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BILATERAL GENERATING FUNCTIONS AND OPERATIONAL METHODS

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Riassunto
Le funzioni generatrici bilaterali sono quelle che coinvolgono prodotti di tipi differenti di polinomi, si dimostra che i metodi operazionali offrono uno strumento potente per derivare queste nuove famiglie di funzioni speciali. Si studiano casi relativi a prodotti di polinomi di Hermite e Laguerre, Hermite e Legendre.... Si propongono infine possibili estensioni del metodo.

Abstract
Bilateral generating functions are those involving products of different types of polynomials. We show that operational methods offer a powerful tool to derive these families of generating functions. We study cases relevant to products of Hermite and Laguerre, Hermite and Legendre.... polynomials and propose further extensions of the method.

Key words: Generating functions, orthogonal polynomials, operational methods, generalized polynomials.

INDEX

BILATERAL GENERATING FUNCTIONS AND OPERATIONAL METHODS

1 INTRODUCTION

One of the by-products of the method of quasi-monomials, proposed in Ref. [1,2], is the possibility of dealing, in a fairly straightforward way, with different classes of generating functions. An example is provided by the Mehler type formula [3]

$$S(x,y;z,w\,|\,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x,y) H_n(z,w) \quad , \tag{1}$$

with $H_n(x,y)$ being Hermite-Kampè de Fèrièt polynomials defined by the series

$$H_n(x,y) = n! \sum_{r=0}^{[n/2]} \frac{y^r x^{n-2r}}{r!(n-2r)!} \quad , \tag{2}$$

by the operational rule [4]

$$H_n(x,y) = e^{y\frac{\partial^2}{\partial x^2}}(x^n) \quad , \tag{3}$$

and by the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x,y) = e^{xt+yt^2} \quad . \tag{4}$$

The use of the identity (3) allows to cast eq. (1) in the form

$$S(x,y;z,w\,|\,t) = e^{y\frac{\partial^2}{\partial x^2}+w\frac{\partial^2}{\partial z^2}} \sum_{n=0}^{\infty} \frac{(xzt)^n}{n!} = e^{y\frac{\partial^2}{\partial x^2}+w\frac{\partial^2}{\partial z^2}} e^{xzt} \quad . \tag{5}$$

Furthermore, since

$$e^{y\frac{\partial^2}{\partial x^2}} f(x) = \frac{1}{2\sqrt{\pi y}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4y}} f(\xi) d\xi \tag{6}$$

we can transform the derivation of the generating function in eq. (1) in the evaluation of a Gaussian integral, thus getting [3]

$$S(x,y;z,w\,|\,t) = \frac{1}{\sqrt{1-4yt^2w}} e^{\frac{xzt+t^2(wx^2+yz^2)}{1-4t^2yw}} \quad . \tag{7}$$

The use of analogous techniques may be useful to state other results concerning e.g. the generating functions associated with Laguerre polynomials, which are quasi monomials under the action of the operators [5] *

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* We remind that the operators $\hat{M}$ and $\hat{P}$ act on a quasi monomial $p_n(x)$ in a multiplicative-like and derivative-like form, namely

$$\hat{M}p_n(x) = p_{n+1}(x)$$

$$\hat{P}p_n(x) = np_{n-1}(x)$$

for further comments see refs. [2,5]

$$\hat{M} = y - \hat{\mathcal{D}}_x^{-1},$$
$$\hat{P} = -\frac{\partial}{\partial x} x \frac{\partial}{\partial x} \quad , \tag{8}$$

where $\hat{\mathcal{D}}_x^{-1}$ is the inverse of the derivative operator and accordingly the Laguerre polynomials can be defined as

$$\mathcal{L}_n(x,y) = \hat{M}^n = \sum_{s=0}^{n} \binom{n}{s} (-1)^s y^{n-s} \hat{\mathcal{D}}_x^{-s} = n! \sum_{s=0}^{n} \frac{(-1)^s y^{n-s} x^s}{(n-s)!(s!)^2} \quad . \tag{9}$$

Equations (8,9) can also be exploited to obtain the following generating function [5]

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{L}_n(x,y) = e^{yt} C_0(xt) \tag{10}$$

with

$$C_n(x) = (x)^{-n/2} J_n(2\sqrt{x}) = \sum_{n=0}^{\infty} \frac{(-1)^r x^r}{r!(n+r)!} \tag{11}$$

being the nth-order Tricomi function [6].

In the forthcoming sections we will prove that the previously outlined procedure offers a useful tool for the derivation of bilateral generating functions involving products of polynomials of different nature (3). We will, indeed, derive generating functions of Mehler type, involving products of Hermite-Laguerre, Hermite-Legendre polynomials, using methods of operational nature.

2 BILATERAL GENERATING FUNCTIONS

The first example we will consider is provided by

$$F(x,y;z,w\,|\,t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x,y)\mathcal{L}_n(z,w) \tag{12}$$

which according to eqs. (3,10) can be rewritten as

$$F(x,y;z,w\,|\,t)=e^{y\frac{\partial^2}{\partial x^2}}e^{wxt}C_0(zxt) \tag{13}$$

It is evident that the evaluation of the infinite sum (12) amounts to the derivation of the integral

$$F(x,y;z,w\,|\,t)=\frac{1}{2\sqrt{\pi y}}\int_{-\infty}^{\infty}e^{-\frac{(x-\xi)^2}{4y}}e^{w\xi t}C_0(z\xi t)d\xi \tag{14}$$

By using integration techniques based on the generating function method (see Appendix) we find the following result

$$F(x,y;z,w\,|\,t)=e^{y(wt)^2+xwt}{}_HC_0(2zt(\frac{x}{2}+ywt),\ y(zt)^2), \tag{15}$$

where

$${}_HC_n(x,y)=\sum_{r=0}^{\infty}\frac{(-1)^r H_r(x,y)}{r!(n+r)!} \tag{16}$$

is the nth-order Hermite-Tricomi function whose properties have been discussed in Refs. [7].

The previously outlined method can be exploited to extend the above result, thus getting e.g.

$$F_l(x,y;z,w\,|\,t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}H_{n+l}(x,y)\mathcal{L}_n(z,w)=A_l(x,y;z,w\,|\,t)e^{y(wt)^2+xwt},$$
$$A_l(x,y;z,w\,|\,t)=\sum_{p=0}^{l}\binom{l}{p}(-2yzt)^p H_{l-p}(x+2ywt,y)({}_HC_p(2zt(\frac{x}{2}+ywt),y(zt)^2)) \tag{17}$$

whose derivation is detailed in Appendix. It is also worth noting that the same result can be achieved by exploiting eqs. (4,8-9) which yields

$$F(x,y;z,w\,|\,t)=e^{xt(w-\hat{\mathcal{D}}_z^{-1})+yt^2(w-\hat{\mathcal{D}}_z^{-1})2}=$$
$$e^{y(wt)^2+wxt}e^{-(xt+2yt^2w)\hat{\mathcal{D}}_z^{-1}+yt^2\hat{\mathcal{D}}_z^{-2}} \tag{18}$$

which, according to the operational identity

$$e^{-x\hat{\mathcal{D}}_z^{-1}+y\hat{\mathcal{D}}_z^{-2}}=({}_HC_0(xz,yz^2)) \tag{19}$$

provides us with a further tool to derive eq. (15).

In Ref. [5] the following polynomials, have been introduced

$${}_2\mathcal{L}_n(x,y)=n!\sum_{s=0}^{[n/2]}\frac{y^{n-2s}(-x)^s}{(n-2s)!(s!)^2} \tag{20}$$

whose nature in between Laguerre and Hermite type polynomials, ensures their monomiality under the action of the operators

$$\hat{M}=y-2\hat{\mathcal{D}}_x^{-1}\frac{\partial}{\partial y}$$
$$\hat{P}_y=\frac{\partial}{\partial y},\ \hat{P}_x=-\frac{\partial}{\partial x}x\frac{\partial}{\partial x} \tag{21}$$

The relevant generating function can be directly inferred from the first of eq. (21) thus finding

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}({}_2\mathcal{L}_n(x,y)) = e^{yt}C_0(xt^2) \tag{22}$$

We can now derive a further bilateral generating function involving Hermite and ${}_2\mathcal{L}_n(x,y)$ polynomials. We find indeed

$${}_2F(x,y;z,w\,|\,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}H_n(x,y){}_2\mathcal{L}_n(z,w) = e^{y\frac{\partial^2}{\partial x^2}}e^{wxt}C_0(zx^2t^2) \tag{23}$$

By following the integration method discussed in Appendix we end up with

$${}_2F(x,y;z,w\,|\,t) = e^{xwt+y(wt)^2}{}_HJ_0(4y\sqrt{z}t(\frac{x}{2y}+wt),4zyt^2)) \tag{24}$$

where

$${}_HJ_n(x,y) = \sum_{r=0}^{\infty} \frac{(-1)^r H_{n+2r}(x,y)}{2^{n+2r}r!(n+2r)!} \tag{25}$$

is a Bessel Hermite Function [7].

An important by product of the identity (24) follows from the fact that ${}_2\mathcal{L}_n(x,y)$ implicitly contain the Legendre polynomials [8], namely

$${}_2\mathcal{L}_n(-\frac{1}{4}(1-w^2),-w) = P_n(w) \tag{26}$$

It is therefore lear that by setting $z=-1/4(1-w^2)$ and by replacing w with $-w$ we are able to derive a bilateral Mehler generating function involving products of Hermite and Legendre polynomials.

3 CONCLUDING REMARKS

It is evident that the method we have proposed offers a fairly efficient mean to evaluate families of bilateral generating functions. A further example involves the polynomials

$$R_n(x,y) = (n!)^2 \sum_{k=0}^{n} \frac{(-1)^{n-k} y^k x^{n-k}}{(k!)^2((n-k)!)^2}, \tag{27}$$

defined in Ref. [9] and quasi monomials under the action of

$$\begin{aligned} &\hat{M} = \hat{D}_y^{-1} - \hat{D}_x^{-1}, \\ &\hat{P}_x = -\frac{\partial}{\partial x} x \frac{\partial}{\partial x}, \ \hat{P}_y = \frac{\partial}{\partial y} y \frac{\partial}{\partial y} \ . \end{aligned} \tag{28}$$

It is relatively easy to prove that

$$\sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} R_n(x,y) = C_0(-yt)C_0(xt) \ . \tag{29}$$

according to the previous relations we can easily state that

$$G(x,y;z,w\,|\,t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} H_n(x,y)R_n(z,w) = e^{y\frac{\partial^2}{\partial x^2}} C_0(-wxt)C_0(zxt) \tag{30}$$

The problem is again reduced to the evaluation of Gaussian integrals, the specific details of the integration method are given in Appendix, and we get

$$G(x,y;z,w\,|\,t) =_H C_{0,0}(-xwt, y(wt)^2; zxt, y(zt)^2\,|\,-2wyzt^2) \ . \tag{31}$$

Where ${}_HC_0(\alpha,\beta;\gamma,\delta|\varepsilon)$ is the (0th, 0th) order of the two-index Tricomi-Bessel functions defined as

$$ {}_{H}C_{m,n}(x,y;z,w\,|\,\tau) = \sum_{(r,s)=0}^{\infty} \frac{(-1)^{r+s} H_{r,s}(x,y;z,w\,|\,\tau)}{r!s!(m+r)!(n+r)!}, \tag{32} $$

with $H_{m,n}(x,y;z,w|\tau)$ being two index Hermite polynomials defined in Appendix and studied in [4] and references therein. This last result is relatively important since it offers an interesting conclusion, based on the fact that the Legendre polynomials are a particular case of the polynomials (27); according to Ref. [7] we have

$$ P_n(x) = R_n\left(\frac{1-x}{2}, \frac{1+x}{2}\right) \tag{33} $$

by setting therefore

$$ \begin{aligned} z &= \frac{1-\zeta}{2} \\ w &= \frac{1+\zeta}{2} \end{aligned} \tag{34} $$

in eq. (30,31), we can derive a bilateral generating function of the type

$$ G(x,y;\zeta\,|\,t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} H_n(x,y) P_n(\zeta). \tag{35} $$

In this paper we have outlined a general method to derive families of generating functions, hardly achievable with ordinary means, in a forthcoming investigation. We will discuss the extension of the method to multiindex polynomials.

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APPENDIX A

Most of the results of the present paper are based on integrals of the type

$$I_n(a,b,c) = \int_{-\infty}^{\infty} e^{-ax^2+bx} J_n(cx)dx \quad , \tag{A.1}$$

where $J_n(x)$ is a cylindrical Bessel function of first kind. By recalling that the generating function of ordinary Bessel fucntions is

$$\sum_{n=-\infty}^{\infty} t^n J_n(x) = e^{\frac{x}{2}(t-\frac{1}{t})} \qquad |t| < \infty, \quad t \neq 0 \quad , \tag{A.2}$$

we can derive the integral (A.1) by following the method proposed in Ref. [10] and summarized below

1) We use the eq. (A.2) to write

$$\sum_{n=-\infty}^{\infty} t^n I_n(a,b,c) = \int_{-\infty}^{\infty} e^{-ax^2} e^{\left[b+\frac{c}{2}(t-\frac{1}{t})\right]x} dx \quad . \tag{A.3}$$

2) From the integral on the r.h.s. we get

$$\sum_{n=-\infty}^{\infty} t^n I_n(a,b,c) = \sqrt{\frac{\pi}{a}}\; e^{\frac{b^2}{4a}} e^{\frac{bc}{4a}(t-\frac{1}{t})} e^{\frac{c^2}{16a}(t-\frac{1}{t})^2} \quad . \tag{A.4}$$

3) By recalling that the Hermite Bessel function is just provided by

$$\sum_{n=-\infty}^{\infty} t^n ({}_H J_n(x,y)) = e^{\frac{x}{2}(t-\frac{1}{t})+\frac{y}{4}(t-\frac{1}{t})^2} \;, {}_H J_n(x,y) = \sum_{r=0}^{\infty} \frac{(-1)^r H_{n+2r}(x,y)}{2^{n+2r} r!(n+r)!} \quad , \tag{A.5}$$

It is therefore evident that

$$I_n(a,b,c)=\sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}}({}_HJ_n(\frac{bc}{2a},\frac{c^2}{4a}))\quad .$$

The same method applies to the derivation of integrals involving the Tricomi functions, characterized by the generating functions

$$\sum_{n=-\infty}^{\infty} t^n C_n(x)=e^{t-\frac{x}{t}},\quad C_n(x)=(x)^{-n/2}J_n(2\sqrt{x}) \tag{A.6}$$

thus finding

$$I_n(a,b,c)=\int_{-\infty}^{\infty} e^{-ax^2}e^{bx}C_n(cx)dx=\sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}}({}_HC_n(\frac{bc}{2a},\frac{c^2}{4a})) \tag{A.7a}$$

$${}_HC_n(x,y)=\sum_{r=0}^{\infty}\frac{(-1)^r H_r(x,y)}{r!(n+r)!} \tag{A.7b}$$

As to eq. (17) we note that its derivation depends on integrals of the type

$$I_{n,l}(a,b,c)=\int_{-\infty}^{\infty} e^{-ax^2}e^{bx}x^l C_n(cx)dx \tag{A.8}$$

whose evaluation requires a slight extension of the generating function method, we can indeed write

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\sum_{l=0}^{\infty} t^n\frac{u^l}{l!}I_{n,l}(a,b,c)&=e^t\int_{-\infty}^{\infty} e^{-ax^2}e^{(b+u-\frac{c}{t})x}dx=\\&=e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}}e^{t-\frac{bc}{2at}+\frac{c^2}{4at^2}}e^{\frac{u^2}{4a}+\frac{bu}{2a}}e^{-\frac{uc}{2at}}\end{aligned} \tag{A.9}$$

Thus finally getting

$$I_{n,l}(a,b,c) = \sqrt{\frac{\pi}{a}}\, l! \sum_{p=0}^{l} \frac{(-\frac{c}{2a})^p C_{n+p}(\frac{bc}{2a},\frac{c^2}{4a}) H_{l-p}(\frac{b}{2a},\frac{1}{4a})}{p!(l-p)!} e^{\frac{b^2}{4a}} \tag{A.10}$$

The case relevant to integrals containing products of Bessel functions, namely

$$I_{m,n}(a,b,c,d) = \int_{-\infty}^{\infty} e^{-ax^2+bx} J_m(cx) J_n(dx)dx \tag{A.11}$$

requires an extension of the method leading to eq. (A.11) which yields

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} u^m v^n I_{m,n}(a,b,c,d) = e^t \int_{-\infty}^{\infty} e^{-ax^2} e^{\left[b+\frac{c}{2}(u-\frac{1}{u})+\frac{d}{2}(v-\frac{1}{v})\right]x} dx$$

so that one finally gets

$$I_{m,n}(a,b,c,d) = \sqrt{\frac{\pi}{a}}\; e^{b^2/4a}\left({}_H J_{m,n}\left(\frac{bc}{2a},\frac{c^2}{4a};\frac{d}{2a},\frac{d^2}{4a} \,\middle|\, \frac{dc}{2a}\right)\right) \tag{A.12}$$

where ${}_H J_{m,n}\left(\frac{bc}{2a},\frac{c^2}{4a};\frac{d}{2a},\frac{d^2}{4a} \,\middle|\, \frac{dc}{2a}\right)$ denotes a two index Hermite Bessel function, defined as

$$\begin{aligned} {}_H J_{m,n}(x,y;z,w \mid \tau) &= \sum_{(r,s)=0}^{\infty} \frac{(-1)^{r+s} H_{m+2r,n+2s}(x,y;z,w \mid \tau)}{2^{m+n+2(r+s)} r!s!(m+r)!(n+s)!}, \\ H_{m,n}(x,y;z,w \mid \tau) &= m!n! \sum_{r=0}^{\min(m,n)} \frac{\tau^r H_{m-r}(x,y) H_{n-r}(z,w)}{r!(m-r)!(n-r)!} \end{aligned} \tag{A.13}$$

being this family of functions characterized by the generating function

$$\sum_{(m,n)=-\infty}^{\infty} u^m v^n \left({}_H J_{m,n}(x,y;z,w\,|\,\tau)\right) = e^{\frac{x}{2}(u-\frac{1}{u})+\frac{y}{4}(u-\frac{1}{u})^2+\frac{z}{2}(v-\frac{1}{v})+\frac{w}{4}(v-\frac{1}{v})^2+\frac{\tau}{4}(u-\frac{1}{u})(v-\frac{1}{v})}$$

(A.14)

The Tricomi version of the functions (A.5) can be cast in the form

$${}_H C_{m,n}(x,y;z,w\,|\,\tau) = \sum_{(r,s)=0}^{\infty} \frac{(-1)^{r+s} H_{r,s}(x,y;z,w\,|\,\tau)}{r!s!(m+r)!(n+s)!} \tag{A.15}$$

and the relevant generating function reads

$$\sum_{(m,n)=-\infty}^{\infty} u^m v^n \left({}_H C_{m,n}(x,y;z,w\,|\,\tau)\right) = e^{(u-\frac{x}{u})+\frac{y}{u^2}+(v-\frac{z}{v})+\frac{w}{v^2}+\tau uv} \tag{A.16}$$

These last relations justify the results contained in the concluding section of the present paper.

REFERENCES

[1] Y. Smirnov and A. Turbiner "Hidden SL2-Algebra of finite difference equations" Mod. Phys. Lett. **A10** (1995) 1795-1801.

[2] G. Dattoli, A. Torre and G. Mazzacurati "Quasi monomials and isospectral problems", Nuovo Cimento **B112** (1997) 133-138.

[3] H.M. Srivastava and H.L. Manocha, "A treatise on generating functions" Ellis Horwood, 1984.

[4] G. Dattoli, S. Lorenzutta, G. Maino and A. Torre "Generalized forms of Bessel functions and Hermite polynomials" Annals of Num. Math. **2** (1995) 221-232.

We must underline that operational formulae of this tupe for ordinary polynomials traces back to the works of M.W. Crofton (1879), L. Stranep (1936) and J.L. Burchnall (1941).

[5] G. Dattoli, A. Mancho and A. Torre, J. Comp. and Appl. Math. **108** (1999) 209–218.

[6] L.C. Andrews "Special functions for engineers and applied mathematicians" Mc Millian New York (1994).

[7] G. Dattoli, A. Mancho, and A. Torre, Rad. Phys. Chem. **59**, 229-237 (2000).

[8] For ordinary Legendre polynomials see Ref. [6] or

N.N. Lebedev "Special functions" Dover publ. (1972).

[9] G. Dattoli, P.E. Ricci and C. Cesarano "A note on Legendre polynomials" Le Matematiche (to be published)

[10] I. Dimovski and S.L. Kalla, Math. Jpn. (1988) 33-45.

G. Dattoli, A. Renieri and A. Torre, "Generalized polynomials and derivation of integrals involving products of Hermite polynomials" Rad. Phys. Chem. **53** (1998) 391-395.

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