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**E. Gava**

**J.F. Morales**

**K.S. Narain**

**and**

**G. Thompson**



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## BOUND STATES OF TYPE I D-STRINGS

E. Gava<sup>1</sup>

*Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Trieste, Italy*  
and

*The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy*

J.F. Morales<sup>2</sup>

*Scuola Internazionale Superiore di Studi Avanzati, Trieste, Italy,*

K.S. Narain<sup>3</sup> and G. Thompson<sup>4</sup>

*The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.*

### Abstract

We study the infra-red limit of the  $O(N)$  gauge theory that describes the low energy modes of a system of  $N$  type I D-strings and provide some support to the conjecture that, in this limit, the theory flows to an orbifold conformal theory. We compute the elliptic genus of the orbifold theory and argue that its longest string sector describes the bound states of D-strings. We show that, as a result, the masses and multiplicities of the bound states are in agreement with the predictions of heterotic-type I duality in 9 dimensions, for all the BPS charges in the lattice  $\Gamma_{(1,17)}$ .

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<sup>1</sup> E-mail: [gava@isolde.sissa.it](mailto:gava@isolde.sissa.it)

<sup>2</sup> E-mail: [morales@sissa.it](mailto:morales@sissa.it)

<sup>3</sup> E-mail: [narain@ictp.trieste.it](mailto:narain@ictp.trieste.it)

<sup>4</sup> E-mail: [thompson@ictp.trieste.it](mailto:thompson@ictp.trieste.it)

## 1 Introduction

The putative duality between Type I string theory and the  $SO(32)$  heterotic string theory [1, 2] requires the existence of Type I D-string bound states. The way to see this is to begin by compactifying on a circle of radius  $R_H$  in the  $X^9$  direction. On the heterotic side the electric charge spectrum in the nine dimensional theory sits in the lattice  $\Gamma_{(1,17)}$ . This lattice arises on taking into account the charges associated with the gauge fields of the Cartan subalgebra of  $SO(32)$ , the  $G_{\mu 9}$  component of the metric and the  $B_{\mu 9}$  component of the Neveu-Schwarz antisymmetric tensor. In particular, the states carrying  $N$  units of  $B_{\mu 9}$  charge correspond to the fundamental heterotic string wrapping  $N$  times around the  $X^9$  circle. This particular charge corresponds to a null vector in  $\Gamma_{(1,17)}$  and scrutiny of the partition function shows that the multiplicity of such BPS states, which arise at the level one oscillator mode of the bosonic sector, is 24. The general BPS spectrum, with all the electric charges, is given by

$$kN - \frac{1}{2}P^2 = N_R - 1, \quad (1.1)$$

where  $P$  is a vector in  $\Gamma_{(0,16)}$ ,  $N_R$  is the bosonic oscillator number and  $k$  is an integer, related to the Kaluza-Klein momentum  $p_9$  carried by the state in the following way:

$$p_9 = \frac{1}{R_H} \left( k + B \cdot P + \frac{1}{2} B^2 N \right), \quad (1.2)$$

where  $B^i$  are the holonomies in the Cartan subalgebra of  $SO(32)$  in the  $X^9$  direction (Wilson lines). The multiplicity is then given by the  $N_R$ 'th oscillator level in the partition function  $\eta^{-24}$ . Furthermore, the mass  $m$  in the string frame is given by

$$m = \left| p_9 + \frac{NR_H}{\alpha'} \right|. \quad (1.3)$$

On the other hand, the Type I theory is related to the heterotic by a strong-weak duality. The coupling constants and metrics are related by

$$\lambda_I = \frac{1}{\lambda_H}, \quad G_{MN}^I = \frac{G_{MN}^H}{\lambda_H}. \quad (1.4)$$

The duality relations imply that a heterotic state labelled by  $N$ ,  $k$  and  $P$  is mapped to a

type I state with Kaluza-Klein momentum  $p_9$  and mass  $m$ , in the string frame, given by:

$$p_9 = \frac{1}{R_I}(k + B \cdot P + \frac{1}{2}B^2 N) \quad (1.5)$$

$$m = \left| p_9 + \frac{NR_I}{\alpha' \lambda_I} \right|, \quad (1.6)$$

where  $R_I$  is the type I radius along the  $X^9$  direction.

The Neveu-Schwarz two-form,  $B_{MN}$ , of the heterotic string is mapped to the Ramond-Ramond antisymmetric tensor field,  $B_{MN}^R$ , of the Type I string. Consequently, the winding modes of the fundamental string on the heterotic side are mapped to the D-string winding modes on the Type I side. Duality, therefore, predicts that a BPS system of  $N$  D-strings, in the Type I theory, each of which is wrapped once around the circle, should have a threshold bound state with multiplicity equal to 24. Furthermore, when there are other charges turned on, the multiplicities of the resulting D-string threshold bound states should reproduce the heterotic multiplicity (1.1).

Our aim in this letter is to establish that this is indeed the case. We begin by reviewing the effective world volume gauge theory of  $N$  type I D-strings and argue that in the infrared limit, this gauge theory flows to an orbifold conformal field theory [3, 4, 5, 6], in analogy with a similar phenomenon in the type II case [7, 8, 9, 10, 11] or in the study of 5-dimensional black holes [12, 13, 14]. We compute the elliptic genus of this conformal field theory and show that the twisted sector corresponding to the longest string reproduces the results expected from heterotic-type I duality. We also argue that the threshold bound states arise precisely in the longest string sector. The above orbifold conformal field theories have also appeared in the study of D-string instantons, which contribute to the  $F^4$  coupling in the type I case, and to  $R^4$  coupling in the type II case [15, 16].

## 2 Type-I D-Strings

We begin by recalling the form of the world volume action which describes the low lying modes of a system of  $N$  D-strings in the type I theory:

$$S = \text{Tr} \int d^2x \left[ -\frac{1}{4g^2} F^2 + (DX_I)^2 + g^2([X_I, X_J])^2 \right. \\ \left. + \Lambda \not{D}\Lambda + S \not{D}S + \sum_{a=1}^{16} \bar{\chi}^a \not{D}\chi^a + g\Lambda\Gamma^I[X_I, S] + \sum_{a=1}^{16} \bar{\chi}^a B^a \chi^a \right]. \quad (2.1)$$

The fields transform in various representations of the gauge group  $O(N)$ .  $X$  and  $S$  transform as second rank symmetric tensors, while  $\Lambda$  and  $\chi$  transform in the adjoint and fundamental representations respectively. There is an  $SO(8)_R$ , R symmetry group, under which  $X$ ,  $S$ ,  $\Lambda$  and  $\chi$  transform as an  $\mathbf{8}_V$  (this is the  $I$  label), an  $\mathbf{8}_S$ , an  $\mathbf{8}_c$  and a singlet, respectively. The  $\chi$  transforms under the  $SO(32)$  in the vector representation with  $\chi^a$  and  $\bar{\chi}^a$  denoting the positive and negative weights. The  $\Lambda$  and  $\chi$  are negative chiral (right-moving) world sheet fermions while the  $S$  are positive chiral (left-moving) fermions. Finally  $B^a$  are the background holonomies (i.e. Wilson lines on the 9-branes) in the Cartan subalgebra of  $SO(32)$ . The Yang-Mills coupling  $g$  is related to the type I string coupling via  $g^2 = \lambda_I/\alpha'$ . The vev's of the  $X$  fields, appearing in the action above, measure the distances between the D-strings in units of  $\sqrt{\alpha'\lambda_I}$ . This fact will be important later, when we compare the spectrum with that of the heterotic theory.

Geometrically the fields appear in the following fashion [2, 17]. The above action arises as the  $Z_2$  projection of the corresponding theory in the type II case. Recall that in the type II situation a system of  $N$  branes has a  $U(N)$  symmetry. Write the hermitian matrices as a sum of real symmetric matrices and imaginary anti-symmetric matrices. The  $Z_2$  projection, for type I D-strings, assigns to the world volume components of the gauge field the anti-symmetric matrices, that is, it projects out the real symmetric part and so reduces the gauge group to  $O(N)$ . On the other hand, the components of the gauge field in the transverse directions,  $X$ , have their imaginary part projected out and so are symmetric matrices transforming as second rank symmetric tensors under the  $O(N)$ . The diagonal components of the  $X$  give the positions for the  $N$  branes. The trace part, which we factor out, represents the center of mass motion.

The  $\chi$  carry the  $SO(32)$  vector label, as they are the lowest modes of the strings which are stretched between the 9-branes and the D-strings.

As mentioned earlier, the winding mode  $N$  of the fundamental heterotic string is mapped, via duality, to  $N$  D-strings on the Type I side that wind around the  $X^9$  circle. Since the former appears as a fundamental BPS state with multiplicity 24, we should find that the system of  $N$  D-strings in type I theory admits threshold bound states with multiplicity 24. In other words, in the  $O(N)$  theory, there should be 24 square integrable

ground states that are 10 dimensional  $N = 1$  vector short BPS super-multiplets. That every ground state appears with 8 bosonic and 8 fermionic modes, necessary to form the  $N = 1$  short vector supermultiplet, follows from the fact that there are zero modes of the  $O(N)$  singlet free field  $S$ , describing the center of mass motion (which has not been included in the action (2.1)). The remaining part of the  $O(N)$  theory described in the action (2.1), therefore, must have, 24 bosonic normalizable ground states as predicted by the heterotic-type I duality. In other words, we want to show that the Witten Index for the above theory is 24.

More generally, in the heterotic theory, we can also turn on other charges, namely the one associated with Kaluza-Klein modes that couple to  $G_{\mu 9}$  and the ones associated with the Cartan subalgebra of the  $SO(32)$  gauge group. These charges can also be excited in the system of  $N$  D-strings in the type I theory. Indeed, one can include states carrying a longitudinal momentum along the string and thereby generate Kaluza-Klein momentum. Similarly, one can generate  $SO(32)$  quantum numbers by suitably applying  $\chi$  modes. The information about the multiplicities of states carrying these extra charges will be contained in the elliptic genus of the above theory (2.1).

Since the Witten index and, more generally, the elliptic genus do not depend on the coupling constant, we can take a limit which is most convenient for our present purposes. We will consider the infra-red limit of the theory, as it has been conjectured in [3, 4, 5, 6], that in this limit the theory flows to a  $(8, 0)$  orbifold superconformal field theory. This is in analogy with a similar conjecture for a system of type IIB D-strings [9]. In the following we give some support to this conjecture by, first, gauge fixing (2.1) and then by performing a formal scaling which yields the orbifold theory directly.

### 3 Type II and the IR Limit

Before discussing the type I theory we make a digression on the type II theory that will prove useful later. Our aim here is to show that with a prudent choice of gauge one can simplify matters considerably. This prepares the way for taking the large coupling limit in a fashion that is, to a large extent, controllable. We will work with a little more generality than is really required and begin with an analysis of  $D$  dimensional Yang-Mills

theory reduced to  $d$  dimensions.  $D$  dimensional vector labels are denoted by  $M, N, \dots$ , those in  $d$  dimensions are denoted by  $\mu, \nu, \dots$  and those in the remaining  $D - d$  (reduced) dimensions by  $I, J, \dots$ . To make contact with the type II  $D$ -brane world volume theories one sets  $D = 10$ .

The  $D$  dimensional Yang-Mills theory has a 'potential' of the form,

$$-\text{tr} \frac{g^2}{4} [A_M, A_N]^2. \quad (3.1)$$

Minimising, in any dimension  $d$ , we learn that we are interested in the fields that live in the Cartan subalgebra. Decompose the Lie-algebra,  $\mathfrak{g}$ , of the gauge group as  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^\perp$ , where  $\mathfrak{k}$  is the preferred Cartan subalgebra and  $\mathfrak{k}^\perp$  is its ortho-complement. It makes good sense, therefore, to perform a non-canonical split,

$$A_M = A_M^{\mathfrak{k}} + A_M^{\mathfrak{k}^\perp}, \quad (3.2)$$

where the superscripts indicate the part of the Lie-algebra that the fields live in. Before proceeding we need to gauge fix. Given the splitting of the algebra, it behoves us to choose the 'background field' gauge<sup>1</sup>

$$D^M(gA^I)gA_M^{\mathfrak{k}^\perp} = 0, \quad (3.3)$$

which preserves the maximal Torus gauge invariance. The ghosts come in as

$$\text{tr} \bar{C}^{\mathfrak{k}} D_M(gA^I) D^M(gA) C^{\mathfrak{k}} + \text{tr} g^2 \bar{C}^{\mathfrak{k}^\perp} \{ [A_M^{\mathfrak{k}}, C^{\mathfrak{k}}]^{\mathfrak{k}^\perp}, A^{M \mathfrak{k}^\perp} \}. \quad (3.4)$$

We choose a Feynman type gauge with a co-efficient chosen to give the most straightforward analysis namely we add

$$\text{tr} -\frac{1}{2} (D^M(gA^I)A_M^{\mathfrak{k}^\perp})^2 \quad (3.5)$$

to the action. With this choice the potential becomes

$$-\text{tr} \frac{g^2}{2} [A_M^{\mathfrak{k}}, A_N^{\mathfrak{k}}]^2 + \dots, \quad (3.6)$$

where the ellipses indicate higher order terms in  $A_M^{\mathfrak{k}^\perp}$  and which, directly, will be seen to be irrelevant.

<sup>1</sup>At this point the connection is  $gA$ , which explains some of the, what appear to be, spurious factors of  $g$ . We are gauge fixing the 'canonical' gauge field and not the one scaled by  $g$  so that the BRST transformations are  $Q(gA_M) = D_M(gA)C$  and  $QC = C^2$ .

We now perform the following sequence of scalings on the fields appearing in a  $N = 1$  super Yang-Mills theory in  $D$  dimensions

$$A_M^\dagger \rightarrow \frac{1}{g} A_M^\dagger, \quad \psi^\dagger \rightarrow \frac{1}{\sqrt{g}} \psi^\dagger, \quad \bar{C}^\dagger \rightarrow \frac{1}{g^2} \bar{C}^\dagger. \quad (3.7)$$

On a torus,  $T^d$ , with periodic boundary conditions on all the fields appearing, this scaling has unit Jacobian. We can now take the  $g \rightarrow \infty$  limit. The action, in this limit is:

$$S = \text{tr} \int d^d x - \frac{1}{4} F_{MN} (A^\dagger)^2 + \psi^\dagger \not{\partial} \psi^\dagger - \frac{1}{2} [A_M^\dagger, A_N^\dagger]^2 + \psi^\dagger \Gamma^M [A_M^\dagger, \psi^\dagger] + [\bar{C}^\dagger, A_M^\dagger] [C^\dagger, A_M^\dagger]. \quad (3.8)$$

All the fields in the  $\mathfrak{k}$  part of the Lie-algebra can be integrated out and clearly give an overall contribution of unity to the path integral. Thus, we are left with a free, supersymmetric, system of Cartan valued fields. By invoking the Weyl symmetry that is left over, one finds that the target space of the theory is  $(\mathbf{R}^{(D-d)r} \times T^{dr})/W$ , where  $r$  is the rank of the group and  $W$  is the Weyl group.

Fixing  $D = 10$ , gives us the Type II world volume theories of parallel  $D$  branes. The limit just described, the strong coupling limit in the gauge theory, when  $d = 2$  and  $D = 10$  (the D-string) gives rise to the orbifold conformal field theory as in [9].

## 4 Type I and the IR Limit

The flat directions of the potential in this case require mutually commuting matrices once more. We denote those  $X$ 's, with a slight abuse of notation, by  $X^\dagger$  (for example one may choose these to be diagonal). A convenient way to proceed is to start with the (complexified)  $SU(N)$  Lie algebra and to split it into a Cartan subalgebra  $\mathfrak{t}$  and into positive and negative roots,  $\mathfrak{k}_+$  and  $\mathfrak{k}_-$ , respectively, that is,  $\mathfrak{k} = \mathfrak{k}_+ \oplus \mathfrak{k}_-$ . The  $Z_2$  projection means that, in this basis, the world volume gauge fields are proportional to the anti-symmetric (imaginary part of  $\mathfrak{k}$ ) generators,  $\mathfrak{m}_- = \mathfrak{k}_+ - \mathfrak{k}_-$ , while the  $X$ 's are proportional to the symmetric generators,  $\mathfrak{t}$  and (real part of  $\mathfrak{k}$ )  $\mathfrak{m}_+ = \mathfrak{k}_+ + \mathfrak{k}_-$ . With these identifications the bosonic parts of the type I and type II theories coincide. We choose the same gauge fixing as in the type II theory, now restricted to the  $\mathfrak{m}_-$  directions,

$$g \partial^\mu A_\mu^{m-} + g^2 [X^\dagger, X^{m+}] = 0 \quad (4.1)$$

and we scale the fields in a similar way, that is

$$A_\mu^{m-} \rightarrow \frac{1}{g} A_\mu^{m-}, \quad X_I^{m+} \rightarrow \frac{1}{g} X_I^{m+}, \quad \Lambda^{m-} \rightarrow \frac{1}{\sqrt{g}} \Lambda^{m-}, \\ S^{m+} \rightarrow \frac{1}{\sqrt{g}} S^{m+}, \quad \bar{C}^{m-} \rightarrow \frac{1}{g^2} \bar{C}^{m-}. \quad (4.2)$$

The remaining fields  $X^\dagger$ ,  $S^\dagger$ ,  $C^{m-}$  and  $\chi$  are unchanged. As before the Jacobian of these scalings is unity if we take periodic boundary conditions for the fermions  $S$  and  $\Lambda$ . There is no such requirement on the  $\chi$ . Consequently the  $g \rightarrow \infty$  limit may be safely taken.

The action now takes the form

$$S = \text{tr} \int d^d x - \frac{1}{2} |\partial_\mu X_I^\dagger|^2 + S^\dagger \not{\partial} S^\dagger - \frac{1}{2} [X_I^\dagger, A_\mu^{m-}]^2 - \frac{1}{2} [X_I^\dagger, X_J^{m+}]^2 + \Lambda^{m-} \Gamma^I [X_I^\dagger, S^{m+}] + [\bar{C}^{m-}, X_I^\dagger] [C^{m-}, X_I^\dagger] + \sum_{a=1}^{16} \bar{\chi}^a \not{\partial} \chi^a + \sum_{a=1}^{16} \bar{\chi}^a B^a \chi^a. \quad (4.3)$$

Formally, since the  $\chi$  fields are chiral, only the right moving part of the gauge field is coupled to it and one can perform the integral over the left moving part of the gauge field which sets the right moving part to zero. Hence, on integrating out the massive modes, one would be left with a completely free theory of the massless modes  $X^\dagger$ ,  $S^\dagger$  and  $\chi$ . The determinant factors would then, at least formally, cancel between the fields of various statistics.

However, the above cancellation of the determinant factors is a bit quick. If correct, it would imply that even if we had started with an anomalous theory we would end up, in the limit, with a well defined superconformal field theory. For example, this would seem to be the case if we simply ignored the  $\chi$  fields altogether. The point is that each fermionic determinant appearing is anomalous. These determinants, when defined in a vector gauge invariant way, involve extra quadratic terms in the gauge field. The presence of these would mean that the functional determinants would not cancel, since the gauge field contribution would not be  $\text{Det}(X^\dagger)^2$ . Happily, the condition that the theory be anomaly free means that the total sum of these extra pieces is zero and this is exactly what is required to make our formal argument above work.

On including the center of mass one gets  $N$  of the  $X$ 's and  $S$ 's, each transforming as a  $\mathbf{8}_V$  and  $\mathbf{8}_S$  of  $SO(8)$  respectively and  $N$   $\chi$ 's each transforming as a fundamental of  $SO(32)$ . The field content is like that of  $N$  copies of the heterotic string in the light-cone

gauge with an effective inverse tension

$$\alpha'_{\text{eff}} = \alpha' \lambda_I. \quad (4.4)$$

The condition (4.1) does not completely fix the gauge, there are still discrete transformations which leave the action invariant. There is the permutation group  $S_N$  which permutes the  $N$  copies of  $(X, S, \chi)$  and which has the interpretation of permuting the  $N$   $D$ -strings. There are also  $O(N)$  transformations which leave invariant  $X$  and  $S$  but which act non-trivially on the  $\chi$ 's by reflection giving rise to a  $Z_2^N$ . The full orbifold group is therefore the semidirect product  $S_N \ltimes Z_2^N$ .

## 5 Orbifold Partition Function

We are interested in calculating the elliptic genus of the orbifold conformal theory. In this case the  $S$  fermions have periodic boundary conditions on the world sheet torus. The elliptic genus for our conformal field theory will be zero due to the fact that the center of mass  $S$  will have zero modes in all the twisted sectors as it is orbifold group invariant. However, the zero modes of the center of mass  $S$  precisely give rise to the 8 bosonic and 8 fermionic transverse degrees of freedom that fill out the 10-dimensional  $N = 1$  vector supermultiplet corresponding to a BPS short multiplet. Our goal here is to calculate the multiplicities of these BPS states, that are clearly governed by the elliptic genus of the remaining conformal field theory that describes the relative motions of the  $D$ -strings. This means that we need to consider only those twisted sectors that have at most the zero modes of the center of mass  $S$ .

Let us briefly review how the orbifold elliptic genus is computed [18, 19]. Each twisted sector of the orbifold corresponds to a conjugacy class of the orbifold group. A general element of the group  $G = S_N \ltimes Z_2^N$  can be denoted by  $(g, \omega)$  where  $g \in S_N$  and  $\omega \in Z_2^N$ . First let us identify the twisted sectors where the  $S$ 's have no other zero modes besides the center of mass one. For this it is sufficient to consider the action of the elements of  $S_N$  since the  $Z_2^N$  part does not act on  $S$  fields. A general conjugacy class  $[g]$  in  $S_N$  is characterized by partitions  $\{N_n\}$  of  $N$  satisfying  $\sum n N_n = N$  where  $N_n$  denotes the

multiplicity of the cyclic permutation  $(n)$  of  $n$  elements in the decomposition of  $g$  as

$$[g] = (1)^{N_1} (2)^{N_2} \dots (s)^{N_s}. \quad (5.1)$$

In the  $[g]$ -twisted sector the fields satisfy the boundary condition:  $(X, S, \chi)(\sigma + 2\pi R_I) = g(X, S, \chi)(\sigma)$  where  $\sigma$  is the coordinate along the string.

In each twisted sector one must project by the centralizer subgroup  $C_g$  of  $g$ , which takes the form:

$$C_g = \prod_{n=1}^s S_{N_n} \ltimes Z_n^{N_n}, \quad (5.2)$$

where each factor  $S_{N_n}$  permutes the  $N_n$  cycles  $(n)$ , while each  $Z_n$  acts within one particular cycle  $(n)$ . In the path integral formulation this projection involves summing over all the boundary conditions along the world-sheet time direction  $t$ , twisted by elements  $h$  of  $C_g$ . We shall denote by  $([g], h)$  the twisted sector with twist  $g$  along the  $\sigma$  direction and twist  $h$  along the  $t$  direction.

We will now show that if  $[g]$  involves cycles of different lengths, say  $(n)^a$  and  $(m)^b$  with  $n \neq m$ , then the corresponding twisted sector does not contribute to the elliptic genus. To see this, we note that there are now at least two sets of zero modes for  $S$ , which can be expressed, by a suitable ordering of indices, as  $(S_1 + S_2 + \dots + S_{na})$  and  $(S_{na+1} + \dots + S_{na+mb})$ , where the two factors  $(n)^a$  and  $(m)^b$  act on the two sets of indices in the obvious way. These zero modes survive the group projection because the centralizer of  $[g]$  does not contain any element that mixes these two sets of indices with each other, thereby giving zero contribution to the elliptic genus. Thus we need only to consider those sectors with  $[g] = (L)^M$  where  $N = LM$ .

The centralizer in the case where  $[g] = (L)^M$  is  $C_g = S_M \ltimes Z_L^M$ . From the boundary condition along  $\sigma$  it is clear that there are  $L$  combinations of  $S$ 's that are periodic in  $\sigma$ . By suitable ordering, they can be expressed as  $S^k = \sum_{i=Lk+1}^{L(k+1)} S_i$  for  $k = 0, \dots, M-1$ . These zero modes have to be projected by the elements  $h$  in the centralizer  $C_g$ . In particular, when  $h$  is the generator of  $Z_M \subset S_M \subset C_g$ , it acts on the zero modes  $S^k$  by cyclic permutation. It is clear, therefore, that only the center of mass combination  $\sum_{k=0}^{M-1} S^k$  is periodic along the  $t$  direction. Hence, this sector contributes to the elliptic genus. More generally any  $h = (e, f) \in C_g = S_M \ltimes Z_L^M$  will satisfy the above criteria

provided  $\epsilon = (M) \in S_M$  and  $f$  is some element of  $Z_L^M$ . The number of such elements  $h$  is  $(M-1)! \times L^M$ .

In particular when  $N$  is prime the sectors that contribute to the elliptic genus are:  $(1, h)$  where  $h \in Z_N$  and  $([g], h)$  with  $[g] = (N)$  and  $h$  in the corresponding centralizer  $Z_N$ . In the following we shall refer to these two types of sectors as the shortest and longest string sectors, respectively.

The full orbifold group  $G$  is specified by an element of  $S_N$  (discussed above) together with an element of  $Z_2^N$  that acts on the  $\chi$ 's. Let us denote a general element by  $(g, \epsilon)$  where  $g \in S_N$  and  $\epsilon \in Z_2^N$ . Now  $S_N$  acts as an automorphism in  $Z_2^N$  by permuting the various  $Z_2$  factors. We denote this action by  $g(\epsilon)$ . Then the semi-direct product is defined in the usual way:  $(g, \epsilon) \cdot (g', \epsilon') = (gg', \epsilon g(\epsilon'))$ . Twisted sectors will now be labelled by a conjugacy class in  $G$ . The relevant sectors, for the elliptic genus computation, as discussed above, are the conjugacy classes  $[g]$  in  $S_N$  of the form  $[g] = (L)^M$  with  $N = LM$ . One can easily verify that the various classes in  $G$  are labelled by  $([g], \epsilon)$  with  $\epsilon = \epsilon_1, \epsilon_2 \dots \epsilon_M$  where each  $\epsilon_i$  is in the quotient subgroup of  $Z_2^L$  by its even subgroup. Combining this with the condition that we have found for  $h$  we may conclude that all the  $\epsilon_i$ 's must be equal (i.e. all  $\epsilon_i$ 's must be either even or odd element of  $Z_2^L$ ) in order for such  $h$  to exist in the centralizer of  $([g], \epsilon)$  in  $G$ . In this case the centralizer is the group of elements of the form  $(h, \alpha)$  where  $h \in C_g$  and  $\alpha \in Z_2^N$  satisfies  $ch(\epsilon) = \alpha g(\alpha)$ . The number of independent such  $\alpha$ 's is  $2^M$  and therefore the order of the centralizer of  $([g], \epsilon)$  is  $M!L^M 2^M$ .

We now proceed to compute the elliptic genus  $\text{tr}(-1)^F e^{-TH + 2\pi i R \tau_1 P_\sigma}$  where  $H$  and  $P_\sigma$  are the Hamiltonian and the longitudinal momentum. The computation for general  $L$  and  $M$  is quite tedious, therefore we will only describe it for the longest string sector (i.e.  $M = 1$  and  $L = N$ ). The centralizer  $C_g = Z_N$  and consists of elements of the form  $h = g^s$  for  $s = 0, 1, \dots, N-1$  and as a result their action is obtained by modular transformations  $\tau_1 \rightarrow \tau_1 + s$  from the  $h = 1$  sector. Thus we can restrict ourselves to  $h = 1$ . The eigenvalues of  $[g]$  are  $\omega^r$  for  $r = 0, 1, \dots, N-1$  where  $\omega = e^{2i\pi/N}$ . As a result the  $N$  copies of  $X$ 's and  $S$ 's come with fractional oscillator modes that are shifted by  $r/N$  in units of  $1/R_L$ . The left moving part of the non-zero mode partition function cancels between  $X$ 's and  $S$ 's. The zero modes appear for the center of mass (i.e.  $r = 0$ ): the

zero modes of  $S$  give rise to the usual 8 bosonic and 8 fermionic degrees of freedom filling out the BPS vector supermultiplet, while the zero modes of the left and right moving  $X$ 's give a factor  $\tau_2^{-4}$ . The right moving  $X$ 's, upon taking the product over  $r$ , give  $1/\eta(q^{\frac{1}{N}})^8$ , upto a zero point shift, where  $q = \exp(2\pi i \tau)$  with  $\tau = \tau_1 + i\frac{T}{2\pi R_L} \equiv \tau_1 + i\tau_2$ .

To include the contribution of the  $\chi$ 's we must specify the group elements in  $Z_2^N$  as well. There are two possible  $\epsilon$ 's that come with  $[g]$ : the even and odd element of  $Z_2^N$ . First, let us consider the situation when the Wilson lines  $B_a$  are set to zero. By taking the product of all the eigenvalues of the twist, these for odd  $N$ , give rise, respectively, to the Neveu-Schwarz and Ramond sectors of the  $SO(32)$  fermions, with  $q$  replaced by  $q^{\frac{1}{N}}$  (again upto zero point shifts). For even  $N$ , on the other hand, the Neveu-Schwarz and Ramond sectors appear for  $\epsilon$  odd and even, respectively. Furthermore the centralizer contains two elements with  $h = 1$  namely  $\alpha = \pm 1$ . These two choices give rise to the usual GSO projection.

Finally, one can compute the zero point shift for the right moving  $X$ 's and  $\chi$ 's and the result is that one actually gets  $1/N$  times the right moving part of the heterotic  $SO(32)$  partition function with  $q$  replaced by  $q^{\frac{1}{N}}$ . Including also other elements  $h \neq 1$ , the final result is:

$$\frac{1}{\tau_2^4 N} \sum_{s=0}^{N-1} Z(\omega^s q^{\frac{1}{N}}), \quad (5.3)$$

where  $Z(q)$  is the right moving part of the  $SO(32)$  heterotic partition function:

$$Z(q) = \frac{1}{\eta(q)^{24}} \sum_{\rho \in \Gamma_{16}} q^{\frac{1}{2}\rho^2}, \quad (5.4)$$

with  $\Gamma_{16}$  is the  $\text{spin}(32)/Z_2$  lattice.

For the sector with  $[g] = ((L)^M, \epsilon)$ , we can again repeat the above steps. Recall that in this case there are only two possible  $\epsilon$  that give non zero contribution to the elliptic genus. These are given by  $\epsilon = \epsilon_1, \epsilon_2 \dots \epsilon_M$ , with all  $\epsilon_i$  either an even or odd element of  $Z_2^L$ . As described above, the order of the centralizer  $C_g$  is  $M!L^M 2^M$ , while the number of elements  $h \in C_g$  that give rise to non-zero trace is  $(M-1)!L^M 2^M$ , and therefore these are the relevant elements for the computation of elliptic genus. However, not all the  $h$ 's of this form give different traces. Indeed, if  $h$  and  $h'$  are in the same conjugacy class in  $C_g$ , they will give the same trace. It is easy to verify that the number of elements in the

centralizer  $\hat{C}_h$  in  $C_g$ , for a relevant  $h$ , is  $2ML = 2N$ . As a result, the number of elements in the conjugacy class of such  $h$  in  $C_g$  is  $|C_g|/|\hat{C}_h| = (M-1)!L^{M-1}2^{M-1}$ . The distinct conjugacy classes, that give non-zero traces, can be labelled by  $[h_i^\pm]$  for  $i = 1, \dots, L$ , and the superscript  $\pm$  refers to the GSO projection on the  $\chi$ 's. Each of these classes appear with a prefactor, which is given by the number of elements in the class divided by the order of  $C_g$ , and is equal to  $1/(2N)$ . The factor  $1/2$ , together with the GSO projection implied in  $\sum_{\pm} \text{tr}[h_i^\pm]$ , for each  $i$ , gives either the scalar or spinor conjugacy classes of  $\text{spin}(32)/\mathbf{Z}_2$ . The scalar and the spinor classes of  $\text{spin}(32)/\mathbf{Z}_2$  appear for the two choices of  $\epsilon$  in  $[g]$ . For each distinct  $i$  in  $\sum_i \text{tr}[h_i^\pm]$ , one gets a different trace, and the elliptic genus, after some tedious algebra, turns out to be:

$$\frac{1}{\tau_2^4} \frac{M^4}{N} \sum_{s=0}^{L-1} Z(e^{2\pi i \frac{s}{L}} q^{\frac{M}{L}}). \quad (5.5)$$

Here, the prefactor  $M^4$  appears due to the zero modes of  $8(M-1)$   $S$ 's in the  $[g]$  twisted sector (excluding the center of mass  $S$ ). Indeed, these contribute to the trace as  $\sqrt{\prod_{j=1}^{M-1} (1 - e^{2\pi i j/M})^8} = M^4$ . Note that when  $M$  and  $L$  are not coprime, the different terms in the sum above are not all related by modular transformations of the type  $\tau \rightarrow \tau + s$ . It is, however, clear that each term, which survives the projection, comes with integer multiplicity, as it should be.

So far we had not turned on the Wilson lines  $B$  in the Cartan subalgebra of  $SO(32)$ . The presence of these Wilson lines twists the boundary conditions for  $\chi$ 's. The determinants for the  $\chi$ 's with these additional twists can be calculated in the standard way and, after including the zero point shifts, one finds that, for the longest string sector, the result is:

$$Z(1, N) = \frac{1}{\tau_2^4} \frac{1}{N} \sum_{s=0}^{N-1} \frac{1}{\eta(\omega^s q^{\frac{1}{N}})^{24}} \sum_{P \in \Gamma_{16}} \omega^{s \frac{P^2}{2}} q^{\frac{1}{2N}(P+NB)^2}, \quad (5.6)$$

and for the intermediate strings:

$$Z(L, M) = \frac{1}{\tau_2^4} \frac{M^4}{N} \sum_{s=0}^{L-1} \frac{1}{\eta(e^{2\pi i \frac{s}{L}} q^{\frac{M}{L}})^{24}} \sum_{P \in \Gamma_{16}} e^{2\pi i \frac{sP^2}{L}} q^{\frac{M}{2L}(P+LB)^2}. \quad (5.7)$$

Note that in the presence of Wilson lines, the different terms in the sum over  $s$  are not obtainable from the  $s=0$  term by modular transformations  $\tau \rightarrow \tau + s$ . This is so, because the Wilson lines, which are turned on only in the  $\sigma$  direction, break the symmetry between the  $\sigma$  and  $t$  directions.

## 6 Longest string versus intermediate or short strings

Even though the computation of the elliptic genus received contributions from both the longest string sector and the intermediate or short string sectors, it is only the longest string sector that corresponds to the threshold bound states of  $N$  D-strings. Consider for example the shortest string sector. It receives contributions only from the sector  $\text{tr} h(-1)^F$  where  $[h] = (N)$ . In this sector only the states having zero relative transverse momenta survive. In position space, therefore, the wavefunction of each of the  $N$  strings is constant along the relative separations. As a result such states are not normalizable. The same argument also applies to the intermediate strings. In this case there are  $M$  groups of strings of length  $L$  each and the wavefunction is constant as a function of the relative separations between these  $M$  groups. This state therefore represents a state of  $M$  strings, each of which is a threshold bound state of  $L$  strings. The analogue of a single particle state appears only in the longest string sector with  $M=1$  and  $L=N$ . This interpretation is also clear intuitively from the orbifold conformal field theory description, since the twisted states in this sector correspond to wavefunctions which are localized at the fixed point.

To see this more clearly, we can compactify one of the transverse directions, say  $X_8$ , on a circle of radius  $r$  and give the system a total momentum  $1/r$  along this direction. Note that this is the minimal unit of quantized momentum. We will now show that only the longest string sector can carry this momentum.

The zero modes for  $X_8^i$  ( $i = 1, \dots, N$ ) for a general twisted sector of relevance labelled by  $(L, M)$  reads, after suitable ordering of indices, as:

$$X_8^i = a^i + \frac{kr}{M} \frac{2\pi t}{T} + \frac{\ell r}{L} \frac{\sigma}{R_L}, \quad (6.1)$$

where  $a^i$  satisfy

$$\begin{aligned} a^{i+L} &= a^i + \frac{2\pi kr}{M}, \\ a^{j+1} &= a^j + \frac{2\pi \ell r}{L}, \quad j = 1, \dots, L-1, \end{aligned} \quad (6.2)$$

and  $k$  and  $\ell$  are arbitrary integers. Note that the integers  $k$  and  $\ell$  are independent of  $i$  because only the center of mass  $X_8$  is a zero mode under the combined actions of the



twists along the  $t$  and  $\sigma$  directions.  $\ell$  here denotes the winding number along  $X_8$  direction. These zero modes contribute to the action as

$$\Delta S = \frac{\pi}{\alpha'_{\text{eff}}} \left[ \frac{N}{r_2} \left( \frac{kr}{M} \right)^2 + N \tau_2 \left( \frac{\ell r}{L} \right)^2 \right]. \quad (6.3)$$

The total momentum is  $\frac{1}{2\pi\alpha'_{\text{eff}}} \int d\sigma \sum_i \partial_t X_8^i = \frac{Nkr}{M\tau_2\alpha'_{\text{eff}}}$ . We can now perform a Poisson resummation in order to go to the Hamiltonian formulation, with the result that the total momentum  $p$  along the  $X_8$  direction is:

$$p = \frac{Mk}{r}, \quad (6.4)$$

with  $k$  some integer and the partition function is

$$\sum_{k,\ell} q^{i\alpha'_{\text{eff}} \frac{p_L^2}{N}} \bar{q}^{i\alpha'_{\text{eff}} \frac{p_R^2}{N}} \sqrt{\tau_2} Z(L, M), \quad (6.5)$$

where  $p_L = M(\frac{\alpha'_{\text{eff}} k}{r} + \ell r)$  and  $p_R = M(\frac{\alpha'_{\text{eff}} k}{r} - \ell r)$ . This shows that the smallest unit of momentum,  $p = 1/r$ , gets contributions only from the  $M = 1$  (i.e. the longest string). From the above partition function we see that this state carries an extra energy given by  $\alpha'_{\text{eff}} p^2 / 2N R_I$  which, as we shall see below, is exactly what is expected from the heterotic string side. When  $p = M/r$ , the  $(L, M)$  sector also contributes to the elliptic genus. However, this is consistent with the interpretation that it corresponds to  $M$  groups of strings, each carrying a momentum  $1/r$ .

## 7 Comparison with the heterotic spectrum

Let us now compare the above results for the bound states of  $N$  type I D-strings wrapped around the  $X^9$  circle with the heterotic spectrum in 9 dimensions. Clearly the relevant states on the heterotic side are the ones carrying a winding number  $N$ . The remaining quantum numbers are the  $U(1)^{16}$  charges of the Cartan subalgebra of  $SO(32)$  and the Kaluza-Klein momentum along the  $X^9$  direction. The  $U(1)^{16}$  charges, on the type I side, can be read off from the  $\Gamma_{16}$  lattice charges that appear in the partition function  $Z(1, N)$ . The Kaluza-Klein momentum  $p_9$ , on the other hand, is the longitudinal momentum  $P_\sigma$  of the D-string system along the  $\sigma$  direction. Given the fact that  $P_\sigma$  is the difference of the left and right Virasoro generators  $L_0 - \bar{L}_0$ , its charge is just given by the coefficient of

$2\pi i R_I \tau_1$  in the partition function. From the expression (5.6) for  $Z(1, N)$  and taking into account the projection implied by the sum over  $s$ , we conclude that

$$\begin{aligned} P_\sigma &= \frac{1}{R_I} (k + B \cdot P + \frac{1}{2} B^2 N), \\ k &= \frac{1}{N} \left( \frac{1}{2} P^2 + N_R - 1 \right) \in \mathbf{Z}, \end{aligned} \quad (7.1)$$

where  $(N_R - 1)/N$  appears from the expansion of  $\eta(q^{\frac{1}{N}})^{-24}$ . Note that the multiplicity of these states is the same as the coefficient of  $q^{N_R - 1}$  in the expansion of  $\eta(q)^{-24}$ . The value of  $k$  is bounded below by  $(-1)$  for  $N = 1$  and by  $0$  for  $N > 1$ , while the value of  $P_\sigma$  is bounded by  $-1/N R_I$ . These two equations are exactly the ones appearing in (1.5) for the Kaluza-Klein momentum and the level matching condition (1.1) for the BPS states. It is also clear that the two multiplicities match, as both are given by the coefficient of  $q^{N_R - 1}$  in the expansion of  $\eta(q)^{-24}$ .

Furthermore, the mass of the bound state is the original mass of  $N$  D-strings wrapped around the circle, plus the energy carried by the excitation, which is given by the coefficient of  $T = R_I \tau_2$  in  $Z(1, N)$ . Since the partition function depends only on  $q$  the latter is equal to  $P_\sigma$ . Thus the total energy is  $\frac{N R_I}{\alpha' \lambda_I} + P_\sigma$ . This is exactly the mass given in (1.6) predicted by the duality upto a sign. As mentioned earlier  $P_\sigma \geq -1/N R_I$ . Therefore, for  $R_I^2 > \alpha' \lambda_I / N^2$  the quantity  $\frac{N R_I}{\alpha' \lambda_I} + P_\sigma$  is positive definite and hence it coincides with the absolute value appearing in (1.6). However, for  $R_I^2 < \alpha' \lambda_I / N^2$  this quantity is negative, for a suitable choice of the Wilson line  $B$ , and the result would not make sense. But this is exactly the region in which the type I perturbation theory breaks down, as argued in [2].

Finally, we consider the situation discussed in the last section where a transverse direction is compactified on a circle of radius  $r_I$  and the system carries a momentum  $k/r_I$ . This does not alter the level matching condition and therefore the multiplicity of the state. Recalling that  $\alpha'_{\text{eff}} = \alpha' \lambda_I$ , we find that the extra energy is  $k^2 \alpha' \lambda_I / 2N R_I r_I^2$ . On the heterotic side the mass for a state with winding number  $N$  along the  $X_9$  direction and carrying momentum  $k/r_H$  along the  $X_8$  direction is given by

$$\frac{1}{\alpha'} \sqrt{N^2 R_H^2 + \left( \frac{k \alpha'}{r_H} \right)^2}. \quad (7.2)$$

By using the duality relations (1.4) and expanding the square root to the leading order in

$\lambda_I$  we find that the extra mass is exactly  $k^2\alpha'\lambda_I/2NR_Ir_I^2$ , in agreement with the prediction of duality.

To conclude, we have analysed in detail the orbifold conformal theory arising in the infra-red limit of the  $O(N)$  gauge theory that describes the low energy modes of a system of  $N$  type I D-strings. We argued that the longest string sector of the orbifold theory describes the bound states of D-strings. An additional support for this identification also comes by compactifying one of the transverse directions. We have shown that this identification gives masses and multiplicities of the bound states in agreement with the predictions of heterotic-type I duality in 9 dimensions, for all the BPS charges in  $\Gamma_{(1,17)}$ . In particular the Kaluza-Klein momentum of the heterotic theory is mapped to the longitudinal momentum of the D-strings.

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