3.2 General Description of Tokamak Ideal MHD Instability I①

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The tokamak equilibrium are usually axisymmetric ones but with non-circular cross-sections. The ideal MHD stability analysis of such system had been done in cylindrical approximation (1D case) in detail[1]. For toroidal configuration, the high n- ballooning mode and the local mode (the Mercier mode) can also be simplified as 1D eigenmode equation[2]. For low mode number large scale perturbations, such as the kink mode, the Alfven eigen-mode, their solutions can be obtained numerically even for circular cross-section case because of the mode coupling between the neighboring modes. For non-circular cross-section case, the solution can only be carried out numerically. The numerical treatment is also based on two procedures: the first is to obtain corresponding eigenmode equation and the second is to use a suitable coordinate system. These two procedures are correlative because the corresponding eigen-mode equations are different in their forms and degree of difficulty to solve. In Ref. [3], it discuss the coordinate system with rectified field lines in analysis of various small scale and large scale perturbations. Analysis in Ref. [3] has the merit of unification and being applicable to non-axisymmetric systems. However, just because the use of non-orthogonal coordinate with rectified field lines, the eigen-mode equation is very complicated with many metric quantities involved, lack of direct physical insight. In addition, the transformation between coordinates with the rectified field lines and that used in obtaining the equilibrium solution is very complicated as well.

In this paper, we introduce the unified tokamak ideal MHD eigen-mode equation by using the shear Alfven approximation[4]. Then we use a local toroidal coordinate system related to the equilibrium[5] and find its relation with rectified field lines. By using these system alternatively, we can obtain the eigen-mode equations both for the small scale and the large scale modes. Then we can analyze or numerically solve these equations for general tokamak ideal MHD stability. In this part of the series analysis, we give the eigen-mode equation for small scale modes (the ballooning mode and the local mode).

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1 The general form of tokamak ideal MHD eigen-mode equation for arbitrary cross-sections [4]

In Ref. [4], we use an orthogonal coordinate system related to the magnetic flux (ψ, χ, ζ), with $\psi(R, Z)$ being the poloidal magnetic flux [(R, ζ, Z) the cylindrical coordinates], the magnetic field is expressed as:

$$\boldsymbol{B} = \nabla\zeta \times \nabla\psi + RB_\zeta \nabla\zeta \tag{1}$$

Introduce:

$$X = RB_\chi \xi_\psi$$
$$U = \frac{\xi_\zeta}{R} - \frac{B_\zeta \xi_\chi}{RB_\chi} \tag{2}$$
$$Z = \xi_\chi / B_\chi$$

we can ignore the variable Z by analyzing the motion along the field lines. Then by using the shear Alfven approximation, i. e., for case of the plasma pressure being not too high, the perturbed magnetic vector potential is dominated by its parallel component $A_\parallel$, correspondingly, variables X, U can be expressed by one variable Φ

$$X = \frac{R}{JB_\zeta}\frac{\partial \Phi}{\partial \chi} \qquad U = \frac{\partial \Phi}{\partial \Psi} \tag{3}$$

where J is the Jacobian. A general normal mode equation is obtained in Eq. [4] as:

$$\nabla_\perp \cdot \left(\frac{nM\omega^2}{B_\zeta^2}\nabla_\perp \Phi\right) = \frac{1}{R}\boldsymbol{e}_\zeta \cdot \nabla \times \left(\frac{R}{B_\zeta}\nabla p_1\right) + \frac{1}{J}\frac{\partial}{\partial \Psi}\left(\frac{j_\parallel}{B}\right)\frac{\partial F}{\partial \chi} - \frac{1}{RB_\zeta}B \cdot \nabla(\Delta_\perp^* F) \tag{4}$$

with

$$F = \frac{R}{B_\zeta}\boldsymbol{B} \cdot \nabla \Phi \tag{5}$$

and p_1 is the perturbed pressure, $j_\parallel$ is the parallel current density, operators with subscript $\perp$ imply operations of gradient and divergence without $\frac{\partial}{\partial \zeta}$, and

$$\Delta_\perp^* = R^2 \nabla_\perp \cdot \left(\frac{\nabla_\perp}{R^2}\right) \tag{6}$$

The eigen-mode Eq. (4) is written in the operator form so that it can conveniently be used in different coordinate systems.

2 Equilibrium in local toroidal system [5]

Assume there is only one magnetic axis in tokamak equilibrium. Consider a local coordinate (r, θ, ζ), its relationship with the cylindrical coordinate with the Z-axis along the symmetric axis is

$$R = R_0 + r\cos\theta$$
$$Z = r\sin\theta \tag{7}$$

Furtherly we introduce a minor radius variable ρ, then the poloidal flux and other magnetic surface functions are all function only of varialbe ρ, every magnetic surface $\psi(R, Z) = \text{const}$ can be expressed as:

$$r = \rho + \sum_{n=1}[a_n(\rho)\cos n\theta + b_n(\rho)\sin n\theta] \tag{8}$$

The minor radius itself is the zero-th coefficient of this Fouries expansion:

$$\rho = \frac{1}{2\pi}\int_0^{2\pi} d\theta(\rho, \theta) \tag{9}$$

The sine part of this expansion will be zero for up-down symmetric system.

To determine the transformation of the (r, θ) coordinates and the (ρ, χ) ones, another restriction on χ could be made. In the treatment, we assume variable χ is a periodical function with period 2π.

3 Eigen-mode equation of small perturbations (ballooning mode and local mode)

For tokamak plasma, the mode varying slowly along the field line while the fast one in perpendicular direction is the most dangerous mode. Two kinds of such modes are generally studied in detail. One is the flute mode near the rational surfaces, another is the ballooning. We firstly derive the ballooning mode equation from the general Eq. (4).

3.1 High n ballooning mode

The general perturbation of such mode can be expressed as:

$$X(\rho, \chi, \zeta) = \bar{X}(\rho, \chi)\exp[-\mathrm{i}nS(\rho, \chi, \zeta)] \tag{10}$$

where $S(\rho, \chi, \zeta)$ is the eikonal, its concrete form is

$$S(\rho,\chi,\zeta) = \zeta - \int_{\chi_0}^{\chi} \mathrm{d}\chi' q_{\mathrm{L}}(\rho,\chi') \tag{11}$$

where q_{L} is the local safety factor. Here the high mode number means $n \gg R/a \gg 1$. $1/n$ can be used as the expansion parameter while the usual small quantity $\varepsilon = a/R$ plays no role in derivation. It is noted that any perturbed quantities when operated by $\partial/\partial\rho$, $\partial/\partial\chi$, $\partial/\partial\zeta$ will upgrade to higher order in n, except the operator $\boldsymbol{B}\cdot\nabla$ which does not induce the change of quantity order. Retain the lowest order in $1/n$, we find that contribution from the $\mathrm{d}j/\mathrm{d}\rho$ term (which is the driving source of the kink mode and the tearing mode) in Eq. (4) is a higher order quantity so that is negligible. The perturbed pressure term becomes

$$p_1 = -\frac{X}{\Psi'(\rho)}\frac{\mathrm{d}p}{\mathrm{d}\rho} = -\mathrm{i}n\frac{\mathrm{d}p/\mathrm{d}\rho}{\Psi'}\Phi \tag{12}$$

$$\boldsymbol{e}\cdot\nabla\times\left(\frac{R}{B_\zeta}\nabla p_1\right) = \frac{|\nabla\rho|}{\rho}\left[\frac{\partial}{\partial\rho}\left(\frac{R}{B_\zeta}\frac{\partial p_1}{\partial\chi}\right) - \frac{\partial}{\partial\chi}\left(\frac{R}{B_\zeta}\frac{\partial p_1}{\partial\rho}\right)\right]$$
$$= n^2\left\{\frac{|\nabla\rho|}{\rho}\frac{P'(\rho)}{\Psi}\left[\left(\int^{\chi}\mathrm{d}\chi'\frac{\partial q_{\mathrm{L}}}{\partial\chi'}\right)\frac{\partial}{\partial\chi}\left(\frac{R}{B_\zeta}\right) - q_{\mathrm{L}}\frac{\partial}{\partial\rho}\left(\frac{R}{B_\zeta}\right)\right]\Phi\right\} \tag{13}$$

This is the main driving source for the ballooning mode (the so-called unfavorable curvature effect). To study the second stability regime, the related part should be written in detail:

$$\frac{\partial}{\partial\rho}\left(\frac{R}{B_\zeta}\right) = \frac{1}{B_\zeta}\left[2\frac{\partial r}{\partial\rho}\cos\theta - \frac{\alpha}{2} + \frac{R(B_\theta^2)'}{B_\zeta^2} + \frac{2RB_\theta^2}{\rho B_\zeta^2}\right] \tag{14}$$

$$\frac{\partial}{\partial\chi}\left(\frac{R}{B_\zeta}\right) = \frac{2}{B_\zeta}\frac{\partial R}{\partial\chi} = \frac{2}{B_\zeta}\left(-r\sin\theta + \frac{\partial r}{\partial\theta}\cos\theta\right)\Big/\left(\frac{\partial\chi}{\partial\theta}\right) \tag{15}$$

where

$$\alpha = -8\pi R(\mathrm{d}p/\mathrm{d}\rho)/B_\zeta^2 \tag{16}$$

In Eq. (14), terms containing α are the main factor leading to the second stability regime.

The initial term becomes

$$\nabla_\perp\cdot\left(\frac{\omega^2}{v_A^2}\nabla_\perp\Phi\right) = n^2 - \left\{\frac{\omega^2}{v_A^2}\left[(\nabla\rho)^2\left(\int_{\chi_0}^{\chi}\mathrm{d}\chi'\frac{\partial q_{\mathrm{L}}}{\partial\rho}\right)^2 + \left(\frac{q_{\mathrm{L}}}{\rho}\right)^2\right]\right\}\Phi \tag{17}$$

In ballooning representation

$$F = \frac{1}{q_{\mathrm{L}}}\frac{\partial\bar{\Phi}}{\partial\chi}\exp(-\mathrm{i}nS) \tag{18}$$

we obtain:

$$\frac{1}{RB_\zeta} B \cdot \nabla(\Delta_1^* F) = -n^2 \left\{ \frac{1}{R^2 \rho^2} \frac{\partial}{\partial \chi} \left[\left(\frac{\rho \Delta \rho}{q_L}\right)^2 \left(\int d\chi' \frac{\partial q_L}{\partial \rho}\right)^2 + (q_1/q_0)^2 \right] \frac{\partial \bar{\Phi}}{\partial \chi} \exp(-inS) \right\} \quad (19)$$

Substituting Eqs. (12) ~ (19) into Eq. (4) results in the ballooning mode equation. This is a second order differential equation for the eigen-function $\bar{\Phi}(\rho, \chi)$ at a given magnetic surface $\rho = c$ within the region $-\infty < \chi < \infty$. This result completely coincides with Ref. [2] and other related papers. In ballooning representation the quantity $\nabla \cdot \boldsymbol{\xi}$ can be replaced by $\nabla_\perp \cdot \boldsymbol{\xi}_\perp$, then the perturbed pressure could produce another term proportional to the curvature, this more complicated situation will discussed elsewhere.

3.2 Local mode

Now we consider the local perturbation around a rational surface $q_0 = m/n$ [3]. In this case the basic perturbation quantity should be written in coordinates with rectified field lines, i. e., in (ρ, ω, ζ):

$$f(\rho, \omega, \zeta) = \exp i(m\omega - n\zeta)\{\bar{f}(\rho) + f_1(\rho, \omega)\} \quad (20)$$

For this kind of perturbation, we have

$$\boldsymbol{B} \cdot \nabla f = i \frac{B_\zeta}{R} \exp[i(m\omega - n\zeta)] \left\{(n - m/q)[\bar{f} + f_1] + \left[\frac{1}{q} \frac{\partial}{\partial \omega} f_1\right]\right\} \quad (21)$$

Near the rational surface $[q(\rho_0) = m/n]$, the safety factor q can be expanded, then express the perturbed quantities in the k-representation:

$$[(n - m/q)\bar{f}]_k = -\frac{q'}{q} \frac{\partial f_k}{\partial k} \quad (22)$$

Note that the scale length of the perturbed quantity is much smaller than the equilibrium one, so that in the basic Eq. (4), except the parallel operator in Eq. (21), the other perpendicular operators only act on the perturbed quantities and results in $\partial/\partial\rho = ik$. The metric factors of system (ρ, ω, ζ) can be expressed in system (ρ, θ, ζ). This transformation can be easily done because Eq. (4) is written in operator form. Then terms in Eq. (4) can be expressed in the following forms in the local representation:

$$F_k = \exp(im\omega - in\zeta) \left[\frac{nq'}{q} \frac{\partial}{\partial k} (\bar{\Phi}_k + \Phi_{1k}) + \frac{\partial \Phi_{1k}}{q \partial \omega} \right] \quad (23)$$

$$\left\{\frac{R}{B_\zeta} \boldsymbol{B} \cdot \nabla(\Delta_1^* F)\right\}_k^{(0)} = \left(\frac{nq'}{q}\right)^2 \left\{\frac{\partial}{\partial k} L^{(0)} \frac{\partial \bar{\Phi}_k}{\partial k} + \left\langle \frac{\partial}{\partial k} L^{(1)} \frac{\partial \bar{\Phi}_{1k}}{\partial k} \right\rangle\right\} \quad (24)$$

$$\left\{Re_\zeta \cdot \nabla \times \left(\frac{R}{B_\zeta} \nabla p_1\right)\right\}_k^{(0)} = \frac{mp_0'}{hq\Psi'} [K^{(0)} \bar{\Phi}_k + \langle K^{(1)} \Phi_{1k}\rangle] - \frac{ip_0'}{hq\Psi'} \langle K^{(1)} \partial \Phi_{1k}/\partial \omega\rangle \quad (25)$$

where $\langle \cdots \rangle$ means averaging over θ, and

$$L^{(0)} = \frac{1}{2\pi} \int_0^{2\pi} d\theta L$$

$$L = -[k^2 g^{11} + 2mkg^{12} + m^2 g^{22}] \quad (26)$$

$$L^{(1)} = L - L^{(0)} \quad (27)$$

$$K = \left[m \frac{\partial}{\partial \rho}\left(\frac{R}{B_\zeta}\right) - k \frac{\partial}{\partial \omega}\left(\frac{R}{B_\zeta}\right)\right] \quad (28)$$

$$h(\rho) = \frac{1}{2\pi} \int_0^{2\pi} d\theta (r\, \partial r/\partial \rho)/R \quad (29)$$

Quantities with superscripts (0) are their flute part or the equilibrium one depending on ρ only. Quantities with superscripts (1) are the ballooning part of the mode or that part of the equilibrium depending on θ. Term containing the current gradient in Eq. (4) appears in higher order, because of the condition $k\rho \gg R/\rho$, or $x/\rho \ll \rho/R$. However, for more large scale perturbations when the current gradient can play its role, the analysis is still applicable, of cause, the derivation is much complicated. At present, this case is not considered.

To determine the contribution from the ballooning part in above formulae, It is needed to solve the ballooning solution Φ_{1k} from the eigen-mode equation and in general, it is rather complicated. For tokamak, further simplification is possible. We note that the uniform part of the eigen value L is larger than the ballooning part by a factor of (R/ρ) or $[\rho/a_n(\rho)]$, then, we find that $\Phi_{1k}/\bar{\Phi}_k$ itself is a small quantity of order $(m/k\rho)$. Mean-while, the eigen value of K is in same order with the ballooning part or even smaller [see Eqs. (14), (15)], then we can determine approximately Φ_{1k} as:

$$\Phi_{1k} = \frac{m p_0' q}{h \Psi' L^{(0)}} K_p \bar{\Phi}_k \tag{30}$$

where

$$K_p = \iint d\theta_1 d\theta_2 K^{(1)} (\partial \omega / \partial \theta)^2 \tag{31}$$

the required eigen-mode equation for the local mode is

$$\frac{\partial}{\partial k} L^{(0)} \frac{\partial \bar{\Phi}_k}{\partial k} + \left[-U(k,n,\rho) + \frac{\omega^2}{\omega_A^2} L^{(0)} \left(\frac{q}{nq'} \right)^2 \right] \bar{\Phi}_k = 0 \tag{32}$$

where

$$U = \frac{q^2 p_0'}{nq'^2 h \Psi'} \left[K^{(0)} + \frac{q^2}{L^{(0)}} \langle K^{(1)} K_p \rangle - i \frac{q}{nL^{(0)}} \langle K^{(1)} \partial K_p / \partial \omega \rangle \right] \tag{33}$$

It can be seen that stability depends on the pressure driven force. Similar to the ballooning mode, stability of the local mode is related to the curvature of the field lines. When only terms proportional to k^2 in $L^{(0)}$ are retained, the eigen function would have a form of $\bar{\Phi}_k \propto k^{-1/2}$, then the necessary condition for stability becomes

$$U + 1/4 > 0 \tag{34}$$

This is the well-known Mercier criterion[3,6].

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