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Debashish Goswami

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THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

TWISTED ENTIRE CYCLIC COHOMOLOGY, J-L-O COCYCLES AND EQUIVARIANT SPECTRAL TRIPLES

Debashish Goswami[1]
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

We study the "quantized calculus" corresponding to the algebraic ideas related to "twisted cyclic cohomology" introduced in [12]. With very similar definitions and techniques as those used in [9], we define and study "twisted entire cyclic cohomology" and the "twisted Chern character" associated with an appropriate operator theoretic data called "twisted spectral data", which consists of a spectral triple in the conventional sense of noncommutative geometry ([1]) and an additional positive operator having some specified properties. Furthermore, it is shown that given a spectral triple (in the conventional sense) which is equivariant under the action of a compact matrix pseudogroup, it is possible to obtain a canonical twisted spectral data and hence the corresponding (twisted) Chern character, which will be invariant under the action of the pseudogroup, in contrast to the fact that the Chern character coming from the conventional noncommutative geometry need not be invariant under the above action.

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[1]goswamid@ictp.trieste.it

1 Introduction

Ordinary and entire cyclic cohomology theory are indeed some of the fundamental ingredients of Connes' noncommutative geometry. A comprehensive account of this theory can be found in [1] and the references cited in that book. Let us briefly recall how this theory is used to define a noncommutative version of the Chern character. First of all, there is a canonical pairing between the K-theory and the ordinary as well as the entire cyclic cohomology. Let $\mathcal{A}$ be a Banach or more generally locally convex topological algebra, $\langle .,. \rangle : K_*(\mathcal{A}) \times H^*(\mathcal{A}) \to C$ and $\langle .,. \rangle_\epsilon : K_*(\mathcal{A}) \times H_\epsilon^*(\mathcal{A}) \to C$ be the canonical pairing (c.f. [1]) between the K-theory and the periodic cyclic cohomology and the pairing between the K-theory and the entire cyclic cohomology of $\mathcal{A}$ respectively. Given a Fredholm module $(\mathcal{H}, F)$ over $\mathcal{A}$, or equivalently a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, one constructs a canonical element $ch_*(\mathcal{H}, F)$, called the chern character, of $H^*(\mathcal{A})$ or $H_\epsilon^*(\mathcal{A})$, depending on whether the Fredholm module is "p-summable" for some $p > 0$ or it is only "Θ-summable", and $*$ will stand for even or odd depending on whether the Fredholm module is even or odd (i.e. equipped with a compatible grading or not). The map $\phi : K_*(\mathcal{A}) \to C$ given by $\phi(.) = \langle , ch_*(\mathcal{H}, F) \rangle$ ($\langle , ch_*(\mathcal{H}, F) \rangle_\epsilon$ for the Θ-summable case) actually takes integer values, and can be obtained as an index of a suitable Fredholm operator.

For the finite summable situation, $ch_*(\mathcal{H}, F)$ is given by (upto a constant) the cocycle $\tau_n(a_0, ...a_n) = Tr_s(a_0[F, a_1]...[F, a_n]), a_j \in \mathcal{A}$, where Tr_s is a kind of graded trace defined in [1]. In terms of the associated spectral triple, under some additional assumption, one gets a canonical Hochschild n-cocycle ϕ_w given by, $\phi_w(a_0, ..a_n) = \lambda_n \Psi(a_0[D, a_1]...[D, a_n])$, where λ_n is a constant and $\Psi(A) := Tr_w(A|D|^{-n}), A \in \mathcal{B}(\mathcal{H})$, where $Tr_w(.)$ denotes the Dixmier trace and n is suitably chosen to have $|D|^{-n}$ in the Dixmier-trace-class. The positive linear functional $\mathcal{A} \in a \mapsto Tr_w(a|D|^{-n})$ is a trace and can be thought of as the "noncommutative volume form" associated with the noncomutative spin geometry encoded by the spectral triple. Furthermore, if $\mathcal{A}$ is equipped with the action of a classical compact group G coming from a unitary representation on $\mathcal{H}$ and the spectral triple is G-equivariant (which means in particular that D commutes with the G-representation on $\mathcal{H}$), then the above-mentioned cocycles are G-invariant in the sense that $\tau_n(g.a_0, ...g.a_n) = \tau(a_0, ..a_n)$ and similar thing is true for ϕ_w. In particular, if $\mathcal{A}$ is chosen to be an appropriate function algebra (containing the smooth functions) on a classical compact Lie group G and the canonical equivariant Dirac operator D is chosen, then the above-mentioned volume form will (upto a constant) coincide with the integral with respect to the Haar measure. The above-mentioned invariance of the cocycles makes it possible to consider its lifting to the algebraic crossed-product in a canonical way.

However, things do drastically change when one replaces classical compact groups by non-commutaive and non-cocommutative comapct quantum groups defined by Woronowicz ([16]). The first major difference is that the canonical Haar state on such quantum groups are no longer tracial, and if one considers an equivariant spectral triple such as the ones constructed by Chakraborty and Pal (c.f. [5]), and constructs the Chern character as mentioned before, it will no longer be invariant under the natural action of the quantum group. In particular, the noncomutative volume form in these cases will not coincide with the Haar state, and in fact need not be even faithful. The simplest and important case of $SU_q(2)$ $(0 < q < 1)$ deserves some discussion in this context. From the explicit description of the K-homology of $SU_q(2)$ in [14], it is easily seen that the Chern character ω (in the notation of the above-mentioned paper) of the 1-summable generator of the odd K-homology is not invariant under the $SU_q(2)$-action. Had it been invariant, one would have $\omega * \tau_{even} = \omega$, which is not the case as it is shown in [14] (here τ_{even} is as in [14] and $*$ denotes the product described in that paper, which is the combination of the shuffle product and the $SU_q(2)$-coproduct). Thus, it is not possible to get an invariant Chern character within the framework of conventional noncommutative geometry, and this explians why the chern character obtained in [5] cannot be invariant. However, even if one forgets the aspect of invariance, there are other strange properties observed in this example. It can be shown (to be discussed in detail elsewhere) that some of the natural properties which one almost always observes for a nice compact manifold such as the classcical $SU(2)$ are not valid. For example, there can be no odd 1-summable spetral triple for $SU_q(2)$ whose Chern character (in HC^1) is nontrivial and which also satisfies the property that the corresponding Dirac operator D_1 has compact resolvents, $|D_1|^{-1}$ (to be interpreted as the inverse of the restriction of $|D_1|$ onto the orthogonal complement of the kernel of D_1) is of Dixmier-trace-class, and repeated commutators with $|D_1|$ is bounded for any element in the finite $*$-algebraic span of the canonical generators of $SU_q(2)$. It is interesting to remark that the equivariant Dirac operator D in [5], whose associated Fredholm module is the generator of the odd K-homology group, has the property that D^{-3} is Dixmier-trace-class, and the Hochschild cohomology class of the associated Chern character (when thought of as an element of HC^3) vanishes, thus this Chern character (which is in HC^3) must be of the form $S\tau$ for some $\tau \in HC^1$ (where S is the periodicity operator used in [1] in the exact couple relation between the Hochschild and cyclic cohomology), and it is not clear whether it is at all possible to obtain this τ as a Chern character coming from some equivariant Dirac operator D' such that $(D')^{-1}$ is of Dixmier-trace-class. Thus, it is not known whether one can describe the Hochschild class corresponding to $\tau \in HC^1$ by a suitable "local" formula involving Dixmier trace (we should note that it is indeed possible to describe the periodic cyclic cohomology class of τ by local formula involving residues, as will be shown in [8]). In any case, all these facts seem to suggest that there may be some incompatibility between the existing framework of noncommutative geometry and the theory of compact quantum groups, since even for the simplest such quantum group a few anomalies seem to occur, specially in context of invariance and the "local" formulae. We feel that the research in this direction (i.e. towards finding a framework in which both noncommutative manifolds and quantum groups are well fitted) is still at a beginning or rather somewhat experimental stage, and it will take more time to reach a conclusive answer. Thus, at this moment, it may be worthwhile to explore some alternative frameworks of noncommutative geometry too, and try to compare the relative advantages and disadvantages of different approaches in the light of various examples at hand. In the present article, we shall focus our attention on one such alternative approach suggested in [12], based on what the authors have called "twisted cyclic cohomology". We shall study the operator theoretic framework for that, and for some natural reason deal with its "entire" version. We would like to point out here that we shall build some amount of general theory only, which in particular will enable one to obtain an invariant (twisted) Chern character in this context, but we shall leave the study of particular examples for future. Thus, apart from the aspect of invariance, we do not know yet whether this alternative framework will help us to understand $SU_q(2)$ and similar models better than the conventional theory; but we hope to take up that issue later on.

Motivated by the fact that the Haar state on typical compact quantum groups are not tracial and other things, the authors of [12] have found it somewhat natural to introduce "twisted cyclic cohomology", which is indeed a module in the cyclic category (see. e.g. [1] or [13]). However, they did not focuss on the "quantized calculus" related the twisted cyclic cohomology, which is our goal in the present article. To be more precise, we shall discuss the twisted analogue of the entire cyclic cohomology and show how one can obtain canonical J-L-O type c.f. [9]) cocycles in this twisted entire cyclic cohomology from a spectral triple and an additional positive operator giving rise to the "twist". In fact, although we shall make a somewhat general theory, our main focuss will be on the examples coming from the quantum group theory and we shall show that a canonical "twisting" operator exists for a given equivariant spectral triple for the action of compact matrix pseudogroups of Woronowicz. Let us remark here that in some special examples of noncommutative manifolds studied in [3] and [4], the conventional theory of noncommutative geometry was shown to be nicely applicable to certain Hopf algebras or associated homogeneous spaces, but those Hopf algebras (e.g. $SL_q(2)$ for q complex of modulus 1) do not come under the framework of topological quantum groups given by Woronowicz and others.

Before we enter into the main results, we should perhaps mention why we are interested in the twisted version of the entire cyclic cohomolgy (hence J-L-O type cocycles) rather than the ordinary cyclic cohomology. This is motivated by our study of the example $SU_q(2)$ ([7], and also [5]), where we have shown that the Haar state can be recaptured by the formula $\frac{Tr(aRe^{-tD^2})}{Tr(Re^{-tD^2})}, a \in SU_q(2)$, where D is the equivariant Dirac operator and R is a suitable positive operator, coming from the modular theory of $SU_q(2)$. It is also shown that there is no finite postiive number d so that $Tr(Re^{-tD^2}) = O(t^{-d})$. This in some sense indicates that the associated Fredholm module is not finite dimensioanl, or in other words in Θ-summable, so that it is natural to construct $J - L - O$ type cocyles in the (twisted) entire cyclic cohomology.

2 Twisted entire cyclic cohomology and J-L-O cycles

2.1 Algebraic aspects

Let us develop the theory for Banach algebras for simplicity, but we note that our results extend to locally convex algebras, which we actually need. The extnesion to the locally convex algebra case follows exactly as remarked in [1, page 370]. So, let $\mathcal{A}$ be a unital Banach algebra, with $\|.\|_*$ denoting its Banach norm, and let σ be a continuous automorphism of $\mathcal{A}$, $\sigma(1) = 1$. For $n \geq 0$, let C^n be the space of continuous $n+1$-linear functionals ϕ on $\mathcal{A}$ which are σ-invariant, i.e. $\phi(\sigma(a_0),..\sigma(a_n)) = \phi(a_0,...a_n) \forall a_0,...a_n \in \mathcal{A}$; and $C^n = \{0\}$ for $n < 0$. We define linear maps $T_n, N_n : C^n \to C^n$, $U_n : C^n \to C^{n-1}$ and $V_n : C^n \to C^{n+1}$ by,

$$(T_n f)(a_0,...a_n) = (-1)^n f(\sigma(a_n), a_0, ...a_{n-1}), N_n = \sum_{j=0}^{n} T_n^j,$$

$$(U_n f)(a_0,...a_{n-1}) = (-1)^n f(a_0, ..., a_{n-1}, 1),$$

$$(V_n f)(a_0,...a_{n+1}) = (-1)^{n+1} f(\sigma(a_{n+1})a_0, a_1, ..., a_n).$$

Let $B_n = N_{n-1}U_n(T_n - I)$, $b_n = \sum_{j=0}^{n+1} T_{n+1}^{-j-1} V_n T_n^j$. Let B, b be maps on the complex $C \equiv (C^n)n$ given by $B|_{C^n} = B_n, b|_{C^n} = b_n$. It is easy to verify (similar to what is done for the untwisted case , e.g. in [1]) that $B^2 = 0$, $b^2 = 0$ and $Bb = -bB$, so that we get a bicomplex ($C^{n,m} \equiv C^{n-m}$) with differentials d_1, d_2 given by $d_1 = (n - m + 1)b : C^{n,m} \to C^{n+1,m}$, $d_2 = \frac{B}{n-m} : C^{n,m} \to C^{n,m+1}$. Furthermore, let $C^e = \{(\phi_{2n})n \in N; \phi_{2n} \in C^{2n} \forall n \in N\}$, and $C^o = \{(\phi_{2n+1})n \in N; \phi_{2n+1} \in C^{2n+1} \forall n \in N\}$. We say that an element $\phi = (\phi_{2n})$ of C^e is a σ-twisted even entire cochain if the radius of convergence of the complex power series $\sum \|\phi_{2n}\| \frac{z^n}{n!}$ is infinity, where $\|\phi_{2n}\| := \sup_{\|a_j\|_* \leq 1} |\phi_{2n}(a_0, ..., a_{2n})|$. Similarly we define σ-twisted odd entire cochains, and let $C^e_\epsilon(\mathcal{A}, \sigma)$ ($C^o(\mathcal{A}, \sigma)$ respectively) denote the set of σ-twisted even (respectively odd) entire cochains. Let $\partial = d_1 + d_2$, and we have the shrot complex $C^e_\epsilon(\mathcal{A}, \sigma) \overset{\partial}{\underset{\partial}{\rightleftarrows}} C^o_\epsilon(\mathcal{A}, \sigma)$. We call the cohomology of this complex the σ-twisted entire cyclic cohomology of $\mathcal{A}$ and denote it by $H^*_\epsilon(\mathcal{A}, \sigma)$.

Proposition 2.1 *Let $\mathcal{A}_\sigma = \{a \in \mathcal{A} : \sigma(a) = a\}$ be the fixed point subalgebra for the automorphism σ. There is a canonical pairing $< .,. >_{\sigma,\epsilon}: K_*(\mathcal{A}_\sigma) \times H^*_\epsilon(\mathcal{A}, \sigma) \to C$.*

The proof is omitted, since it is very similar to the untwisted case, for example, as given in [1]. In fact, this pairing is nothing but the pairing between the K-theory of $\mathcal{A}_\sigma$ and the entire cyclic (untwisted) cohomologies of $\mathcal{A}_\sigma$, as any element in the σ-twisted entire cyclic cohomology of $\mathcal{A}$ can be viewed as an (untwisted) entire cyclic cocycle on $\mathcal{A}_\sigma$ by restriction on $\mathcal{A}_\sigma$. Thus, the arguments for the untwisted case apply to our situation to prove the above proposition.

2.2 Construction of the Chern character using the J-L-O cocycles

We begin with the following definition :

Definition 2.2 *Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{A}^\infty$ be a subalgebra (not necessarily complete) of $\mathcal{B}(\mathcal{H})$, R be a positive (possibly unbounded) operator in $\mathcal{H}$, D be a self-adjoint operator in $\mathcal{H}$ such that the followings hold :*
(i) $[D,a] \in \mathcal{B}(\mathcal{H}) \forall a \in \mathcal{A}^\infty$,
(ii) R commutes with D,
(iii) For any real number s and $a \in \mathcal{A}^\infty$, $\sigma_s(a) := R^{-s}aR^s$ is bounded and belongs to $\mathcal{A}^\infty$. Furthermore, $a \mapsto \sigma_s(a)$ is an automorphism of $\mathcal{A}^\infty$ and for any positive integer n, $\sup_{s\in[-n,n]} \|\sigma_s(a)\| < \infty$.
Then we call the quadruplet $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ an odd R-twisted spectral data. Furthermore, if there is a grading given by $\gamma \in \mathcal{B}(\mathcal{H})$ with $\gamma = \gamma^* = \gamma^{-1}$, and γ commutes with $\mathcal{A}^\infty$ and R, and anticommutes with D, then we say that we are given an even R-twisted spectral data. We say that the given (odd or even) twisted spectral data is Θ-summable if Re^{-tD^2} is trace-class for all $t > 0$.

Let us consider the case of even twisted spectral data only, as the odd case can be treated with obvious and minor modifications as done in the untwisted case. Let us assume that we are given such an even twisted spectral data specified by $\mathcal{A}^\infty, \mathcal{H}, D, R, \gamma$ as in the above definition, and fix any $\beta > 0$. Let $H = D^2$, $A^{(s)} = e^{-sH}Ae^{-sH}$, $A(s) = e^{-sH}Ae^{sH}$ for $s > 0$ and $A \in \mathcal{B}(\mathcal{H})$. Let us denote by $\mathcal{B}$ the set of all $A \in \mathcal{B}(\mathcal{H})$ for which $\sigma_s(A) := R^{-s}AR^s \in \mathcal{B}(\mathcal{H})$ for all real number s, $[D,A] \in \mathcal{B}(\mathcal{H})$ and $s \mapsto \|\sigma_s(A)\|$ is bounded over compact subsets of the real line. We define for $n \in N$ an $n+1$-linear functional F_n^β on $\mathcal{B}$ by the formula

$$F_n^\beta(A_0, ...A_n) = \int_{\sigma_n} Tr(\gamma A_0 A_1(t_1)..A_n(t_n)Re^{-\beta H})dt_1...dt_n,$$

where $\sigma_n = \{(t_1, ...t_n) : 0 \leq t_1 \leq ... \leq t_n \leq \beta\}$. That the above integral exists as a finite quantity follows from the following lemma.

Lemma 2.3 *F_n^β is well defined and one has the estimate*

$$|F_n^\beta(A_0, ..A_n)| \leq \frac{\beta^n}{n!} Tr(Re^{-\beta H}) \Pi_{j=o}^n C_j,$$

where $C_j = \sup_{s\in[-1,1]} \|\sigma_s(A_j)\|$.

Proof :-
The proof is very similar to that of Proposition IV.2 pf [9], so we only sketch the main ideas. We use the same notation $\delta_1, ...\delta_{n+1}$ as in [9], i.e. $\delta_j = \frac{t_j - t_{j-1}}{\beta}$, with $t_0 = 0, t_{n+1} = \beta$. Thus, $(t_1, ..t_n)$ in the integrand can be replaced by $(\delta_1, ...\delta_{n+1})$ with the condition that $\delta_j \geq 0, \sum \delta_j = 1$. Then, as in [9], we have that

$$\begin{aligned} & Tr(A_0 A_1(t_1)..A_n(t_n)Re^{-\beta H}) \\ = \; & Tr\left(\gamma A_0 e^{-\beta\delta_1 H} A_1 e^{-\beta\delta_2 H} ...A_n e^{-\beta\delta_{n+1} H} R^{\sum_j \delta_j}\right) \\ = \; & Tr\left(\gamma\sigma^{-1}(A_0)(Re^{-\beta H})^{\delta_1} \sigma^{-\sum_{j=2}^{n+1}\delta_j}(A_1)(Re^{-\beta H})^{\delta_2} ....\sigma^{-\delta_{n+1}}(A_n)(Re^{-\beta H})^{\delta_{n+1}}\right), \end{aligned}$$

where in the last step we have used the fact that R and H commute, and γ and R also commute. Now, by the generalised Holder's inequality for Schetten ideals the desired estimate follows, noting that $\|(Re^{-\beta H})^{\delta}\|_{\delta^{-1}} = Tr(Re^{-\beta H})^{\delta}$, as $Re^{-\beta H}$ is a positive operator.

For $A \in \mathcal{B}(\mathcal{H})$, let $\dot{A}$ denote $\frac{d}{dt}(A(t))|_{t=0} = -[H, A]$, whenever it exists as a bounded operator. Clearly for A of the form $A = B^{(s)}, s > 0$, $\dot{A} \in \mathcal{B}(\mathcal{H})$.

Lemma 2.4 *Let $A_i, i = 0, 1, ..., n$ be elements of $\mathcal{B}$ such that $\dot{A}_i \in \mathcal{B}(\mathcal{H}) \forall i$. Let $dA := i[D, A]$. Then we have the following :*
(i) For $j = 1, ..., n$, $F^{\beta}_{n+1}(A_0, ..., A_{j-1}, \dot{A}_j, ..., A_{n+1}) = F^{\beta}_n(A_0, ..., A_{j-1}, A_j A_{j+1}, ..., A_{n+1}) - F^{\beta}_n(A_0, ...A_{j-1}A_j, ..., A_{n+1})$.
(ii) $F^{\beta}_{n+1}(\dot{A}_0, A_1, ..., A_{n+1}) = F^{\beta}_n(A_0A_1, ..., A_{n+1}) - F^{\beta}_n(A_1, ..., A_{n+1}\sigma_{-1}(A_0))$.
(iii) $F^{\beta}_{n+1}(A_0, A_1, ..., \dot{A}_{n+1}) = F^{\beta}_n(\sigma_1(A_{n+1})A_0, ..., A_n) - F^{\beta}_n(A_0, A_1, ..., A_nA_{n+1})$.
(iv) $F^{\beta}_n(A_0, ..., A_n) = F^{\beta}_n(\sigma_1(A_n), A_0, ..., A_{n-1})$ and $F^{\beta}_n(\sigma_1(A_0), ..., \sigma_1(A_n)) = F^{\beta}_n(A_0, ..., A_n)$.
(v) $\sum_{j=0}^{n-1} F^{\beta}_n(A_0, ..., A_j, 1, A_{j+1}, ..., A_{n-1}) = \beta F^{\beta}_{n-1}(A_0, ..., A_{n-1})$.
(vi) $F^{\beta}_n(dA_0, ..., dA_n) = \sum_{j=1}^{n}(-1)^j F^{\beta}_n(A_0, dA_1, ..., dA_{j-1}, \dot{A}_j, dA_{j+1}, ..., dA_n)$.

The proof of the above formulae are straightforward and very similar to the analogous formulae derived in [9], hence we omit the proof.

Let us now equip $\mathcal{A}^{\infty}$ with the locally convex topology given by the family of Banach norms $\|.\|_{*,n}, n = 1, 2, ...,$ where $\|a\|_{*,n} := \sup_{s \in [-n,n]}(\|\sigma_s(a)\| + \|[D, \sigma_s(a)]\|)$. Let $\mathcal{A}$ denote the completion of $\mathcal{A}^{\infty}$ under this topology, and thus $\mathcal{A}$ is Frechet space. We shall now construct the Chern character in $H^*_{\epsilon}(\mathcal{A}, \sigma)$, where $\sigma = \sigma_1$, which extends on the whole of $\mathcal{A}$ by continuity.

Theorem 2.5 *Let $\phi^e \equiv (\phi_{2n})$ and $\phi^o \equiv (\phi_{2n+1})$ be defined by*

$$\phi_{2n}(a_0, ..., a_{2n}) = \beta^{-n} F^{\beta}_{2n}(a_0, [D, a_1], ..., [D, a_{2n}]), a_i \in \mathcal{A},$$

$$\phi_{2n+1}(a_0, ..., a_{2n+1}) = \sqrt{2i}\beta^{-n-\frac{1}{2}} F^{\beta}_{2n+1}(\gamma a_0, [D, a_1], ..., [D, a_{2n+1}]), a_i \in \mathcal{A}.$$

Then $(b + B)\phi^e = 0$, $(b + B)\phi^o = 0$, and hence $\psi^e \equiv ((2n)!\phi_{2n}) \in H^e_{\epsilon}(\mathcal{A}, \sigma)$ and $\psi^0 \equiv ((2n + 1)!\phi_{2n+1}) \in H^o_{\epsilon}(\mathcal{A}, \sigma)$.

Proof :-
First we extend the definition of ϕ_{2n}, ϕ_{2n+1} on the whole of multiple copies of $\mathcal{B}$ by the same formula, which is clearly well defined. Let $\mathcal{B}^{\infty}$ denote the unital algebraic span of elements of the form $A^{(s)}$ for $s > 0$ and $A \in \mathcal{A}^{\infty}$. Let us denote by $C^n(\mathcal{B}^{\infty})$ the space of all $n + 1$-linear functionals on $\mathcal{B}^{\infty}$ (without any continuity requirements) and extend the definitions of b and B on the complex $C(\mathcal{B}^{\infty}) \equiv (C^n(\mathcal{B}^{\infty})_n)$ by the same expression as in the case of C^n, i.e. for functionals on $\mathcal{A}$. This is possible because $\sigma = \sigma_1$ is defined on the whole of $\mathcal{B}^{\infty}$. Now, the formulae (i) to (vi) of Lemma (2.4) are applicable for elements of $\mathcal{B}^{\infty}$, and by a straightforward calculation as in [9] we can show that $(b + B)(\phi^*) = 0$ on elements of $\mathcal{B}^{\infty}$. Now, to prove the

same for elemenets of $\mathcal{A}^\infty$, we note that for $A \in \mathcal{A}^\infty$, $A^{(s)} \to A$, $[D, A^{(s)}] = [D, A]^{(s)} \to [D, A]$ and $\sigma_t(A^{(s)}) \to \sigma_t(A) \forall t$, as $s \to 0+$ and the convergence of operators is w.r.t. the strong operator topology of $\mathcal{B}(\mathcal{H})$. By using the fact that $Tr(B_n C) \to Tr(BC)$ if $B_n \to B$ w.r.t. the strong operator topology and C is trace-class, we conclude that the integrand in the definition of $F^{\beta}_{2n}(a_0^{(s)}, [D, a_1^{(s)}], ...[D, a_{2n}^{(s)}])$ converges to that with $a_j^{(s)}$ replaced by a_j (for $a_j \in \mathcal{A}^\infty$), and finally, as $\|a^{(s)}\| \leq \|a\| \forall a \in \mathcal{B}(\mathcal{H})$, an application of the Dominated Convergence Theorem allows us prove that $(b + B)\phi^e = 0$ on elements of $\mathcal{A}^\infty$, and hence by continuity the same thing holds for $\mathcal{A}$. Similarly the odd case can be done. The remaining part of the statement of the theorem is straighforward, and follows exactly in the same way as in [1].

We shall call ψ^* in the above theorem the **Chern character** of the twisted spectral data $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ (or $(\mathcal{A}^\infty, \mathcal{H}, D, R, \gamma)$ for the even case). We remark here by an easy adaptation of the techniques of [9] and [10], we can show that the above chern characters do not depend on our choice of β, namely cohomologous for all $\beta > 0$. Furthermore, invariance of the chern character under some suitable homotopy of the spectral data can possibly be established along the lines of the above mentioned references. We, however, would like to consider those issues elsewhere.

3 Canonical twisted equivariant spectral data arising from actions of compact matrix pseudogroups

In this section we shall show how one can find canonical examples of twisted spectral data from the theory of compact matrix pseudogroups of Woronowicz (c.f. [16]. Let $\mathcal{S}$ be such a compact matrix pseudogroup, with the matrix elements $t^n_{ij}, n = 1, 2, ..., i, j = 1, 2, ..., d_n$, such that for each n, $T_n \equiv ((t^n_{ij}))_{ij=1}^{d_n}$ is a unitary element of $M_{d_n}(C) \otimes \mathcal{S}$, and the coproduct Δ and the antipode κ are given by $\Delta(t^n_{ij}) = \sum_k t^n_{ik} \otimes t^n_{kj}$, $\kappa(t^n_{ij}) = (t^n_{ji})^*$. Let $\mathcal{K} = L^2(\mathcal{S}, h)$ be the GNS-space associated to the faithful Haar state h on $\mathcal{S}$, and we imbed $\mathcal{S}$ in $\mathcal{B}(\mathcal{K})$ in the natural manner. We recall from the theory of quantum groups that a unitary representation of the quantum group $\mathcal{S}$ is given by a separable Hilbert space $\mathcal{H}$ and a unitary element V of $\mathcal{L}(\mathcal{H} \otimes \mathcal{S}) \subseteq \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$ with additional properties (c.f. [15] and other relevant literature on compact quantum groups, and note that $\mathcal{L}(\mathcal{H} \otimes \mathcal{S})$ denotes the C^*-algebra of adjointable linear maps on the Hilbert module $\mathcal{H} \otimes \mathcal{S}$), and we can equivalently think of the representation to be given by a map V' from $\mathcal{H}$ to the Hilbert module $\mathcal{H} \otimes \mathcal{S}$ given by $V'(\xi) = V(\xi \otimes 1)$, where 1 is the identity in $\mathcal{S}$. For $A \in \mathcal{B}(\mathcal{H})$, we define $\delta(a) = V(a \otimes I)V^* \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$. Let us assume that there is a subalgebra $\mathcal{A}^\infty$ of $\mathcal{B}(\mathcal{H})$ such that $\delta(\mathcal{A}^\infty) \subseteq \mathcal{A}^\infty \otimes_{\text{alg}} \mathcal{S}^\infty$, where $\mathcal{S}^\infty$ denotes the algebraic span of the matrix elements of $\mathcal{S}$ and their adjoints. Clearly, $\delta : \mathcal{A}^\infty \to \mathcal{A}^\infty \otimes_{\text{alg}} \mathcal{S}^\infty$ is a coaction of the Hopf algebra $\mathcal{S}^\infty$.

Lemma 3.1 *Let $\phi : \mathcal{S}^\infty \to C$ is a multiplicative linear functional. We define a linear map F_ϕ on $\mathcal{H}$ with the domain consisting of all $\xi \in \mathcal{H}$ such that $V'\xi \in \mathcal{H} \otimes_{\text{alg}} \mathcal{S}^\infty$, and $F_\phi(\xi) = (id \otimes \phi)(V'\xi)$*

for any ξ in the above domain. Then we have that

(i) F_ϕ is densely defined,

(ii) For $a \in \mathcal{A}^\infty$, $aDom(F_\phi) \subseteq Dom(F_\phi)$ and $F_\phi(a\xi) = (\phi * a)F_\phi(\xi)$ for $\xi \in Dom(F_\phi)$, where $(\phi * a) := (id \otimes \phi)(\delta(a))$.

Proof :-

By the general theory $\mathcal{H}$ will be decomposed as $\mathcal{H} = \bigoplus_{n\geq 1, k=1,...,m_n, m_n \leq \infty} \mathcal{H}_{n,k}$, and there exists an orthonormal basis $\{e_j^{n,k}\}_{j=1,...,d_n}$ for $\mathcal{H}_{n,k}$ such that $V' e_j^{n,k} = \sum_i e_i^{n,k} \otimes t_{ij}^n$. It is obvious that $\mathcal{H}_n \equiv \bigoplus_k \mathcal{H}_{n,k} \subseteq Dom(F_\phi)$. This proves (i). For (ii), we first note that for $\xi \in Dom(F_\phi)$ and $a \in \mathcal{A}^\infty$, we have $V'(a\xi) = \delta(a)V'\xi$ (where an element of $\mathcal{A}^\infty \otimes_{\text{alg}} \mathcal{S}$ is naturally acting from left by multiplication on $\mathcal{H} \otimes \mathcal{S}$), which shows that $a\xi \in Dom(F_\phi)$. The fact that $F_\phi(a\xi) = (\phi * a)F_\phi(\xi)$ is verified easily using the multiplicativity of ϕ.

Let us now recall a few facts from the general theory of compact matrix pseudogroups as in [16]. It is known that for each n, there is a unique $d_n \times d_n$ complex matrix F_n with the following properties :

(i) F_n is positive and invertible with $Tr(F_n) = Tr(F_n^{-1}) = M_n > 0$, say.

(ii) If h denotes the Haar state on $\mathcal{S}$, then we have, $h(t_{ij}^n t_{kl}^{n*}) = \frac{1}{M_n}\delta_{ik}F_n(j,l)$, where δ_{ik} is the Kronecker delta.

(iii) For any complex number z, let ϕ_z be the functional on $\mathcal{S}^\infty$ defined by $\phi_z(t_{ij}^n) = (F_n)^z(j,i)$. Then each ϕ_z is multiplicative, $(\phi_z * 1) = 1$ and for any fixed element $a \in \mathcal{S}^\infty$, $z \mapsto \phi_z * a$ is a complex analytic map.

Let us now take $R = F_{\phi_1}$ on $\mathcal{H}$. With this choice, we have the following result. Note that we call an m-linear functional χ on $\mathcal{A}^\infty$ invariant (or simply invariant if no cinfusion arises) if we have

$$\chi(a_{1(1)}, ..., a_{m(1)})a_{1(2)}...a_{m(2)} = \chi(a_1, ..., a_m)1_{\mathcal{S}},$$

where we have used the Swedler notation $\delta(a) = a_{(1)} \otimes a_{(2)}$, with summation implied.

Theorem 3.2 *Assume that $(\mathcal{A}^\infty, \mathcal{H}, D)$ is an odd equivariant spectral triple in the sense of [5], i.e. D is a self adjoint operator on $\mathcal{H}$ which is $\mathcal{S}$-invariant, in the sense that $V'(Dom(D)) \subseteq Dom(D) \otimes_{\text{alg}} \mathcal{S}^\infty$ and $V'D = (D \otimes I)V'$ on $Dom(D)$, and furthermore, $[D, a] \in \mathcal{B}(\mathcal{H})$ for $a \in \mathcal{A}^\infty$. Then $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ is a twisted odd spectral data. Similarly, if we also have an equivariant grading then we obtain a twisted even spectral data. Moreover, if $Re^{-\beta D^2}$ is trace -class for all $\beta > 0$, then the associated Chern characters are invariant.*

Proof :-

Since the resolvent operators $(i\lambda - D)^{-1}$, $\lambda \in R$, of D are equivariant bounded operators, it is clear that $(i\lambda - D)^{-1} = \bigoplus_n (I \otimes B_{n,\lambda})$, where we have identified $\mathcal{H}_n$, which is a direct sum of d_n-dimensional Hilbert spaces, with $\mathcal{H}_{n,1} \otimes C^{m_n}$ ($C^\infty := l^2$), and in this identification we have $B_{n,\lambda} \in \mathcal{B}(C^{m_n})$. Thus, D is also of the same form, with $B_{n,\lambda}$ replaced by say D_n, which is a self adjoint (possibly unbounded) operator on C^{m_n}. Now, by the definiton of R, it is of the from $\bigoplus_n (r_n|_{\mathcal{H}_{n,1}} \otimes I)$, from which it follows that R and D commute. Other conditions in the definition of twisted spectral data are verified easily using the facts about the canonical family of functionals ϕ_z noted before.

Finally, we shall verify the invariance of the associated chern character. To this end, first note that if $Re^{-\beta D^2}$ is trace class, then $e^{-\beta D_n^2}$ is also trace-class for each n, hence in particular has a complete set of eigenvectors in C^{m_n}. Thus, by renaming the orthonormal basis $e_j^{n,k}$ if necessary, we can without loss of generality assume that $e^{-\beta D_n^2} e_j^{n,k} = \lambda_{n,k} e_j^{n,k}$ and $Re_j^{n,k} = \sum_l F_n(j,l) e_l^{n,k}$. Let us now use the embedding of $\mathcal{S}$ in the GNS Hilbert space $\mathcal{K} = L^2(\mathcal{S}, h)$ associated with the Haar state h, and denote by 1 the identity of $\mathcal{S}$ viewed as the canonical cyclic vector in $\mathcal{K}$. We have, $h(b) = \langle 1, b1 \rangle \forall b \in \mathcal{S}$. Let $\chi(A) := Tr(ARe^{-\beta D^2})$ be the normal positive linear functional on $\mathcal{B}(\mathcal{H})$. Since the extension of h on $\mathcal{B}(\mathcal{K})$ given by $h(B) := \langle 1, B1 \rangle, B \in \mathcal{B}(\mathcal{K})$ is also a positive linear normal functional, we can define the positive linear normal functional $(\chi \otimes h)$ on $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$. We claim that $(\chi \otimes h)(V(a \otimes 1)V^*) = \chi(a)$ for any $a \in \mathcal{B}(\mathcal{H})$. The proof of this fact is very similar to a similar result obtained in the special case of $SU_q(2)$ in [7].

Proof of the claim :

We first note that $V^*(e_j^{n,k} \otimes 1) = \sum_i e_i^{n,k} \otimes (t_{ji}^n)^*$. Thus, we have,

$$
\begin{aligned}
&(\chi \otimes h)(V(a \otimes I)V^*) \\
&= \sum_{n,i,j} \langle e_j^{n,i} \otimes 1, V(a \otimes I)V^*(Re^{-\beta D^2}(e_j^{n,i}) \otimes 1) \rangle \\
&= \sum_{n,i,j,r} \langle V^*(e_j^{n,i} \otimes 1), (a \otimes I)V^*(\lambda_{n,i} F_n(j,r) e_r^{n,i} \otimes 1) \rangle \\
&= \sum_{n,i,j,r,k,l} \lambda_{n,i} F_n(j,r) \langle e_k^{n,i} \otimes (t_{jk}^n)^*, (a \otimes I)(e_l^{n,i} \otimes (t_{rl}^n)^*) \rangle \\
&= \sum_{n,i,j,r,k,l} \lambda_{n,i} F_n(j,r) \langle e_k^{n,i}, a e_l^{n,i} \rangle h(t_{jk}^n t_{rl}^{n\,*}) \\
&= \sum_{n,i,j,k,l} \lambda_{n,i} F_n(j,j) \langle e_k^{n,i}, a e_l^{n,i} \rangle \frac{F_n(k,l)}{M_n} \\
&= \sum_{n,i,k} \Big(\sum_j \frac{F_n(j,j)}{M_n}\Big) \lambda_{n,i} \langle e_k^{n,i}, a\Big(\sum_l F_n(k,l) e_l^{n,i}\Big) \rangle \\
&= \chi(a),
\end{aligned}
$$

since $M_n = Tr(F_n)$.

Now, it is easy to see that since $V(D \otimes I) = (D \otimes I)V$ (viewing V as a unitary in $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$), for $a \in \mathcal{A}^\infty$ and $s > 0$, one has $V(ae^{-sD^2} \otimes I)V^* = \delta(a)(e^{-sD^2} \otimes I)$. For $s_0, ..., s_n > 0$, let η be the

$n+1$ linear functional on $\mathcal{A}^\infty$ given by $\eta(a_0, ..., a_n) = Tr(a_0 e^{-s_0 D^2}[D, a_1]e^{-s_1 D^2}...[D, a_n]e^{-s_n D^2}R)$. Clearly, as V commutes with $(D \otimes I)$, we have,

$$\begin{aligned} & V((a_0 e^{-s_0 D^2}[D, a_1]e^{-s_1 D^2}...[D, a_n]) \otimes I)V^* \\ = \; & \delta(a_0).(e^{-s_0 D^2} \otimes I).[(D \otimes I), \delta(a_1)]...[(D \otimes I), \delta(a_n)] \\ = \; & (a_{0(1)} e^{-ts_0 D^2}...[D, a_{n(1)}]) \otimes (a_{0(2)}...a_n(2)), \end{aligned}$$

using the Swedler notation with summation implied. Hence, by what we have proved earlier, it is easy to see that $\eta(a_{0(1)}, ..., a_{n(1)})h(a_{0(2)}...a_{n(2)}) = \eta(a_0, ..., a_n)$. From this the required invariance of the odd chern characters follows, because by using the fact that $h*\lambda = \lambda(1)h$ for any bounded linear functional λ on $\mathcal{S}$ (where $*$ is the convolution product of two linear functionals on $\mathcal{S}$, defined for example in [16]), we get that $\eta(a_{0(1)}, ..., a_{n(1)})\lambda(a_{0(2)}...a_{n(2)}) = \eta(a_0, ..., a_n)\lambda(1)$ for all bounded linear functionals λ on $\mathcal{S}$. Similarly the case of the even chern characters can be treated.

Corollary 3.3 *Let us consider the case $\mathcal{A} = \mathcal{S}$, $\mathcal{A}^\infty = \mathcal{S}^\infty$, and $\mathcal{H} = L^2(h)$, with V is the operator associated with the canonical regular representationn of $\mathcal{S}$ in $L^2(h)$, and let $R = F_{\phi_1}$ in $L^2(h)$. Given any positive operator L on $L^2(h)$ such that $L(t^n_{ij}) = \lambda_n t^n_{ij}$ and $\sum_n M_n \lambda_n < \infty$, we can recover the Haar state by the following formula,*

$$h(a) = \frac{Tr(aRL)}{Tr(RL)}, a \in \mathcal{S}.$$

The proof of the corollary is immediate from the steps in the proof of the Theorem 3.2.

The above corllary generalizes a similar result obrained in [7] for $SU_q(2)$. We have already remarked that for typical nonclassical examples of compact matrix pseudogroups, we are forced to consider the Θ-summable case rather than the finitely summable case. Now, we shall show that this assertion can be made a bit more definitive.

Proposition 3.4 *Let $\mathcal{S}$ be a compact matrix psedogroup with $\mathcal{H}$ be $L^2(h)$ as before, and suppose that the corresponding operator R is not the identity, i.e. F_n is not equal to I. Assume that there is an equivariant spectral triple $(\mathcal{S}^\infty, \mathcal{H}, D)$ satisfying $[|D|, [|D|, a]] \in \mathcal{B}(\mathcal{H}) \forall a \in \mathcal{S}^\infty$. Then, there cannot be any finite positive number p such that $Lim_{t \to 0+}(t^p Tr(Re^{-tD^2})) = C, 0 < C < \infty$, for some suitable Banach limit Lim on the space of bounded functions on R_+, as considered in [6] and elsewhere.*

Proof :-
Suppose that the assertion of the proposition is false, and we are indeed given an equivariant sepctral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ such that $[|D|, [|D|, a]] \in \mathcal{B}(\mathcal{H}) \forall a \in \mathcal{S}^\infty$ and $Lim_{t \to 0+} t^p Tr(Re^{-tD^2}) = C, 0 < C < \infty, p > 0$. Let $\eta(a) = Lim_{t \to 0+} \frac{Tr(aRe^{-tD^2})}{Tr(Re^{-tD^2})}$. We know from our earlier results that

$\eta(a) = h(a) \forall a \in \mathcal{S}$. We now claim that for $a, b \in \mathcal{S}^\infty$, $\eta(ab) = \eta(\sigma(b)a)$, where $\sigma(b) = \phi_1 * b$. Before we prove this claim, we argue how it leads to a contradiction and hence completes the proof. It is known from [16] that $h(ab) = h((\phi_1 * b * \phi_1)a)$, where $(b * \phi) := (\phi \otimes id)(\Delta(b))$. Thus, we get that $(\sigma(b) * \phi_1) = \sigma(b) \forall b \in \mathcal{S}^\infty$, and as σ is an automorphism, $b * \phi_1 = b$. But this is possible only if $F_n = I \forall n$, which is contradictory to the assumption.

So, it is enough to prove that $\eta(ab) = \eta(\sigma(b)a)$. We have,

$$
\begin{aligned}
& R[e^{-tD^2}, a] \\
&= -t \int_0^1 Re^{-tsD^2}[D^2, a]e^{-t(1-s)D^2} ds \\
&= -t \int_0^1 Re^{-tsD^2}(2|D|[|D|, a] + [|D|, [|D|, a]])e^{-t(1-s)D^2} ds.
\end{aligned}
$$

By standard estimates one can now show that t^p times the above expression goes to 0 in trace-norm as $t \to 0+$, and from this the claim is verified.

Remark 3.5 *If we look at the proof of the Theorem 3.2 carefully, we can easily notice that there is indeed some amount of flexibility in the choice of R. In fact, the conclusion of the theorem will be valid if we replace the canonical R chosen by us by some operator of the form* $R_1 = RR'$, *where* R' *is a positive operator having* $e_j^{n,k}$*'s as a complete set of eigenvectors with* $R'(e_j^{n,k}) = \mu_k e_j^{n,k}$, *with* μ_k*'s be such that* $R_1 e^{-\beta D^2}$ *is trace-class for all* $\beta > 0$. *In the context of* $SU_q(2)$, *this flexibility of choice will play a crucial role. However, it should be noted that the conclusion of Proposition 3.4 does no longer hold if we change R.*

Remark 3.6 *The twisted Chern character can be paired with the equivariant K-theory, i.e. the K-theory of the subalgebra* $\mathcal{A}_{inv} \equiv \{a \in \mathcal{A} : \delta(a) = a \otimes 1_{\mathcal{S}}\}$. *In fact, from the special from of* σ, *it is easily seen that* $\mathcal{A}_{inv} \subseteq \mathcal{A}_\sigma$, *hence we can restrict the pairing* $< ., . >_{\sigma,\epsilon}$ *on* $K_*(\mathcal{A}_{inv}) \times H_\epsilon^*(\mathcal{A}, \sigma)$ *to get the desired map. Furthermore, since the equivariant Dirac operator D decomposes into a direct sum of operators* D_π, *indexed by irreducible representations* π *of* $\mathcal{S}$, *and any projection of* $\mathcal{A}_{inv}$ *also naturally respects this decomposition, one can consider the twisted Chern character corresponding to the spectral data given by any fixed* $P_\pi D P_\pi$, *where* P_π *denotes the projection onto the subspace corresponding to* π. *The corresponding pairing assigns to each element of* $K_*(\mathcal{A}_{inv})$ *a complex number depending on* π, *and thus one gets a map from the set of irreducible representations of* $\mathcal{S}$ *to the dual of the K-theory, which may be formally thought of some kind of "character-valued index". However, any attempt to give this a rigorous meaning requires first of all a generalization of equivariant entire cyclic cohomology as discussed, for example, in [11] and related works of other authors, to the framework of compact quantum groups.*

We shall conclude with some discussion on the case of $SU_q(2)$. We recall from section 2 that in general the twisted entire cyclic cohomology $H_\epsilon^*(\mathcal{A}, \sigma)$ pairs with the K-theory of only a subalgebra $\mathcal{A}_\sigma$ of $\mathcal{A}$, and not with that of $\mathcal{A}$. However, it may sometimes turn out that the subalgebra $\mathcal{A}_\sigma$ is large enough to capture the K-theory of $\mathcal{A}$ itself. We shall see that this is indeed the case if we consider $SU_q(2)$. Let us recall the notation of [5], where the generators of $SU_q(2)$ were denoted by α, β, and $u = I_1(\beta^*\beta)(\beta - I) + I$ was choesn to be the generator of $K_1(SU_q(2))$ which is Z. Since the spectrum $\sigma(u)$ is a connected set (which can be easily verified from the litarature on $SU_q(2)$), and u is invertible, it is clear that the map from $K_1(C^*(u))$ to $K_1(SU_q(2))$, induced by the inclusion map, is an isomorphism of the K_1-groups (where $C^*(u)$ denotes the unital C^*-algebra generated by u). Now our aim is to construct an appropriate twisted spectral triple so that the associated fixed point subalgebra $SU_q(2)_\sigma$ will contain u. To do this, we have to refer to the remark 3.5 made earlier. We consider any of the equivariant spectral triple constructed by the authors of [5] and in the associated Hilbert space (which is the canonical regular representation space of $SU_q(2)$) choose R' to be the operator with corresponding eigenvalues $\mu_k = q^{-2k}$, so that the new choice of R actually coincides with that in [7]. It can be easily verified that for σ corresponding to this choice of R, the fixed point subalgebra $SU_q(2)_\sigma$ is the unital $*$-algebra generated by β, so in particular contians u, allowing us to consider the pairing of the twisted Chern character with $K_1(C^*(u))$, and in turn with $K_1(SU_q(2))$ using the isomorphism noted before. The important question is whether we recover the pairing obtained in [5] in our twisted framework, or if our pairing nontrvial. We conjecture that this question has an affirmative answer, but to calculate the pairing we shall need to build some more tools, analogous to the index theorem available for the untwisted or conventional set-up. We however would like to postpone these issues for future works.

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