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Analytic expressions for the integrals with singularities in the Feynman parametrization

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Abstract

~~We derive~~ Analytic expressions for the integrals with singularities in the Feynman parametrization. are derived

1 Introduction

In the calculation of transition amplitudes, the propagation of on-shell intermediate state is expressed mathematically by the presence of singularities in the integrals. This requires an analytic continuation into the complex plane and gives rise to an imaginary part in the amplitude. In the following we consider integrals expressed in the Feynman parametrization taking the form:

$$\lim_{\varepsilon' \to 0^+} \int_a^b \frac{x^m \, dx}{(\alpha' x + \beta' \pm i\varepsilon')^n} \qquad \text{or} \qquad \lim_{\varepsilon' \to 0^+} \int_a^b \frac{x^m \, dx}{(\alpha' x^2 + \beta' x + \gamma' \pm i\varepsilon')^n} \tag{1}$$

with $n = 1, 2$. We will assume that the singularities cannot be located in a or b.

When there is only one singularity, it is always located on the upper part or the lower part of the real axis depending on the sign in front of $i\varepsilon'$. In that case, it is possible to calculate these integrals by choosing a closed integration path as shown on Fig.1 for a singularity with a negative imaginary part. The integration is then performed choosing the new complex variable $z = \frac{a+b}{2} + \frac{b-a}{2} e^{i\theta}$ and the result is obtained from the Cauchy theorem. When the singularity has a positive imaginary part, the closed integration path is chosen as a semicircle located in the lower part of the complex plane. When there are two singularities, the preceding method may be extented if the two solutions have different real parts (as we shall see later, we are only concerned with singularities with opposite signs in the imaginary part) and the integral is calculated by using the theorem of residues.

Thanks to the extreme simplicity of the numerator in our case, it is possible to find analytic expressions for these integrals allowing more accurate results and to save computing time in practical calculations. The corresponding expressions are given in sections 2 to 4.

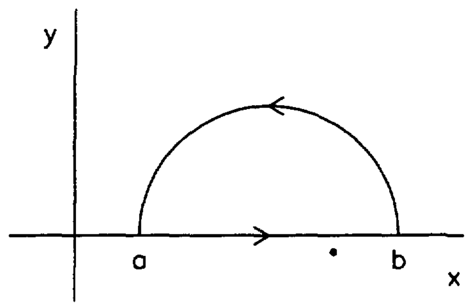

Figure 1: *Integration path corresponding to a singularity with a negative imaginary part.*

Moreover, when there are two singularities with the *same* real part which are located on each side of the real axis, the integration in the complex plane fails and the analytic expressions will be very useful. These integrals correspond to the following expressions:

$$\lim_{\epsilon \to 0^+} \int_a^b \frac{x^m \, dx}{(x - x_0 + i\varepsilon)^n \, (x - x_0 - i\varepsilon)^n} \tag{2}$$

where the singularities are shown on Fig.2. The expressions corresponding to these integrals are given for n=1 in section 5 and their generalization where the numerator is a polynom is given in section 6. In the appendix, all the details of the calculations are shown with the formulae extracted from Ref. [1].

In this report, we give the details of the calculations leading to analytic expressions. The principle of the method is based on the following relation

$$\lim_{\epsilon \to 0^+} \int_a^b \frac{f(x) \, dx}{(x - x_0 \pm i\varepsilon)^n}$$

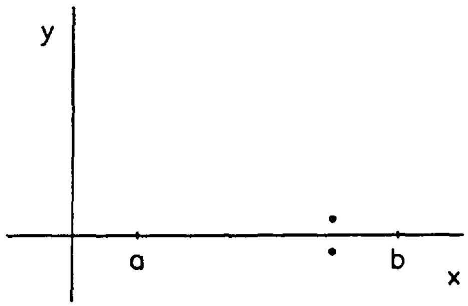

Figure 2: *Singularities with the same real part and opposite imaginary parts.*

$$= \lim_{\varepsilon \to 0^+} \lim_{\eta \to 0^+} \left[\int_a^{x_0-\eta} \frac{f(x)\, dx}{(x - x_0 \pm i\varepsilon)^n} + \int_{x_0-\eta}^{x_0+\eta} \frac{f(x)\, dx}{(x - x_0 \pm i\varepsilon)^n} + \int_{x_0+\eta}^{b} \frac{f(x)\, dx}{(x - x_0 \pm i\varepsilon)^n} \right] \tag{3}$$

2 Singularity of the form $\lim_{\varepsilon' \to 0^+} \int_a^b \frac{x^m\, dx}{\alpha' x + \beta' \pm i\varepsilon'}$

The denominator is written:

$$\alpha' x + \beta' \pm i\varepsilon' = \alpha' \, (x - x_0 \pm i\varepsilon) \ \text{ with } \ x_0 = -\frac{\beta'}{\alpha'} \ \text{ and } \ \varepsilon = \frac{\varepsilon'}{\alpha'} \tag{4}$$

We have two types of integrals which depend on the sign of α'.

1) If $\alpha' > 0$:

$$\lim_{\varepsilon' \to 0^+} \int_a^b \frac{x^m\, dx}{\alpha' x + \beta' \pm i\varepsilon'} = \frac{1}{\alpha'} \lim_{\varepsilon \to 0^+} \int_a^b \frac{x^m\, dx}{x - x_0 \pm i\varepsilon} \tag{5}$$

2) If $\alpha' < 0$

$$\lim_{\varepsilon' \to 0^+} \int_a^b \frac{x^m\, dx}{\alpha' x + \beta' \pm i\varepsilon'} = \frac{1}{\alpha'} \lim_{\varepsilon \to 0^+} \int_a^b \frac{x^m\, dx}{x - x_0 \mp i\varepsilon} \tag{6}$$

We are let to study the integral

$$\lim_{\varepsilon\to 0^+}\int_a^b \frac{x^m\ dx}{x-x_0\pm i\varepsilon} \tag{7}$$

We make a separation in a real part and an imaginary part:

$$\frac{1}{x-x_0\pm i\varepsilon}=\frac{x-x_0}{(x-x_0)^2+\varepsilon^2}\mp i\frac{\varepsilon}{(x-x_0)^2+\varepsilon^2} \tag{8}$$

If we define $z=x-x_0$, the integral is resolved into a real and imaginary parts

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\int_a^b \frac{x^m\ dx}{x-x_0\pm i\varepsilon}\\ &=\lim_{\varepsilon\to 0^+}\Big[\int_{a-x_0}^{b-x_0}\frac{z\ (z+x_0)^m\ dz}{z^2+\varepsilon^2}\mp i\varepsilon\int_{a-x_0}^{b-x_0}\frac{(z+x_0)^m\ dz}{z^2+\varepsilon^2}\ \Big]\end{aligned} \tag{9}$$

The two integrals are even functions of ε. From these above expressions, we can see that the sign of the imaginary part is always opposite to the sign of the prescription $\pm i\varepsilon$.

We perform the following expansion

$$(z+x_0)^m=\sum_{k=0}^{m}\binom{m}{k}x_0^{m-k}\ z^k \qquad \text{with } \binom{m}{k}=\frac{m!}{k!(m-k)!} \tag{10}$$

which gives:

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\int_a^b \frac{x^m\ dx}{x-x_0\pm i\varepsilon}\\ &=\sum_{k=0}^{m}\binom{m}{k}x_0^{m-k}\Big\{\lim_{\varepsilon\to 0^+}\Big[\int_{a-x_0}^{b-x_0}\frac{z^{k+1}\ dz}{z^2+\varepsilon^2}\mp i\varepsilon\int_{a-x_0}^{b-x_0}\frac{z^k\ dz}{z^2+\varepsilon^2}\ \Big]\Big\}\\ &=\sum_{k=0}^{m}\binom{m}{k}x_0^{m-k}\lim_{\varepsilon\to 0^+}\ J_{k+1}^1(\varepsilon)\mp i\sum_{k=0}^{m}\binom{m}{k}x_0^{m-k}\lim_{\varepsilon\to 0^+}\ \varepsilon J_k^1(\varepsilon)\end{aligned} \tag{11}$$

where we have defined

$$J_k^1(\varepsilon)=\int_{a-x_0}^{b-x_0}\frac{z^k\ dz}{z^2+\varepsilon^2} \tag{12}$$

From the appendix A,

$$\lim_{\varepsilon\to 0^+} \varepsilon J_k^1(\varepsilon) = \begin{cases} \pi & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases} \tag{13}$$

we then define:

$$J_k = \lim_{\varepsilon\to 0^+} J_k^1(\varepsilon) \quad k \geq 1 \tag{14}$$

which gives

$$J_1 = \frac{1}{2}\ln\frac{(b-x_0)^2}{(a-x_0)^2}; \qquad J_n = \frac{1}{n-1}\Big[\, (b-x_0)^{n-1} - (a-x_0)^{n-1} \,\Big]; \qquad n \geq 2 \tag{15}$$

and:

$$\lim_{\varepsilon\to 0^+} \int_a^b \frac{x^m\, dx}{x - x_0 \pm i\varepsilon} = \sum_{k=0}^{m} \binom{m}{k} x_0^{m-k}\, J_{k+1} \mp\, i\pi x_0^m \tag{16}$$

In fact, as we now show, when the numerator x^m is replaced by $f(x)$, the imaginary part of the integral is equal to $\mp\pi\ f(x_0)$. From eq. 8:

$$\begin{aligned} &\lim_{\varepsilon\to 0^+} \int_a^b \frac{f(x)\, dx}{x - x_0 \pm i\varepsilon} \\ &= \lim_{\varepsilon\to 0^+} \Big[\int_{a-x_0}^{b-x_0} \frac{z\, f(z+x_0)\, dz}{z^2+\varepsilon^2} \mp i\varepsilon \int_{a-x_0}^{b-x_0} \frac{f(z+x_0)\, dz}{z^2+\varepsilon^2} \,\Big] \end{aligned} \tag{17}$$

So $\mathcal{I}$, the imaginary part of the integral is given by:

$$\mathcal{I} = \Im m\left\{ \lim_{\varepsilon\to 0^+} \int_a^b \frac{f(x)\, dx}{x - x_0 + i\varepsilon} \right\} = \mp\varepsilon \int_{a-x_0}^{b-x_0} \frac{f(z+x_0)\, dz}{z^2+\varepsilon^2} \tag{18}$$

Because $\varepsilon \to 0$ and the numerator has finite values, this integral goes to 0 except when $z \to 0$. So we have

$$\mathcal{I} = \mp \lim_{\varepsilon\to 0^+} \lim_{\eta\to 0^+} \varepsilon \int_{-\eta}^{\eta} \frac{f(z+x_0)\, dz}{z^2+\varepsilon^2} \tag{19}$$

We make a Taylor expansion:

$$f(z+x_0) = f(x_0) + z f^{(1)}(x_0) + \frac{z^2}{2!} f^{(2)}(x_0) + \frac{z^3}{3!} f^{(3)}(x_0) + \cdots \tag{20}$$

Only the value z^k with k even contribute to this integral. But

$$\lim_{\eta\to 0^+} \int_{-\eta}^{\eta} \frac{z^{2n}\, dz}{z^2+\varepsilon^2} = 0 \qquad \text{for } n \geq 1 \tag{21}$$

So, finally:

$$\begin{aligned}
\mathcal{I} &= \mp f(x_0) \lim_{\varepsilon\to 0^+} \lim_{\eta\to 0^+} \varepsilon \int_{-\eta}^{\eta} \frac{dz}{z^2+\varepsilon^2} = \mp f(x_0) \lim_{\varepsilon\to 0^+} \lim_{\eta\to 0^+} \varepsilon \Big[\frac{2}{\varepsilon} \tan^{-1} \frac{\eta}{\varepsilon}\Big] \\
&= \mp 2 f(x_0) \lim_{\eta\to 0^+} \lim_{\varepsilon\to 0^+} \Big[\frac{\pi}{2} - \frac{\varepsilon}{\eta} + \cdots\Big] \\
&= \mp\pi\, f(x_0)
\end{aligned} \tag{22}$$

3 Singularity of the form $\lim_{\varepsilon'\to 0^+} \int_a^b \frac{x^m\, dx}{(\alpha' x + \beta' \pm i\varepsilon')^2}$

As in the previous case, there are two types of integrals which depend on the sign of α'.

1) If $\alpha' > 0$:

$$\lim_{\varepsilon'\to 0^+} \int_a^b \frac{x^m\, dx}{(\alpha' x + \beta' \pm i\varepsilon')^2} = \frac{1}{\alpha'^2} \lim_{\varepsilon\to 0^+} \int_a^b \frac{x^m\, dx}{(x - x_0 \pm i\varepsilon)^2} \tag{23}$$

2) If $\alpha' < 0$

$$\lim_{\varepsilon'\to 0^+} \int_a^b \frac{x^m\, dx}{(\alpha' x + \beta' \pm i\varepsilon')^2} = \frac{1}{\alpha'^2} \lim_{\varepsilon\to 0^+} \int_a^b \frac{x^m\, dx}{(x - x_0 \mp i\varepsilon)^2} \tag{24}$$

We turn now to study the integral

$$\lim_{\varepsilon\to 0^+} \int_a^b \frac{x^m\, dx}{(x - x_0 \pm i\varepsilon)^2} \tag{25}$$

We make a separation in a real and an imaginary part:

$$\frac{1}{(x-x_0 \pm i\varepsilon)^2} = \frac{(x-x_0)^2-\varepsilon^2}{((x-x_0)^2+\varepsilon^2)^2} \mp 2\, i \frac{\varepsilon(x-x_0)}{((x-x_0)^2+\varepsilon^2)^2} \tag{26}$$

so

$$\lim_{\varepsilon\to 0^+} \int_a^b \frac{x^m\, dx}{(x-x_0\pm i\varepsilon)^2}$$

$$= \lim_{\varepsilon\to 0^+} \Big[\int_{a-x_0}^{b-x_0} \frac{(z^2-\varepsilon^2)\,(z+x_0)^m\, dz}{(z^2+\varepsilon^2)^2} \mp 2\, i\varepsilon \int_{a-x_0}^{b-x_0} \frac{z\,(z+x_0)^m\, dz}{(z^2+\varepsilon^2)^2} \Big]$$

$$= \sum_{k=0}^{m} \binom{m}{k} x_0^{m-k} \Big\{ \lim_{\varepsilon\to 0^+} \Big[\int_{a-x_0}^{b-x_0} \frac{z^{k+2}\, dz}{(z^2+\varepsilon^2)^2} - \varepsilon^2 \int_{a-x_0}^{b-x_0} \frac{z^k\, dz}{(z^2+\varepsilon^2)^2} \mp 2i\varepsilon \int_{a-x_0}^{b-x_0} \frac{z^{k+1}\, dz}{(z^2+\varepsilon^2)^2} \Big] \Big\}$$

$$= \sum_{k=0}^{m} \binom{m}{k} x_0^{m-k} \lim_{\varepsilon\to 0^+} \left(J^2_{k+2}(\varepsilon) - \varepsilon^2 J^2_k(\varepsilon) \right) \mp 2i \sum_{k=0}^{m} \binom{m}{k} x_0^{m-k} \lim_{\varepsilon\to 0^+} \varepsilon J^2_{k+1}(\varepsilon) \tag{27}$$

where we have defined

$$J^2_k(\varepsilon) = \int_{a-x_0}^{b-x_0} \frac{z^k\, dz}{(z^2+\varepsilon^2)^2} \tag{28}$$

From the appendix A:

$$\text{for } k \neq 0,\ \lim_{\varepsilon\to 0^+} \varepsilon J^2_k(\varepsilon) = \begin{cases} \frac{\pi}{2} & \text{if } k=2 \\ 0 & \text{if } k \neq 2 \end{cases} \tag{29}$$

so, only the term $k=1$ will contribute to the imaginary part of the integral

we then define:

$$I_k = \lim_{\varepsilon\to 0^+} \left(J^2_{k+2}(\varepsilon) - \varepsilon^2 J^2_k(\varepsilon) \right) \tag{30}$$

which gives:

$$I_0 = \frac{1}{a-x_0} - \frac{1}{b-x_0} \qquad I_1 = \frac{1}{2} \ln \frac{(b-x_0)^2}{(a-x_0)^2} \tag{31}$$

$$I_n = \frac{1}{n-1} \left[(b-x_0)^{n-1} - (a-x_0)^{n-1} \right] \qquad n \geq 2 \tag{32}$$

Then:

$$\lim_{\varepsilon\to 0^+}\int_a^b \frac{x^m\,dx}{(x-x_0\pm i\varepsilon)^2} = \sum_{k=0}^{m}\binom{m}{k} x_0^{m-k}\, I_k \mp i\pi\, m\, x_0^{m-1} \tag{33}$$

As we now show, the imaginary part can be derived for any function $f(x)$. From the equation (26), we have,

$$\mathcal{I} = \Im m\left\{\lim_{\varepsilon\to 0^+}\int_a^b \frac{f(x)\,dx}{(x-x_0\pm i\varepsilon)^2}\right\} = \mp 2\lim_{\varepsilon\to 0^+}\varepsilon\int_a^b \frac{f(x)(x-x_0)\,dx}{((x-x_0)^2+\varepsilon^2)^2} \tag{34}$$

With the arguments given in the preceding section, we write:

$$\mathcal{I} = \mp 2\lim_{\varepsilon\to 0^+}\varepsilon\lim_{\eta\to 0^+}\int_{x_0-\eta}^{x_0+\eta} \frac{f(x)(x-x_0)\,dx}{((x-x_0)^2+\varepsilon^2)^2} = \mp 2\lim_{\varepsilon\to 0^+}\lim_{\eta\to 0^+}\varepsilon\int_{-\eta}^{\eta}\frac{f(z+x_0)z\,dz}{(z^2+\varepsilon^2)^2} \tag{35}$$

We make a Taylor expansion:

$$f(z+x_0) = f(x_0) + zf^{(1)}(x_0) + \frac{z^2}{2!}f^{(2)}(x_0) + \frac{z^3}{3!}f^{(3)}(x_0) + \cdots \tag{36}$$

Only the terms z^k with k odd in the expansion can contribute to this integral. But

$$\lim_{\eta\to 0^+}\int_{-\eta}^{\eta}\frac{z^{2n}\,dz}{(z^2+\varepsilon^2)^2} = 0 \qquad \text{for } n\geq 2 \tag{37}$$

and only the term z in the expansion will contribute. We then have

$$\begin{aligned}
\mathcal{I} &= \mp 2f^{(1)}(x_0)\lim_{\varepsilon\to 0^+}\varepsilon\lim_{\eta\to 0^+}\int_{-\eta}^{\eta}\frac{z^2\,dz}{(z^2+\varepsilon^2)^2} = \mp 2f^{(1)}(x_0)\lim_{\varepsilon\to 0^+}\varepsilon\lim_{\eta\to 0^+}\Big(-\frac{\eta}{\eta^2+\varepsilon^2}+\frac{1}{\varepsilon}\tan^{-1}\frac{\eta}{\varepsilon}\Big)\\
&= \mp 2f^{(1)}(x_0)\lim_{\varepsilon\to 0^+}\lim_{\eta\to 0^+}\tan^{-1}\frac{\eta}{\varepsilon}\\
&= \mp 2f^{(1)}(x_0)\lim_{\eta\to 0^+}\lim_{\varepsilon\to 0^+}\tan^{-1}\frac{\eta}{\varepsilon} = mp2f^{(1)}(x_0)\lim_{\eta\to 0^+}\lim_{\varepsilon\to 0^+}[\frac{\pi}{2}-\frac{\varepsilon}{\eta}+\cdots]\\
&= \mp f^{(1)}(x_0)\,\pi
\end{aligned} \tag{38}$$

4 Singularity of the form $\lim_{\varepsilon'\to 0^+}\int_a^b \frac{x^m\ dx}{(\alpha' x^2+\beta' x+\gamma'\pm i\varepsilon')^n}\quad n=1,2$

The denominator reads:

$$(\alpha' x^2+\beta' x+\gamma'\pm i\varepsilon')^n=\alpha'^n\ (x^2+\beta x+\gamma\pm i\varepsilon)^n\qquad \beta=\frac{\beta'}{\alpha'}\qquad \gamma=\frac{\gamma'}{\alpha'}\qquad \varepsilon=\frac{\varepsilon'}{\alpha'}\quad (39)$$

1) If $\alpha'>0$

$$\lim_{\varepsilon'\to 0^+}\int_a^b \frac{x^m\ dx}{(\alpha' x^2+\beta' x+\gamma'\pm i\varepsilon')^n}=\frac{1}{\alpha'^n}\lim_{\varepsilon\to 0^+}\int_a^b \frac{x^m\ dx}{(x^2+\beta x+\gamma\pm i\varepsilon)^n}\qquad (40)$$

2) If $\alpha'<0$

$$\lim_{\varepsilon'\to 0^+}\int_a^b \frac{x^m\ dx}{(\alpha' x^2+\beta' x+\gamma'\pm i\varepsilon')^n}=\frac{1}{\alpha'^n}\lim_{\varepsilon\to 0^+}\int_a^b \frac{x^m\ dx}{(x^2+\beta x+\gamma\mp i\varepsilon)^n}\qquad (41)$$

In both cases, we have to calculate the following integrals:

$$\lim_{\varepsilon\to 0^+}\int_a^b \frac{x^m\ dx}{(x^2+\beta x+\gamma\pm i\varepsilon)^n}\qquad n=1,2\qquad (42)$$

In this expression, β, γ and ε are real numbers.

The discriminant of the denominator is a complex number written Δ.

$$\Delta=\beta^2-4(\gamma\pm i\varepsilon)=\delta\mp i4\varepsilon;\qquad \delta=\beta^2-4\gamma\qquad (43)$$

The real and imaginary parts of Δ can be expressed explicitely in function of β, γ and ε.

$$\Delta=\rho\ e^{i\theta};\qquad \rho=\sqrt{\delta^2+16\varepsilon^2};\qquad \cos\theta=\frac{\delta}{\rho};\qquad \sin\theta=\mp\frac{4\varepsilon}{\rho};\qquad (44)$$

The denominator has two solutions denoted x_+ and x_-:

$$x_+=\frac{-\beta+\sqrt{\Delta}}{2};\qquad x_-=\frac{-\beta-\sqrt{\Delta}}{2};\qquad (45)$$

When ε goes to 0, the sqare root of the complex discriminant we are interested in is given by:

$$\sqrt{\Delta}=\sqrt{\rho}\ e^{i\frac{\theta}{2}}=\sqrt{\rho}\ (\cos\frac{\theta}{2}+i\sin\frac{\theta}{2})\qquad (46)$$

$$\frac{1}{\rho} = \frac{1}{\sqrt{\delta^2 + 16\varepsilon^2}} \approx \frac{1}{\delta}\left(1 - \frac{8\varepsilon^2}{\delta^2}\right) \tag{47}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} \approx \sqrt{1 - \frac{4\varepsilon^2}{\delta^2}} \approx 1 - \frac{2\varepsilon^2}{\delta^2} \tag{48}$$

$$\sin\frac{\theta}{2} = \frac{1}{2}\,\frac{\sin\theta}{\cos\frac{\theta}{2}} \approx \mp\frac{2\varepsilon}{\delta} \tag{49}$$

which gives:

$$\sqrt{\Delta} \approx \sqrt{\delta}(1 \mp i\frac{2\varepsilon}{\delta}) = \sqrt{\delta} \mp i\frac{2\varepsilon}{\sqrt{\delta}} \tag{50}$$

The two solutions x_+ and x_- have opposite imaginary parts. We now define

$$\tilde{\varepsilon} = \frac{2\varepsilon}{\sqrt{\delta}} \tag{51}$$

then the two solutions read:

$$x_+ = \frac{-\beta + \sqrt{\delta}}{2} - i\tilde{\varepsilon}; \qquad x_- = \frac{-\beta - \sqrt{\delta}}{2} + i\tilde{\varepsilon}; \tag{52}$$

The real part of these values are given by:

$$x_+^R = \frac{-\beta + \sqrt{\delta}}{2}; \qquad x_-^R = \frac{-\beta - \sqrt{\delta}}{2}; \tag{53}$$

The denominator is now written:

$$x^2 + \beta x + \gamma \pm i\varepsilon = (x - x_+)(x - x_-) = (x - x_+^R \pm i\tilde{\varepsilon})\,(x - x_-^R \mp i\tilde{\varepsilon}) \tag{54}$$

Finally, we obtain:

$$\lim_{\varepsilon \to 0^+} \int_a^b \frac{x^m\,dx}{(x^2 + \beta x + \gamma \pm i\varepsilon)^n} = \lim_{\tilde{\varepsilon} \to 0^+} \int_a^b \frac{x^m\,dx}{(x - x_+^R \pm i\tilde{\varepsilon})^n\,(x - x_-^R \mp i\tilde{\varepsilon})^n} \tag{55}$$

Generally, we have two singularities with opposite signs for the imaginary part. They are shown on Fig. 3.

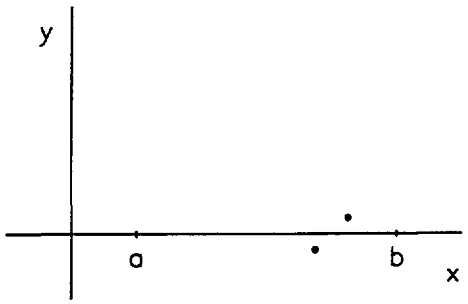

Figure 3: *Singularities corresponding to* $(x - x_+^R \pm i\tilde{\varepsilon})^n \; (x - x_-^R \mp i\tilde{\varepsilon})^n$

To calculate these integrals, we perform a decomposition of in elementary fractions.

Calculation for n=1.

$$\frac{1}{(x - x_+^R \pm i\tilde{\varepsilon})\,(x - x_-^R \mp i\tilde{\varepsilon})} = \frac{A}{x - x_+^R \pm i\tilde{\varepsilon}} + \frac{B}{x - x_-^R \mp i\tilde{\varepsilon}} \tag{56}$$

we find

$$A = \frac{1}{x_+^R - x_-^R \mp 2i\tilde{\varepsilon}} = \frac{1}{\sqrt{\delta} \mp 2i\tilde{\varepsilon}} \approx \frac{1}{\sqrt{\delta}} \qquad B = -A \tag{57}$$

Finally:

$$\lim_{\varepsilon \to 0^+} \int_a^b \frac{x^m\, dx}{x^2 + \beta x + \gamma \pm i\varepsilon} = \frac{1}{\sqrt{\delta}} \lim_{\tilde{\varepsilon} \to 0^+} \int_a^b \frac{x^m\, dx}{x - x_+^R \pm i\tilde{\varepsilon}} - \frac{1}{\sqrt{\delta}} \lim_{\tilde{\varepsilon} \to 0^+} \int_a^b \frac{x^m\, dx}{x - x_-^R \mp i\tilde{\varepsilon}} \tag{58}$$

Calculation for n=2.

$$\frac{1}{(x - x_+)^2\,(x - x_-)^2} = \frac{A_2}{(x - x_+)^2} + \frac{B_2}{(x - x_-)^2} + \frac{A_1}{x - x_+} + \frac{B_1}{x - x_-} \tag{59}$$

The final result is given by:

$$
\begin{aligned}
&\lim_{\varepsilon\to 0^+}\int_a^b \frac{x^m\,dx}{(x^2+\beta x+\gamma\pm i\varepsilon)^2}\\
&= \frac{1}{\delta}\lim_{\tilde\varepsilon\to 0^+}\int_a^b \frac{x^m\,dx}{(x-x_+^R\pm i\tilde\varepsilon)^2} - \frac{2}{\delta^{3/2}}\lim_{\tilde\varepsilon\to 0^+}\int_a^b \frac{x^m\,dx}{x-x_+^R\pm i\tilde\varepsilon}\\
&\quad + \frac{1}{\delta}\lim_{\tilde\varepsilon\to 0^+}\int_a^b \frac{x^m\,dx}{(x-x_-^R\mp i\tilde\varepsilon)^2} + \frac{2}{\delta^{3/2}}\lim_{\tilde\varepsilon\to 0^+}\int_a^b \frac{x^m\,dx}{x-x_-^R\mp i\tilde\varepsilon}
\end{aligned}
\tag{60}
$$

As it is seen, the equations 58, 60 become infinite when $\sqrt{\delta}=x_+^R-x_-^R$ goes to zero. This case corresponds to pinch singularities and is developed in the next section.

5 Singularity of the form $\lim_{\varepsilon\to 0^+}\int_a^b \frac{x^m\,dx}{(x-x_0+i\varepsilon)\,(x-x_0-i\varepsilon)}$

In this type of integrals, the two poles have the *same* real part which leads to new expressions.

$$
\begin{aligned}
\lim_{\varepsilon\to 0^+}\int_a^b \frac{x^m\,dx}{(x-x_0+i\varepsilon)\,(x-x_0-i\varepsilon)} &= \lim_{\varepsilon\to 0^+}\int_{a-x_0}^{b-x_0} \frac{(z+x_0)^m\,dz}{z^2+\varepsilon^2}\\
&= \sum_{k=0}^{m}\binom{m}{k} x_0^{m-k}\lim_{\varepsilon\to 0^+} J_k^1(\varepsilon)
\end{aligned}
\tag{61}
$$

We see that this integral is purely real. From the appendix A, we know that the $k=0$ term diverges:

$$
\lim_{\varepsilon\to 0^+} J_0^1(\varepsilon) = \frac{\pi}{\varepsilon} \tag{62}
$$

As previously defined,

$$
J_k = \lim_{\varepsilon\to 0^+} J_k^1(\varepsilon) \quad k\geq 1 \tag{63}
$$

which gives:

$$\lim_{\varepsilon \to 0^+} \int_a^b \frac{x^m \, dx}{(x - x_0 + i\varepsilon)\,(x - x_0 - i\varepsilon)} = x_0^m \lim_{\varepsilon \to 0^+} \frac{\pi}{\varepsilon} + \sum_{k=1}^{m} \binom{m}{k} x_0^{m-k} \, J_k \tag{64}$$

The last term is defined only for $m > 0$ and the coefficients J_k are given explicitely by:

$$J_1 = \frac{1}{2} \ln \frac{(b - x_0)^2}{(a - x_0)^2}; \qquad J_n = \frac{1}{n-1} \Big[\, (b - x_0)^{n-1} - (a - x_0)^{n-1} \,\Big]; \qquad n \geq 2 \tag{65}$$

6 Singularity of the form $\lim_{\varepsilon \to 0^+} \int_a^b \frac{P(x) \, dx}{(x - x_0 + i\varepsilon)\,(x - x_0 - i\varepsilon)}$

From the relations in the previous section, if the numerator is any polynomial $P(x)$, all these integrals will be divergent except if $P(x_0) = 0$. Let us assume that this property is verified. We write

$$P(x) = \sum_{i=0}^{N} \alpha_i \, x^i \tag{66}$$

which gives:

$$P(x) = P(x) - P(x_0) = \sum_{i=1}^{N} \alpha_i \, (x^i - x_0^i) \tag{67}$$

From the preceding section, we have ($m > 0$):

$$\lim_{\varepsilon \to 0^+} \int_a^b \frac{(x^m - x_0^m) \, dx}{(x - x_0 + i\varepsilon)\,(x - x_0 - i\varepsilon)} = \sum_{k=1}^{m} \binom{m}{k} x_0^{m-k} \, J_k^1 \tag{68}$$

If there are ℓ Feynman graphs with the same denominator, with the numerators $P_1(x)$, $\ldots P_\ell(x)$, we only need $P(x) = P_1(x) + \ldots P_\ell(x)$ to be equal to 0 for $x = x_0$. We now write

$$\begin{aligned} P(x) &= P(x) - P(x_0) = P_1(x) + \ldots + P_\ell(x) - [P_1(x_0) + \ldots + P_\ell(x_0)] \\ &= P_1(x) - P_1(x_0) + \ldots + P_\ell(x) - P_\ell(x_0) \end{aligned} \tag{69}$$

Then it is possible to calculate each integral with the modified numerators $P_1(x) - P_1(x_0)$, $\ldots P_\ell(x) - P_\ell(x_0)$

A Expression of the elementary integrals

A.1 Calculation of $J_k^1(\varepsilon)$

The integral $J_k^1(\varepsilon)$ is defined as:

$$J_k^1(\varepsilon) = \int_{a-x_0}^{b-x_0} \frac{z^k\,dz}{z^2+\varepsilon^2}$$

A.1.1 Calculation of $J_0^1(\varepsilon)$

$$\int \frac{dz}{z^2+\varepsilon^2} = \frac{1}{\varepsilon}\tan^{-1}\frac{z}{\varepsilon} \tag{A.1}$$

$$\lim_{\varepsilon\to 0^+}\int_{a-x_0}^{b-x_0} \frac{dz}{z^2+\varepsilon^2} = \lim_{\varepsilon\to 0^+}\lim_{\eta\to 0^+}\Big[\int_{a-x_0}^{-\eta}\frac{dz}{z^2+\varepsilon^2}+\int_{-\eta}^{\eta}\frac{dz}{z^2+\varepsilon^2}+\int_{\eta}^{b-x_0}\frac{dz}{z^2+\varepsilon^2}\Big] \tag{A.2}$$

$$\int_{a-x_0}^{-\eta}\frac{dz}{z^2+\varepsilon^2} = \frac{1}{\varepsilon}\Big(\tan^{-1}\frac{-\eta}{\varepsilon}-\tan^{-1}\frac{a-x_0}{\varepsilon}\Big) = \frac{1}{\varepsilon}\Big(-\tan^{-1}\frac{\eta}{\varepsilon}+\tan^{-1}\frac{x_0-a}{\varepsilon}\Big) \tag{A.3}$$

$$\int_{-\eta}^{\eta}\frac{dz}{z^2+\varepsilon^2} = \frac{1}{\varepsilon}\Big(\tan^{-1}\frac{\eta}{\varepsilon}-\tan^{-1}\frac{-\eta}{\varepsilon}\Big) = \frac{2}{\varepsilon}\tan^{-1}\frac{\eta}{\varepsilon} \tag{A.4}$$

$$\int_{\eta}^{b-x_0}\frac{dz}{z^2+\varepsilon^2} = \frac{1}{\varepsilon}\Big(\tan^{-1}\frac{b-x_0}{\varepsilon}-\tan^{-1}\frac{\eta}{\varepsilon}\Big) \tag{A.5}$$

$$\lim_{\varepsilon\to 0^+}\int_{a-x_0}^{b-x_0}\frac{dz}{z^2+\varepsilon^2} = \frac{1}{\varepsilon}\Big(\tan^{-1}\frac{b-x_0}{\varepsilon}+\tan^{-1}\frac{x_0-a}{\varepsilon}\Big) \tag{A.6}$$

$$\tan^{-1}x = \frac{\pi}{2}-\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{5x^5}+\frac{1}{7x^7}+\cdots \tag{A.7}$$

$$\lim_{\varepsilon\to 0^+}\int_{a-x_0}^{b-x_0}\frac{dz}{z^2+\varepsilon^2} = \lim_{\varepsilon\to 0^+} J_0^1(\varepsilon)$$
$$= \frac{1}{\varepsilon}\Big[\pi-\varepsilon\,\Big(\frac{1}{b-x_0}+\frac{1}{x_0-a}\Big)+\frac{\varepsilon^3}{3}\,\Big(\frac{1}{(b-x_0)^3}+\frac{1}{(x_0-a)^3}\Big)-\frac{\varepsilon^5}{5}\,\Big(\frac{1}{(b-x_0)^5}+\frac{1}{(x_0-a)^5}\Big)\Big] \tag{A.8}$$

A.1.2 Calculation of $J_1^1(\varepsilon)$

$$\int \frac{z\ dz}{z^2+\varepsilon^2} = \frac{1}{2}\ \ln(z^2+\varepsilon^2) \tag{A.9}$$

$$\lim_{\varepsilon\to 0+} \int_{a-x_0}^{b-x_0} \frac{z\ dz}{z^2+\varepsilon^2} = \lim_{\varepsilon\to 0+} \lim_{\eta\to 0+} \Big[\int_{a-x_0}^{-\eta} \frac{z\ dz}{z^2+\varepsilon^2} + \int_{-\eta}^{\eta} \frac{z\ dz}{z^2+\varepsilon^2} + \int_{\eta}^{b-x_0} \frac{z\ dz}{z^2+\varepsilon^2} \Big] \tag{A.10}$$

$$\int_{a-x_0}^{-\eta} \frac{z\ dz}{z^2+\varepsilon^2} = \frac{1}{2}\ \Big(\ln(\eta^2+\varepsilon^2) - \ln((a-x_0)^2+\varepsilon^2) \Big) \tag{A.11}$$

$$\int_{-\eta}^{\eta} \frac{z\ dz}{z^2+\varepsilon^2} = 0 \qquad \text{odd function} \tag{A.12}$$

$$\int_{\eta}^{b-x_0} \frac{z\ dz}{z^2+\varepsilon^2} = \frac{1}{2}\ \Big(\ln((b-x_0)^2+\varepsilon^2) - \ln(\eta^2+\varepsilon^2) \Big) \tag{A.13}$$

$$\int_{a-x_0}^{b-x_0} \frac{z\ dz}{z^2+\varepsilon^2} = \frac{1}{2}\ \ln \frac{(b-x_0)^2+\varepsilon^2}{(a-x_0)^2+\varepsilon^2} \tag{A.14}$$

$$\ln \frac{(b-x_0)^2+\varepsilon^2}{(a-x_0)^2+\varepsilon^2} = \ln\left((b-x_0)^2+\varepsilon^2\right) - \ln((a-x_0)^2+\varepsilon^2) \tag{A.15}$$

$$\ln\left((b-x_0)^2+\varepsilon^2\right) = \ln(b-x_0)^2 + \ln\Big(1+\frac{\varepsilon^2}{(b-x_0)^2}\Big) \tag{A.16}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots \tag{A.17}$$

$$\ln\left((b-x_0)^2+\varepsilon^2\right) = \ln(b-x_0)^2 + \frac{\varepsilon^2}{(b-x_0)^2} - \frac{\varepsilon^4}{2(b-x_0)^4} + \frac{\varepsilon^6}{3(b-x_0)^6} \tag{A.18}$$

$$\ln \frac{(b-x_0)^2+\varepsilon^2}{(a-x_0)^2+\varepsilon^2} = \ln \frac{(b-x_0)^2}{(a-x_0)^2} + \varepsilon^2\Big(\frac{1}{(b-x_0)^2} - \frac{1}{(a-x_0)^2}\Big) - \frac{\varepsilon^4}{2}\Big(\frac{1}{(b-x_0)^4} - \frac{1}{(a-x_0)^4}\Big) \tag{A.19}$$

$$\lim_{\varepsilon\to 0+}\int_{a-x_0}^{b-x_0}\frac{z\,dz}{z^2+\varepsilon^2}=\lim_{\varepsilon\to 0+} J_1^1(\varepsilon)$$

$$=\frac{1}{2}\ln\frac{(b-x_0)^2}{(a-x_0)^2}+\frac{1}{2}\left[\varepsilon^2\Big(\frac{1}{(b-x_0)^2}-\frac{1}{(a-x_0)^2}\Big)-\frac{\varepsilon^4}{2}\Big(\frac{1}{(b-x_0)^4}-\frac{1}{(a-x_0)^4}\Big)\right] \quad (A.20)$$

A.1.3 Calculation of $J_k^1(\varepsilon)$, $k \geq 2$

$$z^2 = z^2+\varepsilon^2-\varepsilon^2 \quad (A.21)$$

$$\frac{z^2}{z^2+\varepsilon^2} = 1-\frac{\varepsilon^2}{z^2+\varepsilon^2} \quad (A.22)$$

$$z^3 = z\,z^2 = z\,(z^2+\varepsilon^2-\varepsilon^2) \quad (A.23)$$

$$\frac{z^3}{z^2+\varepsilon^2} = z-\frac{\varepsilon^2\,z}{z^2+\varepsilon^2} \quad (A.24)$$

$$z^4 = z^2\,z^2 = z^2\,(z^2+\varepsilon^2-\varepsilon^2) \quad (A.25)$$

$$\frac{z^4}{z^2+\varepsilon^2} = z^2-\frac{\varepsilon^2\,z^2}{z^2+\varepsilon^2} \quad (A.26)$$

$$J_2^1(\varepsilon)=\int_{a-x_0}^{b-x_0}\frac{z^2\,dz}{z^2+\varepsilon^2}=b-x_0-(a-x_0)-\varepsilon^2\,J_0^1(\varepsilon) \quad (A.27)$$

$$J_3^1(\varepsilon)=\int_{a-x_0}^{b-x_0}\frac{z^3\,dz}{z^2+\varepsilon^2}=\frac{1}{2}[(b-x_0)^2-(a-x_0)^2]-\varepsilon^2\,J_1^1(\varepsilon) \quad (A.28)$$

$$J_4^1(\varepsilon)=\int_{a-x_0}^{b-x_0}\frac{z^4\,dz}{z^2+\varepsilon^2}=\frac{1}{3}[(b-x_0)^3-(a-x_0)^3]-\varepsilon^2\,J_2^1(\varepsilon) \quad (A.29)$$

More generally, the recurrence relation is given by

$$J_k^1(\varepsilon)=\frac{1}{k-1}[(b-x_0)^{k-1}-(a-x_0)^{k-1}]-\varepsilon^2\,J_{k-2}^1(\varepsilon) \quad (A.30)$$

$$
\begin{aligned}
\lim_{\varepsilon\to 0^+}\int_{a-x_0}^{b-x_0}\frac{z^2\,dz}{z^2+\varepsilon^2} &= \lim_{\varepsilon\to 0^+} J_2^1(\varepsilon)\\
= \quad & b-x_0-(a-x_0)\\
& -\varepsilon\left[\pi-\varepsilon\,(\frac{1}{b-x_0}+\frac{1}{x_0-a})+\frac{\varepsilon^3}{3}\,(\frac{1}{(b-x_0)^3}+\frac{1}{(x_0-a)^3})-\frac{\varepsilon^5}{5}\,(\frac{1}{(b-x_0)^5}+\frac{1}{(x_0-a)^5})\right]
\end{aligned}
\tag{A.31}
$$

$$
\begin{aligned}
\lim_{\varepsilon\to 0^+}\int_{a-x_0}^{b-x_0}\frac{z^3\,dz}{z^2+\varepsilon^2} &= \lim_{\varepsilon\to 0^+} J_3^1(\varepsilon)\\
= \quad & \frac{1}{2}[(b-x_0)^2-(a-x_0)^2]\\
& -\varepsilon^2\left\{\frac{1}{2}\ln\frac{(b-x_0)^2}{(a-x_0)^2}+\frac{1}{2}\left[\varepsilon^2\Big(\frac{1}{(b-x_0)^2}-\frac{1}{(a-x_0)^2}\Big)-\frac{\varepsilon^4}{2}\Big(\frac{1}{(b-x_0)^4}-\frac{1}{(a-x_0)^4}\Big)\right]\right\}
\end{aligned}
\tag{A.32}
$$

$$
\begin{aligned}
\lim_{\varepsilon\to 0^+}\int_{a-x_0}^{b-x_0}\frac{z^4\,dz}{z^2+\varepsilon^2} &= \lim_{\varepsilon\to 0^+} J_4^1(\varepsilon)\\
= \quad & \frac{1}{3}[(b-x_0)^3-(a-x_0)^3]-\varepsilon^2(\,b-x_0-(a-x_0)\,)\\
& +\varepsilon^3\left[\pi-\varepsilon\,(\frac{1}{b-x_0}+\frac{1}{x_0-a})+\frac{\varepsilon^3}{3}\,(\frac{1}{(b-x_0)^3}+\frac{1}{(x_0-a)^3})-\frac{\varepsilon^5}{5}\,(\frac{1}{(b-x_0)^5}+\frac{1}{(x_0-a)^5})\right]
\end{aligned}
\tag{A.33}
$$

A.2 Calculation of $J_k^2(\varepsilon)$

The integral $J_k^2(\varepsilon)$ is defined as:

$$J_k^2(\varepsilon) = \int_{a-x_0}^{b-x_0} \frac{z^k\,dz}{(z^2+\varepsilon^2)^2}$$

A.2.1 Calculation of $J_0^2(\varepsilon)$

$$\int \frac{dz}{(z^2+\varepsilon^2)^2} = \frac{1}{2\varepsilon^2}\,\frac{z}{z^2+\varepsilon^2} + \frac{1}{2\varepsilon^3}\tan^{-1}\frac{z}{\varepsilon} \tag{A.34}$$

$$\lim_{\varepsilon\to 0^+}\int_{a-x_0}^{b-x_0} \frac{dz}{(z^2+\varepsilon^2)^2} = \lim_{\varepsilon\to 0^+}\lim_{\eta\to 0^+}\left[\int_{a-x_0}^{-\eta}\frac{dz}{(z^2+\varepsilon^2)^2} + \int_{-\eta}^{\eta}\frac{dz}{(z^2+\varepsilon^2)^2} + \int_{\eta}^{b-x_0}\frac{dz}{(z^2+\varepsilon^2)^2}\right] \tag{A.35}$$

$$\int_{a-x_0}^{-\eta}\frac{dz}{(z^2+\varepsilon^2)^2} = \frac{1}{2\varepsilon^2}\left[\frac{-\eta}{\eta^2+\varepsilon^2} - \frac{a-x_0}{(a-x_0)^2+\varepsilon^2}\right] + \frac{1}{2\varepsilon^3}\left(-\tan^{-1}\frac{\eta}{\varepsilon} + \tan^{-1}\frac{x_0-a}{\varepsilon}\right) \tag{A.36}$$

$$\int_{-\eta}^{\eta}\frac{dz}{(z^2+\varepsilon^2)^2} = \frac{1}{\varepsilon^2}\,\frac{\eta}{\eta^2+\varepsilon^2} + \frac{1}{\varepsilon^3}\tan^{-1}\frac{\eta}{\varepsilon} \tag{A.37}$$

$$\int_{\eta}^{b-x_0}\frac{dz}{(z^2+\varepsilon^2)^2} = \frac{1}{2\varepsilon^2}\left[\frac{b-x_0}{(b-x_0)^2+\varepsilon^2} - \frac{\eta}{\eta^2+\varepsilon^2}\right] + \frac{1}{2\varepsilon^3}\left(\tan^{-1}\frac{b-x_0}{\varepsilon} - \tan^{-1}\frac{\eta}{\varepsilon}\right) \tag{A.38}$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \tag{A.39}$$

$$\lim_{\varepsilon\to 0^+}\int_{a-x_0}^{b-x_0}\frac{dz}{(z^2+\varepsilon^2)^2} = \frac{1}{2\varepsilon^2}\left[\frac{b-x_0}{(b-x_0)^2+\varepsilon^2} - \frac{a-x_0}{(a-x_0)^2+\varepsilon^2}\right] + \frac{1}{2\varepsilon^3}\left(\tan^{-1}\frac{x_0-a}{\varepsilon} + \tan^{-1}\frac{b-x_0}{\varepsilon}\right) \tag{A.40}$$

$$\lim_{\varepsilon\to 0^+} \int_{a-x_0}^{b-x_0} \frac{dz}{(z^2+\varepsilon^2)^2} = \lim_{\varepsilon\to 0^+} J_0^2(\varepsilon)$$

$$= \frac{1}{2\varepsilon^2}\Big[\frac{1}{b-x_0} - \frac{1}{a-x_0} - \varepsilon^2\Big(\frac{1}{(b-x_0)^3} - \frac{1}{(a-x_0)^3}\Big) + \varepsilon^4\Big(\frac{1}{(b-x_0)^5} - \frac{1}{(a-x_0)^5}\Big)\Big]$$

$$+\frac{1}{2\varepsilon^3}\Big[\pi - \varepsilon\,(\frac{1}{b-x_0} + \frac{1}{x_0-a}) + \frac{\varepsilon^3}{3}\,(\frac{1}{(b-x_0)^3} + \frac{1}{(x_0-a)^3}) - \frac{\varepsilon^5}{5}\,(\frac{1}{(b-x_0)^5} + \frac{1}{(x_0-a)^5})\Big] \tag{A.41}$$

A.2.2 Calculation of $J_1^2(\varepsilon)$

$$\int \frac{z\,dz}{(z^2+\varepsilon^2)^2} = -\frac{1}{2}\,\frac{1}{z^2+\varepsilon^2} \tag{A.42}$$

$$\lim_{\varepsilon\to 0^+} \int_{a-x_0}^{b-x_0} \frac{z\,dz}{(z^2+\varepsilon^2)^2} = \lim_{\varepsilon\to 0^+}\lim_{\eta\to 0^+}\Big[\int_{a-x_0}^{-\eta} \frac{z\,dz}{(z^2+\varepsilon^2)^2} + \int_{-\eta}^{\eta} \frac{z\,dz}{(z^2+\varepsilon^2)^2} + \int_{\eta}^{b-x_0} \frac{z\,dz}{(z^2+\varepsilon^2)^2}\Big] \tag{A.43}$$

$$\int_{a-x_0}^{-\eta} \frac{z\,dz}{(z^2+\varepsilon^2)^2} = -\frac{1}{2}\,(\frac{1}{\eta^2+\varepsilon^2} - \frac{1}{(a-x_0)^2+\varepsilon^2}) \tag{A.44}$$

$$\int_{-\eta}^{\eta} \frac{z\,dz}{(z^2+\varepsilon^2)^2} = 0 \qquad \text{(odd function)} \tag{A.45}$$

$$\int_{\eta}^{b-x_0} \frac{z\,dz}{(z^2+\varepsilon^2)^2} = -\frac{1}{2}\,(\frac{1}{(b-x_0)^2+\varepsilon^2} - \frac{1}{\eta^2+\varepsilon^2}) \tag{A.46}$$

$$\lim_{\varepsilon\to 0^+} \int_{a-x_0}^{b-x_0} \frac{z\,dz}{(z^2+\varepsilon^2)^2} = -\frac{1}{2}\Big[\frac{1}{(b-x_0)^2+\varepsilon^2} - \frac{1}{(a-x_0)^2+\varepsilon^2}\Big] \tag{A.47}$$

$$\lim_{\varepsilon\to 0^+} \int_{a-x_0}^{b-x_0} \frac{z\,dz}{(z^2+\varepsilon^2)^2} = \lim_{\varepsilon\to 0^+} J_1^2(\varepsilon)$$

$$= -\frac{1}{2}\Big[\frac{1}{(b-x_0)^2} - \frac{1}{(a-x_0)^2} - \varepsilon^2\Big(\frac{1}{(b-x_0)^2} - \frac{1}{(a-x_0)^2}\Big) + \varepsilon^4\Big(\frac{1}{(b-x_0)^4} - \frac{1}{(a-x_0)^4}\Big)\Big] \tag{A.48}$$

A.2.3 Calculation of $J_k^2(\varepsilon)$, $k \geq 2$

$$\int \frac{z^2\,dz}{(z^2+\varepsilon^2)^2} = -\frac{1}{2}\,\frac{z}{z^2+\varepsilon^2} + \frac{1}{2\varepsilon}\tan^{-1}\frac{z}{\varepsilon} \tag{A.49}$$

$$\lim_{\varepsilon\to 0^+}\int_{a-x_0}^{b-x_0}\frac{z^2\,dz}{(z^2+\varepsilon^2)^2} = \lim_{\varepsilon\to 0^+}\lim_{\eta\to 0^+}\Big[\int_{a-x_0}^{-\eta}\frac{z^2\,dz}{(z^2+\varepsilon^2)^2} + \int_{-\eta}^{\eta}\frac{z^2\,dz}{(z^2+\varepsilon^2)^2} + \int_{\eta}^{b-x_0}\frac{z^2\,dz}{(z^2+\varepsilon^2)^2}\Big] \tag{A.50}$$

$$\int_{a-x_0}^{-\eta}\frac{z^2\,dz}{(z^2+\varepsilon^2)^2} = -\frac{1}{2}\Big(\frac{-\eta}{\eta^2+\varepsilon^2} - \frac{a-x_0}{(a-x_0)^2+\varepsilon^2}\Big) + \frac{1}{2\varepsilon}\Big(\tan^{-1}\frac{-\eta}{\varepsilon} - \tan^{-1}\frac{a-x_0}{\varepsilon}\Big) \tag{A.51}$$

$$\int_{-\eta}^{\eta}\frac{z^2\,dz}{(z^2+\varepsilon^2)^2} = -\frac{1}{2}\Big(\frac{\eta}{\eta^2+\varepsilon^2} - \frac{-\eta}{\eta^2+\varepsilon^2}\Big) + \frac{1}{2\varepsilon}\Big(\tan^{-1}\frac{\eta}{\varepsilon} - \tan^{-1}\frac{-\eta}{\varepsilon}\Big) = -\frac{\eta}{\eta^2+\varepsilon^2} + \frac{1}{\varepsilon}\tan^{-1}\frac{\eta}{\varepsilon} \tag{A.52}$$

$$\int_{\eta}^{b-x_0}\frac{z^2\,dz}{(z^2+\varepsilon^2)^2} = -\frac{1}{2}\Big(\frac{b-x_0}{(b-x_0)^2+\varepsilon^2} - \frac{\eta}{\eta^2+\varepsilon^2}\Big) + \frac{1}{2\varepsilon}\Big(\tan^{-1}\frac{b-x_0}{\varepsilon} - \tan^{-1}\frac{\eta}{\varepsilon}\Big) \tag{A.53}$$

$$\lim_{\varepsilon\to 0^+}\int_{a-x_0}^{b-x_0}\frac{z^2\,dz}{(z^2+\varepsilon^2)^2} = -\frac{1}{2}\Big[\frac{b-x_0}{(b-x_0)^2+\varepsilon^2} - \frac{a-x_0}{(a-x_0)^2+\varepsilon^2}\Big] + \frac{1}{2\varepsilon}\Big(\tan^{-1}\frac{x_0-a}{\varepsilon} + \tan^{-1}\frac{b-x_0}{\varepsilon}\Big) \tag{A.54}$$

$$\begin{aligned}
\lim_{\varepsilon\to 0^+}\int_{a-x_0}^{b-x_0}\frac{z^2\,dz}{(z^2+\varepsilon^2)^2} &= \lim_{\varepsilon\to 0^+} J_2^2(\varepsilon) \\
&= -\frac{1}{2}\Big[\frac{1}{b-x_0} - \frac{1}{a-x_0} - \varepsilon^2\Big(\frac{1}{(b-x_0)^3} - \frac{1}{(a-x_0)^3}\Big) + \varepsilon^4\Big(\frac{1}{(b-x_0)^5} - \frac{1}{(a-x_0)^5}\Big)\Big] \\
&\quad + \frac{1}{2\varepsilon}\Big[\pi - \varepsilon\Big(\frac{1}{b-x_0} + \frac{1}{x_0-a}\Big) + \frac{\varepsilon^3}{3}\Big(\frac{1}{(b-x_0)^3} + \frac{1}{(x_0-a)^3}\Big) - \frac{\varepsilon^5}{5}\Big(\frac{1}{(b-x_0)^5} + \frac{1}{(x_0-a)^5}\Big)\Big]
\end{aligned} \tag{A.55}$$

$$\int \frac{z^3\, dz}{(z^2+\varepsilon^2)^2} = \frac{\varepsilon^2}{2}\frac{1}{z^2+\varepsilon^2} + \frac{1}{2}\ \ln(z^2+\varepsilon^2) \tag{A.56}$$

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \int_{a-x_0}^{b-x_0} \frac{z^3\, dz}{(z^2+\varepsilon^2)^2} = \lim_{\varepsilon\to 0^+}\ J_3^2(\varepsilon) \\
=\ & \frac{\varepsilon^2}{2}\Big[\ \frac{1}{(b-x_0)^2} - \frac{1}{(a-x_0)^2} - \varepsilon^2\Big(\frac{1}{(b-x_0)^2} - \frac{1}{(a-x_0)^2}\Big) + \varepsilon^4\Big(\frac{1}{(b-x_0)^4} - \frac{1}{(a-x_0)^4}\Big)\ \Big] \\
& + \frac{1}{2}\ln\frac{(b-x_0)^2}{(a-x_0)^2} + \frac{1}{2}\Big[\varepsilon^2\Big(\frac{1}{(b-x_0)^2} - \frac{1}{(a-x_0)^2}\Big) - \frac{\varepsilon^4}{2}\Big(\frac{1}{(b-x_0)^4} - \frac{1}{(a-x_0)^4}\Big)\ \Big]
\end{aligned} \tag{A.57}$$

$$z^4 = (z^2+\varepsilon^2-\varepsilon^2)^2 = (z^2+\varepsilon^2)^2 - 2\varepsilon^2(z^2+\varepsilon^2) + \varepsilon^4 \tag{A.58}$$

$$\int_{a-x_0}^{b-x_0} \frac{z^4\, dz}{(z^2+\varepsilon^2)^2} = b - x_0 - (a-x_0) - 2\varepsilon^2 J_0^1(\varepsilon) + \varepsilon^4 J_0^2(\varepsilon) \tag{A.59}$$

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \int_{a-x_0}^{b-x_0} \frac{z^4\, dz}{(z^2+\varepsilon^2)^2} \\
=\ & b - x_0 - (a-x_0) \\
& -2\varepsilon\ \Big[\pi - \varepsilon\ \Big(\frac{1}{b-x_0} + \frac{1}{x_0-a}\Big) + \frac{\varepsilon^3}{3}\ \Big(\frac{1}{(b-x_0)^3} + \frac{1}{(x_0-a)^3}\Big) - \frac{\varepsilon^5}{5}\ \Big(\frac{1}{(b-x_0)^5} + \frac{1}{(x_0-a)^5}\Big)\Big] \\
& + \frac{\varepsilon^2}{2}\Big[\ \frac{1}{b-x_0} - \frac{1}{a-x_0} - \varepsilon^2\Big(\frac{1}{(b-x_0)^3} - \frac{1}{(a-x_0)^3}\Big) + \varepsilon^4\Big(\frac{1}{(b-x_0)^5} - \frac{1}{(a-x_0)^5}\Big)\ \Big] \\
& + \frac{\varepsilon}{2}\ \Big[\pi - \varepsilon\ \Big(\frac{1}{b-x_0} + \frac{1}{x_0-a}\Big) + \frac{\varepsilon^3}{3}\ \Big(\frac{1}{(b-x_0)^3} + \frac{1}{(x_0-a)^3}\Big) - \frac{\varepsilon^5}{5}\ \Big(\frac{1}{(b-x_0)^5} + \frac{1}{(x_0-a)^5}\Big)\Big]
\end{aligned} \tag{A.60}$$

$$\lim_{\varepsilon\to 0^+}\int_{a-x_0}^{b-x_0}\frac{z^4\,dz}{(z^2+\varepsilon^2)^2}=\lim_{\varepsilon\to 0^+}J_4^2(\varepsilon)$$

$$= \quad b-x_0-(a-x_0)-\frac{3\pi}{2}\varepsilon$$

$$+\,2\,\varepsilon^2\Big(\,\frac{1}{b-x_0}-\frac{1}{a-x_0}\,\Big)-\varepsilon^4\Big(\,\frac{1}{(b-x_0)^3}-\frac{1}{(a-x_0)^3}\,\Big)+\frac{4\varepsilon^6}{5}\Big(\,\frac{1}{(b-x_0)^5}-\frac{1}{(a-x_0)^5}\,\Big) \tag{A.61}$$

$$z^5=z(z^2+\varepsilon^2-\varepsilon^2)^2=z[\,(z^2+\varepsilon^2)^2-2\varepsilon^2(z^2+\varepsilon^2)+\varepsilon^4\,] \tag{A.62}$$

$$\int_{a-x_0}^{b-x_0}\frac{z^5\,dz}{(z^2+\varepsilon^2)^2}=\frac{1}{2}(\,(b-x_0)^2-(a-x_0)^2\,)-2\varepsilon^2J_1^1(\varepsilon)+\varepsilon^4J_1^2(\varepsilon) \tag{A.63}$$

$$\lim_{\varepsilon\to 0^+}\int_{a-x_0}^{b-x_0}\frac{z^5\,dz}{(z^2+\varepsilon^2)^2}$$

$$= \quad \frac{1}{2}\,[\,(b-x_0)^2-(a-x_0)^2\,]$$

$$-2\varepsilon^2\left\{\,\frac{1}{2}\ln\frac{(b-x_0)^2}{(a-x_0)^2}+\frac{1}{2}\left[\varepsilon^2\left(\frac{1}{(b-x_0)^2}-\frac{1}{(a-x_0)^2}\right)-\frac{\varepsilon^4}{2}\left(\frac{1}{(b-x_0)^4}-\frac{1}{(a-x_0)^4}\right)\right]\right\}$$

$$-\frac{\varepsilon^4}{2}\left[\,\frac{1}{(b-x_0)^2}-\frac{1}{(a-x_0)^2}-\varepsilon^2\Big(\frac{1}{(b-x_0)^2}-\frac{1}{(a-x_0)^2}\Big)+\varepsilon^4\Big(\frac{1}{(b-x_0)^4}-\frac{1}{(a-x_0)^4}\Big)\,\right] \tag{A.64}$$

$$\lim_{\varepsilon \to 0^+} \int_{a-x_0}^{b-x_0} \frac{z^5 \, dz}{(z^2+\varepsilon^2)^2} = \lim_{\varepsilon \to 0^+} J_5^2(\varepsilon)$$

$$= \frac{1}{2} [\, (b-x_0)^2 - (a-x_0)^2 \,] \, - \varepsilon^2 \, \ln \frac{(b-x_0)^2}{(a-x_0)^2}$$

$$-\frac{3\varepsilon^4}{2} \left(\frac{1}{(b-x_0)^2} - \frac{1}{(a-x_0)^2} \right) + \frac{\varepsilon^6}{2} \left[\frac{1}{(b-x_0)^4} - \frac{1}{(a-x_0)^4} - \left(\frac{1}{(b-x_0)^2} - \frac{1}{(a-x_0)^2} \right) \right] \tag{A.65}$$

References

[1] H. B. Dwight, Tables of integrals and other mathematica data, Macmillan, 1961