3.3 General Description of Tokamak Ideal MHD Instability Ⅱ①

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Ideal MHD stability analysis is a basic theoretical tool to study the stability of the magnetic confinement plasma. Based on this framework, it is possible to introduce other physical effects, to analyze or find some new instabilities. In this paper, the ideal MHD shear Alfven equation of motion[1] is used to study one of the important mode (i. e., the toroidal Alfven mode) . This is a peculiar mode in toroidal confinement system, it relates closely with the Alfven continua, but is quite different from the former in property. It exists in the gap of two continua which split each other due to toroidicity effect and has discrete spectrum and global structure. The early discovered one is the toroidal Alfven eigen-mode, called the TAE mode (Toroidicity-induced Alfven mode)[2,3], attention has attracted to this mode for long time due to the possible excitation of such mode by fusion alpha particles (now experimentally observed[4]), later, the non-circularity of the tokamak equilibrium, especially when its elongation effect surpasses the toroidicity, is found to be responsible for a new branch of mode with discrete spectrum. It exists in the gap between two Alfven continua that corresponding to two poloidal mode numbers with integer difference. The typical one is called the EAE mode (ellipticity-induced Alfven mode)[5]. Recently, experiments on toroidal eigen-modes and their effects on energetic particles have been an important area in tokamak physics.

1 General form of toroidal mode coupling

Now we use coordinate system with rectified magnetic field lines (ρ, ω, ζ) to extend the eigen mode equation. This equation used as the starting equation, though derived from an orthogonal coordinate system, is written in operator form so that it is applicable in non-orthogonal system directly.

This ideal MHD shear Alfven equation of motion has the form[1,6]:

$$\frac{R^2}{R_0^2}\nabla_\perp\cdot\left(\hat{\omega}^2\frac{R^2}{R_0^2}\nabla_\perp\Phi\right)=R\boldsymbol{e}_\zeta\cdot\nabla\times\left(\frac{R}{B_\zeta}\nabla p_1\right)+\frac{\partial}{\partial\rho}\left(\frac{j_\parallel}{B}\right)\frac{R^2}{J}\frac{\partial F}{\partial\chi}-\frac{R}{B_\zeta}\boldsymbol{B}\cdot\nabla(\Delta_\perp^* F) \tag{1}$$

where $\hat{\omega}=\omega/\omega_A$, $\omega_A=\frac{B_{\zeta0}}{\sqrt{4\pi\rho_m}R_0}$, the

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function F and Φ has the following relation:

$$F=\frac{R}{B_\zeta}\boldsymbol{B}\cdot\nabla\Phi \tag{2}$$

For Alfven waves, their real frequency $\omega\approx\omega_A\gg k_\parallel c_s$, the divergence part in perturbed pressure is a small quantity in ξ/R, can be neglected.

In some previous publications[2], derivation procedure of the eigen-mode equation is rather complicated, meanwhile the resultant TAE mode equation is too simple to retain all physical effects. In Ref. [3], this derivation is accomplished by considering the electromagnetic variables and current continuity, results in a similar equation as the equation (1), however, some physical effects not retained yet.

Now we introduce magnetic coordinate system. At first, we write the equilibrium magnetic surfaces in coordinates (ρ,θ,ζ), assume an up-down symmetry, we have

$$r(\rho,\theta)=\rho+\sum_{n=1}a_n(\rho)\cos n\theta \tag{3}$$

To extend Eq. (1) in coordinates with rectified field lines (ρ,ω,ζ), the perturbed quantity has the form:

$$X(\rho,\omega,\zeta)=\sum_m X_m(\rho)\exp[\mathrm{i}(m\omega-n\zeta)] \tag{4}$$

For Alfven eigen-mode with poloidal mode number m, only coupling between two neighboring modes is important which we denote as m, $m+1$ or m, $m+2$, the toroidal mode number n is kept fixed. Let

$$l_m=m/q-n \tag{5}$$

then

$$F_m=\mathrm{i}\,l_m\Phi_m \tag{6}$$

The Jacobian of coordinates (ρ,ω,ζ) can be written as

$$J=(\nabla\rho\times\nabla\omega\cdot\nabla\zeta)^{-1}=R^2\rho h(\rho)/R_0 \tag{7}$$

where $h=1+O(\varepsilon^2)$ and ε denotes small quantity of order a/R or a_n/ρ. We assume the particle density is a function of the minor radius so that the Alfven frequency is a function of minor radius as well. Expanding the initial term in Eq. (1), according to regulations of differential operation in non-orthogonal system and noting the definitions of perpendicular divergence and gradient, expressing corresponding metric factors into their Fourier series, to the order of ε, we have

$$\left[\frac{R^2}{R_0^2}\nabla_\perp\cdot\left(\hat{\omega}^2\frac{R^2}{R_0^2}\nabla_\perp\Phi\right)\right]_m=\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\,\hat{\omega}^2\frac{\mathrm{d}}{\mathrm{d}\rho}\Phi_m\right)-\frac{m^2}{\rho^2}\hat{\omega}^2\Phi_m+\mathrm{I}_{m+1} \tag{8}$$

$$\mathrm{I}_{m+1}=\left(\frac{2\rho}{R_0}+g_1^{11}\right)\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\,\hat{\omega}^2\frac{\mathrm{d}\Phi_{m+1}}{\mathrm{d}\rho}\right)+\hat{\omega}^2g^a\frac{\mathrm{d}\Phi_{m+1}}{\mathrm{d}\rho}+\hat{\omega}^2(m+1)\left[\frac{-2(m+1)}{R_0\rho}+\mathrm{i}g_1^{12}(\ln\hat{\omega}^2)'+\mathrm{i}g^b\right]\Phi_{m+1} \tag{9}$$

In Eq. (1), the main term in its right part is the third term

$$\left[-\frac{R}{B_\zeta}\boldsymbol{B}\cdot\nabla(\Delta_\perp^*F)\right]_m=S_m+S_{m+1} \tag{10}$$

$$S_m=\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\,l_m^2\frac{\mathrm{d}\Phi_m}{\mathrm{d}\rho}\right)+\left[-\frac{m^2}{\rho^2}l_m^2\Phi_m+l_m\left(\frac{l_m'}{\rho}+l_m''\right)\right]\Phi_m \tag{11}$$

$$S_{m+1}=\frac{g_1^{11}}{\rho}\frac{d}{d\rho}\left(\rho l_m l_{m+1}\frac{d\Phi_{m+1}}{d\rho}\right)+$$
$$[g_1^{11}(l_m l'_{M+1}-l_{m+1}l'_m)+g^a l_m l_{m+1}]$$
$$\frac{d\Phi_{m+1}}{d\rho}+[-m(m+1)g_1^{22}+$$
$$i(m+1)g^b l_m l_{m+1}]\Phi_{m+1} \quad (12)$$

where

$$g^a=g_{1\rho}^{11}+g_{1\omega}^{12}+i(2m+1)g_1^{12} \quad (13)$$
$$g^b=g_{1\omega}^{22}+g_{1\rho}^{12}+g_1^{12}/\rho \quad (14)$$

Another two terms can be written as:

$$\left[Re\cdot\nabla\times\left(\frac{R}{B_\zeta}p_1\right)\right]_m=\frac{R_0}{\rho}\left[\frac{-(m+1)^2p'_0}{q\Psi'}\right.$$
$$\frac{\partial}{\partial\rho}\left(\frac{R}{B_\zeta}\right)\Phi_{m+1}-i\frac{\partial}{\partial\omega}\left(\frac{R}{B_\zeta}\right)\frac{\partial}{\partial\rho}$$
$$\left.\left(\frac{(m+1)p'_0}{\Psi'q}\Phi_{m+1}\right)\right] \quad (15)$$

$$\left[\frac{\partial}{\partial\rho}\left(\frac{j_\parallel}{B}\right)\frac{R^2}{J}\frac{\partial F}{\partial\chi}\right]_m=\frac{-(m+1)^2l_{m+1}R_0}{\rho}$$
$$\frac{\partial}{\partial\rho}\left(\frac{j_\parallel}{B}\right)\Phi_{m+1} \quad (16)$$

To expand terms related with the curvature of the magnetic field lines in Eq. (15) we find that the main parts are those proportional to $\cos\theta$, $\sin\theta$, implying that the pressure gradient appears in the coupling terms with sidebands.

To the first order, the current gradient becomes:

$$\frac{\partial}{\partial\rho}\left(\frac{j_\parallel}{B}\right)=\frac{1}{B_{\zeta0}}j'_{\zeta0}+\left(\frac{\alpha q}{\rho}\right)'\cos\theta \quad (17)$$

Combining Eqs. (8) ~ (17), we obtain the set of equations describing the coupling of toroidal shear Alfven modes:

$$\hat{L}_m\Phi_m+\hat{M}_{m+1}\Phi_{m+1}=0 \quad (18)$$
$$\hat{L}_m\Phi_{m+1}+\hat{M}_m\Phi_m=0 \quad (19)$$

where

$$\hat{L}_m\Phi_m=\frac{1}{\rho}\frac{d}{d\rho}\left[\rho(l_m^2-\hat{\omega}^2)\frac{d}{d\rho}\Phi_m\right]+$$
$$\left[-\frac{m^2}{\rho^2}(l_m^2-\hat{\omega}^2)+l_m(l''_m+l'_m)-\right.$$
$$\left.\frac{ml_mR_0}{\rho B_{\zeta0}}j'_{\zeta0}\right]\Phi_m \quad (20)$$

with $l'_m=-mq'/q^2$. By using

$$\frac{mR_0j'}{\rho B_{\zeta0}}=\frac{2l'_m}{\rho}\left(l''_m+\frac{l'_m}{\rho}\right) \quad (21)$$

Eq. (20) can be rewritten as:

$$\hat{L}_m\Phi_m=\frac{1}{\rho}\frac{d}{d\rho}\left[\rho(l_m^2-\hat{\omega}^2)\frac{d}{d\rho}\Phi_m\right]-$$
$$\left[\frac{m^2}{\rho^2}(l_m^2-\hat{\omega}^2)+\frac{2l_ml'_m}{\rho}\right]\Phi_m \quad (22)$$

$$M_{m+1}\Phi_{m+1}=S_{m+1}-I_{m+1}+$$
$$\frac{(m+1)^2\alpha}{2R_0\rho}\Phi_{m+1}+\left(\frac{m+1}{2\rho}\right)\left(\frac{\alpha\Phi_{m+1}}{\rho}\right)$$
$$(23)$$

The form of $L_{m+1}\Phi_{m+1}$ is similar to Eq. (22) with subscripts replaced by $m+1$, the form of $M_m\Phi_m$ is similar to Eq. (23), but the last term is negative.

In above equations, all coupling effects of same order are retained so that it is more correct than that in published literatures. Of course, in practice, further simplification is possible.

2 TAE mode

We first consider the cylindrical case, rewrite the mode equation $L_m\Phi_m=0$ into an eigen equation about the displacement function, let

$$\Phi=\rho\xi \quad (24)$$

we obtain

$$\frac{d}{d\rho}\left[\rho^3(l_m^2-\hat{\omega}^2)\frac{d\xi_m}{d\rho}\right]-$$

$$(m^2-1)(l_m^2-\hat{\omega}^2)\xi_m-\rho(\hat{\omega}^2)'\xi_m=0 \quad (25)$$

This means that condition

$$\omega^2=\omega_A^2 l_m^2 \quad (26)$$

determines the Alfven continua. For uniform density, the spectrum varies with the minor radius and is determined by the safety factor profile. Two neighboring spectrums intersect each other near the point where $q=m/n+1/2$. Toroidal coupling makes split of these two spectrums near this point, the TAE mode exists at the gap.

Any further discussion needs more information about the equilibrium. Though the above results are generally for configuration with non-circular cross-section, for TAE mode, however, the main effects are from the toroidicity and the Shafranov shift of an equal circular configuration. All metric elements are related with these two effects only. Use the equation satisfied by the displacement function $a_1(\rho)$[8], denoting by the usual averaged poloidal ratio of pressure and the internal conductance, we obtain

$$a_1'=-\frac{\rho}{R_0}\left(\bar{\beta}_p+\frac{\bar{l}_i}{2}\right) \quad (27)$$

Substituting these results into Eqs. (22), (23), we obtain the general form of TAE mode as:

$$\hat{L}_m\Phi_m-\frac{a_1'}{\rho}\frac{d}{d\rho}\left(\rho l_m l_{m+1}\frac{d\Phi_{m+1}}{d\rho}\right)-\left(\frac{2\rho}{R_0}-a_1'\right)\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\hat{\omega}^2\frac{d\Phi_{m+1}}{d\rho}\right)\right]=0 \quad (28)$$

$$\hat{L}_{m+1}\Phi_{m+1}-\frac{a_1'}{\rho}\frac{d}{d\rho}\left(\rho l_m l_{m+1}\frac{d\Phi_m}{d\rho}\right)-\left(\frac{2\rho}{R_0}-a_1'\right)\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\hat{\omega}^2\frac{d\Phi_m}{d\rho}\right)\right]=0 \quad (29)$$

These equations consistent with that in Refs. [2, 3, 6, 7], with more effects retained.

3 Alfven eigen-mode induced by non-circularity[4]

Non-circularity can induce mode coupling. Ellipticity can induce coupling between the mode number m and $m+2$ while the triangularity can induce coupling between mode with m and $m+3$. Generally, couplings between mode with m and $m+1$ exist at the same time. However, these couplings happen at different mode frequency and different space points so that when we consider coupling between mode number m and $m+2$, we can put other coupling aside. If we only keep the second derivatives in coupling operators, as we have done for the TAE mode, a similar derivation will give the corresponding equation of mode coupling. Especially, for the EAE mode induced by the ellipticity, we have

$$\hat{L}_m\Phi_m-\frac{a_2'}{\rho}\frac{d}{d\rho}\left[\rho(l_m l_{m+2}-\hat{\omega}^2)\frac{d\Phi_{m+2}}{d\rho}\right]=0 \quad (30)$$

$$\hat{L}_{m+2}\Phi_{m+2}-\frac{a_2'}{\rho}\frac{d}{d\rho}\left[\rho(l_m l_{m+2}-\hat{\omega}^2)\frac{d\Phi_m}{d\rho}\right]=0 \quad (31)$$

The mode frequency is near

$$\omega_0=(m+1)\omega_A/n \quad (32)$$

The way to determine the eigen frequency and the eigen function in Eqs. (30), (31) is similar to that used in TAE analysis, and somewhat more simpler[5].

Similarly, we can discuss the Alfven eigen mode equation induced by the triangularity. Its form is similar to Eqs. (30), (31), it needs only some replacements of a'_2 by a'_3, Φ_{m+2} by Φ_{m+3}, the mode frequency is near

$$\omega_0 = \left[\left(m + \frac{3}{2}\right)/n\right]\omega_A \tag{33}$$

In fact, the ellipticity is larger than the triangularity for most practical geometry so that the EAE mode is more important.

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3.4 Ion Flow at the Presheath Entrance in Tokamak Scrape-off-layer

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In controlled nuclear fusion devices like tokamaks, plasma particles are confined by closed magnetic flux surfaces. Outside the last closed flux surface (LCFS), plasma is in direct contact with a solid surface in the scrape-off-layer (SOL). In the simplest picture, the particles are removed by transport along the magnetic field in the SOL to the solid surface. Such flow results from the pressure gradient which arises along $\boldsymbol{B}$ due to the fact that the solid surface is a sink for charged particles, which depresses the local pressure.

At the interface between the plasma and the solid surface, the quasineutral plasma is shielded from a negative absorbing wall by a thin positive space charge region (sheath) with a thickness of several Debye lengths. For a collisionless sheath, Bohm[1] has derived a criterion that the ions must enter the sheath region with a high directed velocity $v_z \geq c_s$, here c_s is the ion acoustic velocity. Conse-