the relations for α and η, we obtain $h(s,u,v) = Q_v(s,u,v) - Q(s,\theta,\zeta) - q(s)I(s)\lambda(s,u,v)/[J(s) - q(s)I(s)]$ and $k(s,u,v) = q(s)Q_v(s,u,v) - Q(s,\theta,\zeta) - J(s)\lambda(s,u,v)/[J(s) - q(s)I(s)]$ where the function $Q(s,\theta,\zeta) = Q[s,\theta(s,u,v),\zeta(s,u,v)]$ must still be specified. Invoking the expression for β, we obtain that $\nu_c[s,\theta_c(s,u,v),\zeta_c(s,u,v)] = \nu_v(s,u,v) + J'(s)h(s,u,v) - I'(s)k(s,u,v)$ which we combine with the condition that the covariant radial component of $\boldsymbol{B}$ must vanish in the canonical coordinate system to derive the differential equation

$$\left.\frac{\partial Q_c}{\partial s}\right|_{\theta_c,\zeta_c} + \frac{J'(s) - q(s)I'(s)}{J(s) - q(s)I(s)}Q_c = \nu_v(s,u,v) + \frac{J'(s) - q(s)I'(s)}{J(s) - q(s)I(s)}Q_v(s,u,v) + q(s)\frac{I'(s)J(s) - J'(s)I(s)}{J(s) - q(s)I(s)}\lambda(s,u,v)\,. \qquad (8)$$

The resolution of this equation for the transformation from an arbitrary coordinates system to canonical coordinates is an extremely cumbersome procedure that makes the applicability of this formulation virtually impractical.

Boozer magnetic coordinates [3] are defined for 3D systems in which the unperturbed magnetic field forms perfect nested magnetic flux surfaces. These coordinates are also canonical if the time dependent portion of the magnetic field is constrained to have the form [4] $\delta\boldsymbol{B} = \boldsymbol{\nabla}\times[\Upsilon(s,\vartheta,\phi,t)\boldsymbol{B}]$. This form is adequate to describe that radial component of any perturbed magnetic field [4]. Furthermore, the Boozer coordinates satisfy $\lambda(s,\vartheta,\phi) = 0$ (straight field lines) and $Q(s,\vartheta,\phi) = 0$ which allows a straightforward determination of the periodic functions h and k and consequently the mapping from VMEC-like coordinates. The vector potential is $\boldsymbol{A} = \Phi(s)\boldsymbol{\nabla}\vartheta - \psi(s)\boldsymbol{\nabla}\phi + \Upsilon(s,\vartheta,\phi,t)\boldsymbol{B}$. The canonical momenta are $P_\vartheta = e[\Phi(s) + \rho_c\mu_0 J(s)]$ and $P_\phi = -e[\psi(s) + \rho_c\mu_0 I(s)]$ where the effective gyroradius is $\rho_c = p_\| /(eB) + \Upsilon$ [5]. Inverting these relations, we have $s = s(P_\vartheta, P_\phi)$ and $\rho_c = \rho_c(P_\vartheta, P_\phi)$. Applying the Hamiltonian formalism, we obtain the equations of motions

$$\dot{s} = \frac{\mu_0 I(s)}{D_b}\left[\left.\frac{\partial\chi}{\partial\vartheta}\right|_{s,\phi,t} + \frac{1}{\gamma}\left(\frac{\mu}{e} + \frac{eB}{m_0}\rho_\|^2\right)\left.\frac{\partial B}{\partial\vartheta}\right|_{s,\phi} - \frac{eB^2\rho_\|}{\gamma m_0}\left.\frac{\partial\Upsilon}{\partial\vartheta}\right|_{s,\phi,t}\right]$$

$$+\frac{\mu_0 J(s)}{D_b}\left[\frac{\partial\chi}{\partial\phi}\bigg|_{s,\vartheta,t}+\frac{1}{\gamma}\left(\frac{\mu}{e}+\frac{eB}{m_0}\rho_{\|}^2\right)\frac{\partial B}{\partial\phi}\bigg|_{s,\vartheta}-\frac{eB^2\rho_{\|}}{\gamma m_0}\frac{\partial\Upsilon}{\partial\phi}\bigg|_{s,\vartheta,t}\right], \quad (9)$$

$$\dot{\vartheta} = -\frac{eB^2\rho_{\|}}{\gamma m_0 D_b}\left[\psi'(s)+(\rho_{\|}+\Upsilon)\mu_0 I'(s)+\mu_0 I(s)\frac{\partial\Upsilon}{\partial s}\bigg|_{\vartheta,\phi,t}\right]$$
$$-\frac{\mu_0 I(s)}{D_b}\left[\frac{\partial\chi}{\partial s}\bigg|_{\vartheta,\phi,t}+\frac{1}{\gamma}\left(\frac{\mu}{e}+\frac{eB}{m_0}\rho_{\|}^2\right)\frac{\partial B}{\partial s}\bigg|_{\vartheta,\phi}\right], \quad (10)$$

$$\dot{\phi} = -\frac{eB^2\rho_{\|}}{\gamma m_0 D_b}\left[\Phi'(s)+(\rho_{\|}+\Upsilon)\mu_0 J'(s)+\mu_0 J(s)\frac{\partial\Upsilon}{\partial s}\bigg|_{\vartheta,\phi,t}\right]$$
$$-\frac{\mu_0 J(s)}{D_b}\left[\frac{\partial\chi}{\partial s}\bigg|_{\vartheta,\phi,t}+\frac{1}{\gamma}\left(\frac{\mu}{e}+\frac{eB}{m_0}\rho_{\|}^2\right)\frac{\partial B}{\partial s}\bigg|_{\vartheta,\phi}\right], \quad (11)$$

$$\dot{\rho_{\|}} = -\frac{1}{D_b}\left[\psi'(s)+(\rho_{\|}+\Upsilon)\mu_0 I'(s)+\mu_0 I(s)\frac{\partial\Upsilon}{\partial s}\bigg|_{\vartheta,\phi,t}\right]\left[\frac{\partial\chi}{\partial\vartheta}\bigg|_{s,\phi,t}+\frac{1}{\gamma}\left(\frac{\mu}{e}+\frac{eB}{m_0}\rho_{\|}^2\right)\frac{\partial B}{\partial\vartheta}\bigg|_{s,\phi}\right]$$
$$-\frac{1}{D_b}\left[\Phi'(s)+(\rho_{\|}+\Upsilon)\mu_0 J'(s)+\mu_0 J(s)\frac{\partial\Upsilon}{\partial s}\bigg|_{\vartheta,\phi,t}\right]\left[\frac{\partial\chi}{\partial\phi}\bigg|_{s,\vartheta,t}+\frac{1}{\gamma}\left(\frac{\mu}{e}+\frac{eB}{m_0}\rho_{\|}^2\right)\frac{\partial B}{\partial\phi}\bigg|_{s,\vartheta}\right]$$
$$-\frac{\mu_0}{D_b}\left[I(s)\frac{\partial\Upsilon}{\partial\vartheta}\bigg|_{s,\phi,t}+J(s)\frac{\partial\Upsilon}{\partial\phi}\bigg|_{s,\vartheta,t}\right]\left[\frac{\partial\chi}{\partial s}\bigg|_{\vartheta,\phi,t}+\frac{1}{\gamma}\left(\frac{\mu}{e}+\frac{eB}{m_0}\rho_{\|}^2\right)\frac{\partial B}{\partial s}\bigg|_{\vartheta,\phi}\right]-\frac{\partial\Upsilon}{\partial t}\bigg|_{s,\vartheta,\phi}, \quad (12)$$

where

$$D_b = \mu_0[\psi'(s)J(s)-\Phi'(s)I(s)]\Big[1+\mu_0\rho_c[J(s)I'(s)-I(s)J'(s)]/[\psi'(s)J(s)-\Phi'(s)I(s)]\Big].$$

In conclusion, we have extended the formulation of the guiding centre drift orbits in canonical coordinates valid for arbitrary time dependent electric and magnetic fields to relativistic particles. However, the transformation from a general flux coordinate system to these canonical coordinates is too cumbersome for parctical applications. The transformation to Boozer magnetic coordinates, on the other hand, is straightforward and this coordinate system is canonical for perturbed magnetic fields constrined to the form $\delta\boldsymbol{B} = \boldsymbol{\nabla}\times(\Upsilon\boldsymbol{B})$. The relativistic guiding centre drift equations of motion are explicitly derived in this system.

References

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