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THE PRECISE ORDER OF UNITS OF BURNSIDE RINGS OF SOME BOUNDED FINITE ABELIAN GROUPS

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Abstract

We determine the precise order of $B(G)^*$, for $G = \oplus_i G_i$, a bounded abelian 2-group, where G_i is a direct sum of r copies of a cyclic group of order 2^n. The cases $r = 1$ and $r = k$, for some natural number k, are respectively considered in this paper.

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1 INTRODUCTION

For G a finite group, the Burnside ring $B(G)$ of G, as introduced by A. Dress [2] is the Grothendieck group of the category of finite G-sets with multiplication given by direct product.

Tammo tom Dieck in [1] constructed congruences between fixed point numbers to determine the order of units of Burnside rings of various finite groups while Matsuda introduced the structure matrix method to determine the order of units of Burnside rings for various finite groups with many normal subgroups.

Our principle aim is to use these two procedures to determine the precise value of the order of units of Burnside rings for $G := C_{2^n}$, and $G := \underbrace{C_{2^n} \oplus C_{2^n} \oplus ... \oplus C_{2^n}}_{r-times}$ respectively.

More precisely, using the congruence method, due to tom Dieck, we proved first the following result:

__Theorem 3.4:__

Let $G := C_{2^n}$ and $H_i \leq G$ with $1 := H_0 \leq H_1 \leq ... \leq H_n := G$. Let $\gamma(H_i) \in \{\pm 1\}$ for $i = 0, 1, ..., n-1$ then

$$\gamma(H_i) + \gamma(H_{i+1}) + 2\gamma(H_{i+2}) + ... + 2^{j-1}\gamma(H_{i+j}) + ... + 2^{n-i-2}\gamma(H_{n-1}) + 2^{n-i-1}\gamma(H_n) \cong 0(2^{n-i})$$

for all $i = 0, 1, ..., n-1$ if and only if

$$\gamma(H_0) = \gamma(H_1) = ... = \gamma(H_{n-1}) = \pm\gamma(H_n).$$

__Remark:__

Theorem 3.4 implies that

$|B(G)^*| = 2^2$

Finally, using Matsuda's approach, we proved the following:

__Claim 4.2__

Let $G := \underbrace{C_{2^n} \oplus C_{2^n} \oplus ... \oplus C_{2^n}}_{k-times}$, k a natural number greater than 1, then we have $|B(G)^*| = 2^{2^k}$

2 Preliminaries with Notations

In this paper we use the following notations:

1 the unit element of G

(H) the conjugacy class of a subgroup H of G

$\Phi(G)$ the set of conjugacy classes of all subgroups of G

For a G-set X and for each $x \in X$, the set

$G_x := \{g \in G | gx = x\}$ is the isotropy subgroup at a point x of a G-set X,

$X^G = \{x \in X | gx = x \ \ \forall g \in G\}$ is the set of fixed points of a G-set X

$|X|$ is the cardinal number of a set X,

$[X]$ is the element of $B(G)$ represented by a finite G-set X,

$1_{B(G)}$ is the unit element [point] of $B(G)$,
$N(F)$ is the normalizer of a subgroup F of G in G,
R^* is the unit group of a ring R,
$\mathbb{Z}$ is the ring of rational integers,
$\mathbb{Z}_2$ is the set $\{1, -1\}$,
$\mathbb{Z}'_2$ is the set $\{0, -2\}$.

The following is a summary, for the reader's convenience, of elementary facts about the Burnside ring of a finite group and its units which will be used in the sequel, most of which are standard materials taken directly from Matsuda[3] and are stated without proof;

Theorem 2.1:[3]

Let G be a finite group and $B(G)$ the Burnside ring of G. Then we have the following

(1) $B(G)$ is a commutative ring and a free $\mathbb{Z}$-module generated by the set $\{[G/F] | (F) \in \Phi(G)\}$.

(2) Let $\gamma_F : B(G) \to \mathbb{Z}$ be a map defined by

$$\gamma_F([G/H]) = |(G/H)^F|, \text{ where } (H), (F) \in \Phi(G).$$

Then γ_F is a ring homomorphism. Moreover,

$$\gamma = \prod\nolimits_{(F)\in\Phi(G)} \gamma_F : B(G) \to \mathbb{Z}^{|\Phi(G)|}$$

is an injective ring homomorphism.

(3) For each finite G-set X, $[X]$ has the following representation in $B(G)$.

$$[X] = \sum\nolimits_{(F)\in\Phi(G)} \lambda_F[G/F], \text{ where } \lambda_F = |\{x | x \in X \text{ and } (G_x) = (F)\}|/|G/F|$$

(4) For an element $\alpha \in B(G)$, the following three statements are equivalent

(i) $\alpha \in B(G)^*$

(ii) $\alpha^2 = 1_{B(G)}$

(iii) $\gamma(\alpha) \in \mathbb{Z}_2^{|\Phi(G)|}$

Theorem 2.2:[1]

The Burnside ring $B(G)$ can be viewed as a subring of $Map(\Phi(G), \mathbb{Z})$, where $\gamma \in Map(\Phi(G), \mathbb{Z})$ is contained in $B(G)$ if and only if

$$\sum_{(K)} |N(H)/N(H) \cap N(K)||(K/H)^*|\gamma((K)) \cong 0 \ mod|N(H)/H| \text{ for all } (H) \in \Phi(G),$$

where the sum is over $N(H)$-conjugate classes (K) such that H is normal in K and K/H is cyclic, and $(K/H)^*$ is the set of generators of K/H.

Definition 2.3:

A subset S of $\Phi(G)$ is called a basic subset if S satisfies the following two conditions:

(i) $G \in S$, $< 1 > \in S$ and , for $(H) \in S$, H is a normal subgroup of G.

(ii) If $(H), (F) \in S$, then $(H \cdot F), (H \cap F) \in S$, where $H \cdot F$ is a subgroup of G generated by H and F.

Now, for each $H \neq G$ in S, put

$$S(H) = \{(F) \in \Phi(G) | F \supset H, \text{ and } H = H' \text{ if } F \supset H' \supset H \text{ and } H' \in S\}$$

a non-empty set. Next, define a partial order on $\Phi(G)$ by setting $(K) \leqq (P)$ if K is conjugate in G to a subgroup of P.

Further define, with respect to this partial order, a bijection

$$t(S(H)) : S(H) \to \{1, \ldots, |S(H)|\}$$

satisfying

$$(K) \nleqq (P) \text{ if } t(S(H))((K)) < t(S(H))((P)).$$

Finally, we have the following theorem:

Theorem 2.4:[3]

Let S be a basic subset of $\Phi(G)$. Then we have

$|B(G)^*| = 2(\prod_{(H)\in S-\{G\}} |M^{-1}_{t(S(H))}(\mathbb{Z}'^{|S(H)|}_2) \cap \mathbb{Z}^{|S(H)|}|)$, where

$$M_{t(S(H))} = (a_{j,i}(t(S(H)))) = (\gamma_P([G/K])) \text{ is the } |S(H)| \times |S(H)|$$

structure matrix of $B(G)$ over $S(H)$ subordinate to $t(S(H))$ and where $t(S(H))((P)) = j$ and $t(S(H))((K)) = i$

Theorem 2.5:[3]

Let $\Phi(G)$ be the set of conjugate classes of all subgroups of G, then we have

$$|B(G)^*| = |M_t^{-1}(\mathbb{Z}'^{|\Phi(G)|}_2) \cap \mathbb{Z}^{|\Phi(G)|}|,$$

where M_t is the $|\Phi(G)| \times |\Phi(G)|$ structure matrix of $B(G)$ over $\Phi(G)$ subordinate to a bijection t defined on $\Phi(G)$.

Theorem 2.6:[3]

If G is a finite abelian group, then we have $|B(G)^*| = 2^{m+1}$, where

$m = |\{H | H \text{ is a subgroup of } G \text{ with } |G/H| = 2\}|$.

Theorem 2.7:[1]

If G is a finite group of odd order, then we have $|B(G)^*| = 2$.

3 Units of Burnside ring of Abelian 2-group of exponent n and rank 1

Lemma 3.1:

Let $[G:1]=2^n$ then we have for each unique subgroup H_j of G, $[G:H_j]=2^{n-j}$.

Proof:

To see this , let $i=2^n=[G:1]$. We can enumerate all divisors of i in an increasing sequence of numbers, say,

$$i_0 := 1, i_1 := 2, i_2 := 4, i_3 := 8, \ldots, i_n := 2^n.$$

Clearly, since G is cyclic, for each divisor i_j of i, there is a unique subgroup H_j of G such that $|H_j| = 2^j$, and hence $[G:H_j]=2^{n-j}$.$\square$

Lemma 3.2:

Let a denote a generator of G and put $a_j := a^{2^{n-j}}$ so that

$$H_0 :=< a_0 >, H_j :=< a_j >, j \neq 0, j = 1, 2, \ldots, n$$

with

$$1 :=< a_0 > \leq < a_1 > \leq \cdots \leq < a_n > = < a > = G.$$

Then we have the following list of distinct conjugate classes

$$Cl(G) = \{< a_0 >, < a_1 >, \cdots, < a_n >\}.$$

Proof:

This is trivial because for all j , $N_G(< a_j >) = G$.$\square$

Lemma 3.3:

Let A_i be set of generators of $H_i, i = 0, 1, 2, \ldots n$, then we have

$$|A_0| = 1, |A_1| = 1, \ldots, |A_{n-1}| = 2^{n-2} \text{ and } |A_n| = 2^{n-1}.$$

Proof:

Let g be an arbitrary element of G, then $g = a^k$ for all k. It also follows from above lemma that $< g >= H_j$ for some j, that is,$< a^k >=< a^{2^{n-j}} >$. So we can rewrite each member in $Cl(G)$ in terms of its set of generators in the following way:

$$
\begin{array}{lll}
A_0 & := & \{a^{2^n}\} \\
A_1 & := & \{a^{2^{n-1}}\} \\
\vdots & \vdots & \vdots \\
A_{n-1} & := & \{a^2, a^6, \ldots, a^{4n-6}, a^{4n-2}\} \\
A_n & := & \{a, a^3, \ldots, a^{2n-3}, a^{2n-1}\}
\end{array}
$$

and hence the result follows. □

Now, since $|N(H)/N(H) \cap N(K)| = 1$ in this case, applying theorem 2.2 we obtain the congruences

$$
\begin{array}{cccccccccc}
\gamma(H_0)+ & \gamma(H_1)+ & 2\gamma(H_2)+ & 4\gamma(H_3)+ & \cdots + & 2^{n-2}\gamma(H_{n-1})+ & 2^{n-1}\gamma(G) & \cong & 0(2^n) \\
 & \gamma(H_1)+ & \gamma(H_2)+ & 2\gamma(H_3)+ & \cdots + & 2^{n-3}\gamma(H_{n-1})+ & 2^{n-2}\gamma(G) & \cong & 0(2^{n-1}) \\
 & & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
 & & & & & \gamma(H_{n-1})+ & \gamma(G) & \cong & 0(2)
\end{array}
$$

Theorem 3.4:

Let $\gamma(H_i) \in \{\pm 1\}$ for $i = 0, \ldots, n-1$ then

$$\gamma(H_i) + \gamma(H_{i+1}) + 2\gamma(H_{i+2}) + \cdots + 2^{j-1}\gamma(H_{i+j}) + \cdots + 2^{n-i-2}\gamma(H_{n-1}) + 2^{n-i-1}\gamma(H_n) \cong 0(2^{n-i})$$

for all $i = 0, 1, \ldots, n-1$ if and only if

$$\gamma(H_0) = \gamma(H_1) = \cdots = \gamma(H_{n-1}) = \pm\gamma(H_n).$$

Proof:

To see " $\Longleftarrow$ " is easy, since

$$\gamma(H_i) + \gamma(H_{i+1}) + 2\gamma(H_{i+2}) + \cdots + 2^{n-i-2}\gamma(H_{n-1}) = 2^{n-i-1}\gamma(H_n)$$

and by assumption we must have that

$$\gamma(H_i) + \gamma(H_{i+1}) + 2\gamma(H_{i+2}) + \cdots + 2^{n-i-1}\gamma(H_n) \cong 0(2^{n-i}) \text{ for all } i.$$

To see " $\Longrightarrow$ " we use induction on $n - i$:

For $n - i = 0 \Longrightarrow i = n$ it is easy to see that $\gamma(H_0) = \gamma(H_n)$

Similarly for $i = n - 1$

Now assume that the induction hypothesis is true for $i < n - 1$, that is, $n - i > 1$, so that we have

$$\gamma_0 := \gamma(H_{i+1}) = \gamma(H_{i+2}) = \cdots = \gamma(H_{n-1}) = \pm\gamma(H_n)$$

Then we obtain by hypothesis

$$\gamma(H_i) + (2^{n-i-1} - 1)\gamma_0 \pm 2^{n-i-1}\gamma(H_n) \cong 0(2^{n-i}).$$

This implies,

$$\gamma(H_i) + 2^{n-i-1}(\gamma_0 \pm \gamma(H_n)) - \gamma_0 \cong 0(2^{n-i}).$$

But since $(\gamma_0 \pm \gamma(H_n))$ is either 0 or ± 2 we get that $2^{n-i-1}(\gamma_0 \pm \gamma(H_n)) \cong 0(2^{n-i})$ and $\gamma(H_i) - \gamma_0 \cong 0(2^{n-i})$, also since $n - i > 1, \gamma(H_i) = \{\pm 1\}, \gamma_0 = \{\pm 1\}$ we cannot get that $+1 \not\cong -1(4)$ for instance, so it follows that $\gamma(H_i) = \gamma_0$ and the proof is complete.□

Remark:

The above theorem 3.4 implies that $|B(G)^*| = 2^2$.

4 Units of Burnside ring of Abelian 2-group of exponent n and rank $r > 1$

Lemma 4.1:

Let $G := \underbrace{C_{2^n} \oplus C_{2^n} \oplus \cdots \oplus C_{2^n}}_{r-times}, n \geq 2$ and $H \leq G$. Then the number of G/H such that $|G/H| = 2$ is $2^r - 1$.

Proof:

Let $G := \underbrace{C_{2^n} \oplus C_{2^n} \oplus \cdots \oplus C_{2^n}}_{r-times}, n \geq 2$.

Clearly, in this case $|G| = 2^{nr}$ and for $H \leq G$ with $|G/H| = 2$, we have that $|H| = 2^{nr-1}$ and H can be described by a multiple of any two or more of the forms:

$$\begin{array}{ccccccccc} C_{2^n} & \oplus & C_{2^n} & \oplus & \cdots & \oplus & C_{2^n} & \oplus & 1 \\ & & & & \vdots & & & & \\ C_{2^n} & \oplus & 1 & \oplus & C_{2^n} & \oplus & \cdots & \oplus & C_{2^n} \\ 1 & \oplus & C_{2^n} & \oplus & \cdots & & & \oplus & C_{2^n} \end{array}$$

We obtain a matrix representation of each subgroup base as follows:

Starting with the leading first row of the first matrix as a^2, we allow that above the $a's$ on the diagonal, there can only be $1's$, while above the a^2 there can be only members of set $\{1, a\}$. That is, the first subgroup base is a $k \times k$-matrix

$$\begin{pmatrix} a^2 & 1 & 1 & \dots & 1 \\ 1 & a & 1 & \dots & 1 \\ 1 & 1 & a1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & a \end{pmatrix}$$

The second and third subgroup bases are $k \times k$-matrices:

$$\begin{pmatrix} a & 1 & 1 & \dots & 1 \\ 1 & a^2 & 1 & \dots & 1 \\ 1 & 1 & a1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & a \end{pmatrix} \text{ and } \begin{pmatrix} a & a & 1 & \dots & 1 \\ 1 & a^2 & 1 & \dots & 1 \\ 1 & 1 & a1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & a \end{pmatrix} \text{respectively.}$$

Continuing with this rule we stop at the $k \times k$-matrix

$$\begin{pmatrix} a & 1 & 1 & \dots & \epsilon \\ 1 & a & 1 & \dots & \epsilon \\ 1 & 1 & a1 & \dots & \epsilon \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & a^2 \end{pmatrix} \text{where}$$

$\epsilon \in \{1, a\}$.

a^2 being in row and column k, we have $k-1$ elements that can belong to $\{1, a\}$, which gives a total of 2^{k-1} combinations. Hence, the number of subgroups of order 2^{nr-1} is $\sum_{k=1}^{r} 2^{k-1} = 2^r - 1$.□

Claim 4.2:

Let $G := \underbrace{C_{2^n} \oplus C_{2^n} \oplus \cdots \oplus C_{2^n}}_{r-times}$ then we have $|B(G)^*| = 2^{2^r}$

Proof:

$\Phi(G)$ is basic and by definition $\Phi(G)(H) = \{(H)\}$ for each $(H) \in \Phi(G)$. If $(H) \in \Phi(G) - \{G\}$, then the stucture matrix of $B(G)$ over $\Phi(G)(H)$ is the 1×1-matrix $(|G/H|)$. But this 1×1-matrix satisfies theorem 2.4 only if $|G/H| = 2$ and in this case from Lemma 4.1 above we obtain the value m of theorem 2.6 and the result follows.□

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