Informes Técnicos Ciemat **1051**
Febrero, 2005

Plasma Dispersion Functions for Complex Frequencies

Institute of Plasma Physics.
NSC- KhIpt

Laboratorio Nacional de Fusión.
CIEMAT

S. S. Pavlov

F. Castejón

Asociación EURATOM / CIEMAT para Fusión - 93

Departamento de Fusión y Física de Partículas Elementales

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CLASIFICACIÓN DOE Y DESCRIPTORES

S70
WAVE PROPAGATION; TOKAMAK DEVICES; MAGNETIC FIELDS; STELLARATORS; PLASMA DIAGNOSTICS; STABILITY

Plasma dispersion functions for complex frequencies

S. S. Pavlov[1], F. Castejón[2]

1) Institute of Plasma Physics. NSC-KhIPT. Akademicheskaya 1. 61108,Kharkov, Ukraine.
2) Laboratorio Nacional de Fusión.
Asociación EURATOM-CIEMAT PARA Fusión. 28040, Madrid (Spain)

Abstract

Plasma dispersion functions for complex wave propagation frequency in the weak relativistic regime for arbitrary longitudinal refractive index are estimated and presented in this work. These functions, that are known as Shkarofsky functions in the case of real frequency, are estimated using a new method that avoids the singularities that appear in previous calculations shown in the preceding literature. These results can be used to obtain the properties of plasma instabilities in the weakly relativistic regime.

Funciones de dispersion del plasma para frecuencias compplejas

Resumen

En este trabajo se calculan y presentan las funciones de dispersión del plasma para la propagación de ondas con frecuencia compleja e índice de refracción longitudinal arbitrario, en el régimen débilmente relativista. Estas funciones, conocidas como funciones de Shkarovsky para el caso de frecuencia real, se estiman usando un nuevo método que evita las singularidades aparecidas en cálculos previos mostrados en publicaciones anteriores. Estos resultados son útiles para estudiar las propiedades de las inestabilidades en plasmas débilmente relativistas.

1. Introduction.

The description of dispersion and absorption of electron cyclotron (EC) waves in magneto-active plasmas requires to take into account relativistic effects [1] for moderated temperatures in the cuasi-perpendicular propagation with respect to the static magnetic field and for high temperatures in the oblique propagation regime. The dielectric tensor for homogeneous plasma with Maxwellian electron distribution was derived by Trubnikov [2] in two equivalent integral forms. However, the use of those forms for practical proposes presented some difficulties as a consequence of the singularity of the integrals that appear when describing dispersion properties of the plasma. For EC waves perpendicular to magnetic field these difficulties were overcome by Dnestrovskii, Kostomarov and Skrydlov. They calculated the weakly relativistic plasma dispersion functions (PDF) for those waves [3] in terms of non-relativistic PDF [4]. Those functions are known as Dnestrovskii functions. Later, Shkarofsky generalized these results for the case of quasi-perpendicular propagation [5]. He also introduced the weakly relativistic PDFs for any angle of propagation, which are known as Shkarofsky functions.

Further generalization for the case of arbitrary propagation was suggested by Airoldi and Orefice [6] on the basis of expanding of dielectric tensor in the parameters $\mu = (c/V_{T0})^2$ and $\lambda = (k_\perp \rho_e)^2$, where $V_{T0} = (T/m_0)^{1/2}$, T is electron temperature, m_0 is the rest mass of electron, $k_\perp$ is the perpendicular wave number, $\rho_e = V_{T0}/\Omega_c$ is the electron Larmor radius, and $\Omega_c = eB/m_0 c$ is the electron cyclotron frequency. The authors suggested also the method to calculate Shkarofsky functions as a convolution of the classical Maxwellian distribution with Dnestrovskii functions of integer order, which can be expressed in terms of exponential integral function [7]. Krivenski and Orefice suggested in the work [8] a useful method to calculate Shkarofsky functions introducing the recursive properties of PDFs and write the first two ones through the non-relativistic PDFs of 1/2 and 3/2 orders.

The use of these two last methods to estimate Shkarofsky functions in complex region is hindered by some difficulties. The method of Airoldi and Orefice presents the problem of performing analytical continuation in complex region of the convolution of classical Maxwellian distribution with Dnestrovskii functions of complex argument. In the method of Krivenski and Orefice the difficulties happens because the recursion of Shkarofsky functions does not converge for all values of the argument, even on real axis [9].

The works by Robinson [10-12] are devoted totally to review the properties of the weakly relativistic PDFs. In particular, some properties of Shkarofsky functions in complex region are discussed in the work [10] and their qualitative behavior is illustrated in a diagram of modulus-argument in terms of Dnestrovskii function with index $5/2$. However, the structure and symmetry properties of Shkarofsky functions in complex region are not studied in detail in that work.

In the case of non-Maxwellian electron distribution functions, the description of dispersion properties of plasma also leads to singular integrals. For such a case it is useful, from the point of view of numerical calculations, to obtain integral forms in which singularities either were absent or weakened. So, some integral forms in which singularities have a weak character are given in the work [12].

In the present work, a new method to calculate Shkarofsky functions is given on the basis of obtaining the integrals without singularities. This method provides a straightforward calculation of Shkarofsky functions of any half-integer index $q \geq 3/2$, valid for all the Riemann surface, and the study of this surface. This method may also be used for the numerical calculation of the singular integrals that appear for non-Maxwellian distribution functions and for integral transformations of Stieltjes type. The estimation of PDFs for complex frequency could allow one to treat with plasma instabilities by writing the dispersion relation of these instabilities.

2. Calculation of the weakly relativistic PDF in real region.

The weakly relativistic PDFs, or Shkarofsky functions of index q, are defined by the contour integral [5]

$$\mathcal{F}_q(z,a) = -i\int_0^\infty \frac{dt}{(1-it)^q}\exp\left[izt - \frac{at^2}{1-it}\right],$$

(1)

where $z = \mu(1 - n\Omega_c/\omega)$, $a = \mu N_{//}^2/2$, ω is the angular wave frequency, and $N_{//} = k_{//}c/\omega$ is the longitudinal refractive index. An alternative integral form of those functions was obtained by Robinson [11] and is given here:

$$\mathcal{F}_q(z,a) = a^{(1-q)/2}e^{-a}\int_0^\infty du\frac{u^{(q-1)/2}e^{-u}I_{q-1}\left(2\sqrt{au}\right)}{u+z-a},$$

(2)

where $I_{q-1}(x)$ is the modified Bessel function of index $q-1$. Those forms differ in the type of integral transformation. While expression (1) is related to the Laplace transformation, the form (2) is related to the transformation of Stieltjes [13]. We shall start from the form (2) and consider that the half-integer index q satisfies $q \geq 3/2$ from now on, since the dielectric tensor contains only Shkarofsky functions of such orders. For the moment, we shall consider that argument z takes real values. Then, if $z \leq a$, the integral (2) is divergent at the pole $u = a - z$ and must be understood as the Principal Part of integral, in the sense of Cauchy. The contour of integration in that case is chosen to pass above the pole to reproduce the well-known result for $\mathrm{Im}\mathcal{F}_q(z,a)$ [5].

The difficulties to calculate Shkarofsky functions are connected with the calculation of the Principal Part of integral, i.e. of $\mathrm{Re}\mathcal{F}_q(z,a)$. Let us consider this problem in more details to understand the peculiarities of this calculation. First of all, let us study the problem of calculating the principal part of the integral of Stieltjes type in the real regio

$$\varphi(b) = \int_0^\infty dt\frac{f(t)}{t-b},$$

(3)

being $f(t)$ a regular function that satisfies the conditions $f(t) = O\left(t^{1+\varepsilon}\right)$, when $t \to 0$ and $\varepsilon \geq 0$, and $\int_0^\infty f(t)dt < \infty$. If $b > 0$ the contour of integration passes above the pole as in expression (2). Then, according to Sohotskii-Plenelj's formula, one has:

$$\varphi(b) = -i\pi f(b) + \mathrm{P}\int_0^\infty dt \frac{f(t)}{t-b}.$$

As a consequence of the oddness of the function $1/(t-b)$ with respect to the point b, the principal part of the integral may be calculated using any of the infinite number of equivalent expressions

$$\mathrm{P}\int_0^\infty \frac{f(t)}{t-b}dt = \int_0^c \frac{f(t)}{t-b}dt + \int_c^b \frac{f(t) - f[(2b-c)-t]}{t-b}dt - \int_{-\infty}^c \frac{f[(2b-c)-t]}{t-b}dt,$$

for every value of c that satisfies $0 \leq c < b$. The main difference between these forms is the fact that the closer the point c to 0 the smaller the value of first and third integrands are. Moreover, those integrals do not present any singularity.

The second integral is also nonsingular at the point b since, when $t \to b$, the integrand goes to $2f'(b)$. Thus, the most suitable form for numerical calculations is that in which c takes the value 0, although all these forms are equivalent. In this case, first integral disappears, and the integrand of third integral is the most limited. We have the following nonsingular form:

$$\mathrm{P}\int_0^\infty \frac{f(t)}{t-b}dt = \int_0^b \frac{f(t) - f(2b-t)}{t-b}dt - \int_{-\infty}^0 \frac{f(2b-t)}{t-b}dt$$

(4)

It is easy to see that, due to the assumptions we have done for function $f(t)$ when $t \to 0$, both integrals in the right hand side of Eq. (4) have no singularities at the point b if $b \to 0$.

We can use now expression (4) for calculating Shkarofsky functions $\mathcal{F}_q(z,a)$. According to the original integral form (2), we shall have:

$$\mathcal{F}_q(z,a) = -i\pi f(a-z,a) + \int_0^{a-z} \frac{f(u,a) - f[2(a-z)-u,a]}{u+z-a} du - \int_{-\infty}^{0} \frac{f[2(a-z)-u,a]}{u+z-a} du \ ,$$

where

$$f(u,a) = \begin{cases} e^{-a-u}\, I_{q-1}\left[2(au)^{1/2}\right] (u/a)^{(q-1)/2} & ,u > 0 \\ 0 & ,u \le 0 \end{cases}$$

This integral form has no singularity at the pole and may be used for estimating Shkarofsky functions of any half-integer index $q \ge 3/2$, since the integrand corresponds to the conditions assumed for function $f(t)$ for these indexes. For this purpose, any standard program may be used to calculate the integrals with finite and infinite limits.

It is also easy to see that the expression (4) may be used for the numerical calculation of the similar singular integrals obtained for non-Maxwellian distribution functions and of any integral of Stieltjes type, provided that the integrand satisfies the conditions stated above.

3. Calculation of Shkarofsky functions in the complex plane.

The time evolution of distribution function for complex frequency can be useful for studying both instabilities and collisional dumping in the frame of initial value problem. The knowing of Shakarofsky functions of complex argument z is necessary to perform such estimates. These functions must be continued analytically in complex region, which presents some difficulties. In the two methods commented in the Introduction of this paper, the problem of calculating the singular integrals is not solved by removing the singularity but by the transformation into another one. That new singularity is described by more simple integrals or by differential equations that are already studied and the solutions tabulated.

So, in the method of Airoldi and Orefice, the calculation of Shkarofsky functions of half-
integer index is done in terms of the calculation of Dnestrovskii functions of integer index using the following integral convolution

$$\mathcal{F}_{q+\frac{1}{2}}(z,a) = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} dx \exp\left(-x^2\right) F_q\left(z + x^2 - 2a^{\frac{1}{2}}x\right).$$

(5)

The calculation of Dnestrovskii functions may be, in turn, obtained from the calculation of the exponential integral function $E_1(z)$ [9], which should be analytically continued into low semi-plane. And, although the function $E_1(z)$ is tabulated in the upper semi-plane and may be continued analytically into lower semi-plane, the usage of the convolution (5) for calculating Shkarofsky functions in complex region presents some difficulties since, on one hand, the variable of integration x is not single-significantly expressed trough complex argument z and, on the other hand, Dnestrovskii function $F_q(z)$ is not limited in the lower semi-plane.

In the method of Krivenski and Orefice, Shkarofsky functions should be calculated on the basis of the recurrent relation:

$$a\mathcal{F}_{q+2}(z,a) = 1 + (a - z)\mathcal{F}_q(z,a) - q\mathcal{F}_{q+1}(z,a),$$

(6)

Where the calculation of Shkarofsky function of any half-integer index is finally obtained in terms of the two functions $\mathcal{F}_{1/2}(z,a)$ and $\mathcal{F}_{3/2}(z,a)$, which are expressed trough the non-relativistic PDFs $\mathcal{F}_{1/2}(z,a) = -iZ^{+}/(z-a)^{1/2}$, $\mathcal{F}_{3/2}(z,a) = -Z^{-}/a^{1/2}$, where $Z^{\pm} = \frac{1}{2}\left\{Z\left[a^{1/2} + i(z-a)^{1/2}\right] \pm Z\left[-a^{1/2} + i(z-a)^{1/2}\right]\right\}$,

$Z^{\pm} = \frac{1}{2}\left\{Z\left[a^{1/2} + i(z-a)^{1/2}\right] \pm Z\left[-a^{1/2} + i(z-a)^{1/2}\right]\right\}$ and $Z(x)$ is the non-relativistic PDF. In this method, some difficulties appear already for real values of argument z, since the recursion relation (6) does not converge for every value of a and z, for given q [9].

Calculating Shkarofsky functions on the basis of an integral of Stieltjes type (2) and using the non-singular form for the principal part of integral (4), allows one to continue analytically those functions on the whole Riemann surface. It is also possible to explore the properties of that surface on the basis of the well-known theory of analytical continuation of Cauchy integral. We shall use the complex analysis theorem about analytical continuation of Cauchy integral given on any contour C. This theorem is a generalization of Landau prescription for continuation of Cauchy integral by means of deforming the contour of integration in the particular case when the contour is secluded and passing trough infinity on the case of any regular contour in complex plane. Following the theorem, if the integrand inside the Cauchy integral along any regular contour C

$$F(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta$$

(7)

satisfies the condition

$$\int_C \frac{|f(\zeta)|}{1+|\zeta|}|d\zeta| < \infty,$$

then the function $F(z)$ is given by $F^{+}(z)$, on left side of the contour C and by $F^{-}(z)$, on the opposite side of the contour C and the relation $F^{+}(z) = F(z)^{-} + f(z)$ gives the analytical continuation of function $F(z)$ through contour C [14].

The contour to calculate the integral form (2) goes along real axis from 0 to ∞ and, following the Landau prescription, must be chosen to pass above the pole, if $z < a$. Let us consider that the integral form (2) defines the original analytical element on the real axis for

$z < a$, which should be continued on the whole Riemann surface, and let us find out the structure of this surface. Since the integrand contains a function that depends on the double-valued square root, then it is necessary to define the sign of $\sqrt{a}$ and $\sqrt{u}$. For a given half-integer index $q \geq 3/2$ it is easy to show that the function $f(u,a) = e^{-a-u}(u/a)^{q-1/2} I_{q-1}\left[2(au)^{1/2}\right]$ inside the sign of integral is even in $\sqrt{a}$ and odd in $\sqrt{u}$. Consequently, the sign before $\sqrt{a}$ does not have any influence on the result of integration and it is natural to choose "+" for the sign of $\sqrt{u}$, since it is hold that $u \geq 0$ during the integration. Obviously, we shall have the original analytical element choosing the opposite sign and, consequently, the whole analytical continuation for opposite sign.

Following the mentioned theorem, if $\mathrm{Re}\, z > a$, Shkarofsky functions are analytically continued into the whole semi-plane by means of the analytical continuation of the function under integral (2)

$$\mathcal{F}_q(z,a) = \int_0^\infty du \frac{f(u,a)(u + \mathrm{Re}\, z - a)}{(a - \mathrm{Re}\, z - u)^2 + (\mathrm{Im} z)^2} - i \int_0^\infty du \frac{f(u,a)\mathrm{Im} z}{(a - \mathrm{Re}\, z - u)^2 + (\mathrm{Im} z)^2}.$$

(8)

In fact, since $\mathrm{Im} z \Big/ \left[(a - \mathrm{Re}\, z - u)^2 + (\mathrm{Im} z)^2\right] \to \pi\delta(a - \mathrm{Re}\, z - u)$ when $\mathrm{Im} z \to 0$, the second integral in (8) converges to $\pi f(a - \mathrm{Re}\, z, a)$. But $f(u,a) = 0$ for $u \leq 0$ and, consequently, $\mathcal{F}_q^+(z,a) = \mathcal{F}_q^-(z,a)$ when $\mathrm{Im}\, z = 0$, i.e. the function $\mathcal{F}_q(z,a)$ is continuous when crossing real axis. Accordingly to the properties of Cauchy integral, $F^{(n)}(z) = \frac{1}{2\pi i} \int_C \frac{f^{(n)}(\zeta)}{\zeta - z} d\zeta$, all derivatives of $\mathcal{F}_q(z,a)$ will be continuous on real axis, too.

If $\mathrm{Re}\, z < a$, Shkarofsky functions can be analytically continued above the real axis with the same formula (8). In this way we get the function $\mathcal{F}_q^+(z,a)$. Then the function $\mathcal{F}_q^-(z,a)$, which is the analytical continuation of function $\mathcal{F}_q(z,a)$ below real axis, will be given by the formula $\mathcal{F}_q^-(z,a) = \mathcal{F}_q^+(z,a) + 2\pi i f(a - z, a)$, where $f(u,a) = e^{-a-u}(u/a)^{(q-1)/2} I_{q-1}\left[2(au)^{1/2}\right]$ and taking into account the direction of integration. The functions under integrals in (8) happen to be continuous on the line $\mathrm{Re}\, z = a$ $(\mathrm{Im} z > 0)$, consequently the function $\mathcal{F}_q(z,a)$ is continuous on this ray above the real axis. Taking into account that the functions

$f^{(n)}(u,a)(\operatorname{Re} z - a + u) \Big/ \left[(a - \operatorname{Re} z - u)^2 + (\operatorname{Im} z)^2\right]$ and $f^{(n)}(u,a)\operatorname{Im} z \Big/ \left[(a - \operatorname{Re} z - u)^2 + (\operatorname{Im} z)^2\right]$,

(9)

are continuous on the same line, one may conclude that $\mathcal{F}_q^{(n)}(z,a)$, all the derivatives of $\mathcal{F}_q(z,a)$, are continuous , accordingly to the properties of Cauchy integral. Thus, the function $\mathcal{F}_q(z,a)$ in the semi-plane $\operatorname{Re} z > a$ is defined by the analytical continuation of $\mathcal{F}_q(z,a)$ of arguments belonging to the semi-plane $\operatorname{Re} z < a$ through the ray $\operatorname{Re} z = a$ $(\operatorname{Im} z > 0)$.

Thus, the original analytical function, given on real axis, may be analytically continued on the whole complex plane with cutting along the line $\operatorname{Re} z = a$ $(\operatorname{Im} z < 0)$. We will call this continuation first branch of the function $\mathcal{F}_q(z,a)$. The second branch of $\mathcal{F}_q(z,a)$ is presented below.

The function $\mathcal{F}_q^{+}(z,a)$, defined in the first quadrant of complex plane $(\operatorname{Re} z > a, \operatorname{Im} z > 0)$, is analytically continued to the second quadrant $(\operatorname{Re} z < a, \operatorname{Im} z > 0)$ through the ray $\operatorname{Re} z = a\,(\operatorname{Im} z > 0)$.

Similarly, the function $\mathcal{F}_q^{-}(z,a)$, defined in the third quadrant $(\operatorname{Re} z < a, \operatorname{Im} z < 0)$ will be analytically continued to the fourth quadrant $(\operatorname{Re} z > a, \operatorname{Im} z < 0)$ through the ray $\operatorname{Re} z = a$ $(\operatorname{Im} z < 0)$.

If we perform analytical continuation of that function further into 2nd quadrant using the same formula $\mathcal{F}_q^{\,-}(z,a) = \mathcal{F}_q^{+}(z,a) + 2\pi i\, f(a - z,a)$, i.e. $\mathcal{F}_q\,(z,a) = \mathcal{F}_q^{+}(z,a) + 2\pi i\, f(a - z,a)$ in this quadrant. Then the first branch from the 4$^{\text{th}}$ quadrant, which is $\mathcal{F}_q^{\,-}(z,a)$, will be analytically continued through the ray $\operatorname{Re} z = a$ $(\operatorname{Im} z < 0)$ on the whole semi-plane as far as the ray $\operatorname{Re} z = a$ $(\operatorname{Im} z > 0)$ and we obtain in this way half of the second branch, consisting of function $\mathcal{F}_q^{\,-}(z,a)$ in the 3rd quadrant and of function $\mathcal{F}_q^{+}(z,a) + 2\pi i\, f(a - z,a)$ in the 2$^{\text{nd}}$ quadrant.

The other half part of that branch is the function $\mathcal{F}_q^{+}(z,a) + 2\pi i\, f(a - z,a)$ in the 1st quadrant and the function $\mathcal{F}_q^{-}(z,a) - 2\pi i\, f(a - z,a)$ in the 4$^{\text{th}}$ quadrant because it follows that $\mathcal{F}_q^{+}(z,a) = \mathcal{F}_q^{\,-*}(z,a)$ for $\operatorname{Re} z > a$ from expression (8), and then those functions are continued analytically each other trough real axis. Moreover, since $f(u,a)$ is an analytical function in argument u, the whole half branch will analytically continue the former halve through the ray $\operatorname{Re} z = a$ $(\operatorname{Im} z > 0)$, and will transit analytically into first branch through the ray $\operatorname{Re} z = a$ $(\operatorname{Im} z < 0)$, which is the function $\mathcal{F}_q^-(z,a) - 2\pi i\, f(a-z,a)$ in the 3d quadrant.

So, the first branch is defined by $\mathcal{F}_q^+(z,a)$ in the 1st and 2nd quadrants, $\mathcal{F}_q^-(z,a) - 2\pi i\, f(a-z,a)$ in the 3rd and $\mathcal{F}_q^-(z,a)$ in the 4th ones. And the second branch is given by $\mathcal{F}_q^+(z,a) + 2\pi i\, f(a-z,a)$ in the 1st and 2nd quadrants, $\mathcal{F}_q^-(z,a)$ and $\mathcal{F}_q^-(z,a) - 2\pi i\, f(a-z,a)$ in the 3rd and 4th quadrants, respectively.

Thus, we have obtained a double-valued analytical function defined on the whole complex plane excepting the points $z = a$ and $z = \infty$, since it is known that Shkarofsky function $\mathcal{F}_q(z,a)$ and its derivatives till $(q-1/2)th$ are continuous at the point $z = a$, and the $(q-1/2)th$ derivative has a single pole at that point [9].

It holds that $f(a-z,a) = 0$ and $f^{(n)}(a-z,a) = 0$ when $a \to \infty$, for $\operatorname{Re} z = a$ and $\operatorname{Im} z < 0$, so these branches are separating in that case and the first branch tends to the non-relativistic PDF $Z(z)$, and the second branch tends to the function $Z^*(z)$, which is obtained if the contour is passing below the pole [5]. Consequently, accordingly to the prescription of Landau, the second branch should not be taken in consideration and the first branch must be considered as the principal branch.

Obviously, in the case of choosing the sign "-" before $\sqrt{u}$, in the original analytical element, one would get the analytical function defined on the same double connected domain only with opposite sign. Analogically, the principal branch of the two existing ones will be the branch that tends to the function $-Z(z)$ and, consequently, it makes sense for negative longitudinal refractive index. Consequently, the procedure of analytical continuation of the Cauchy integral (2) leads to two double-valued functions that differ only in the sign. The principal branches of those functions corresponding to the signs "+" and "-" before $\sqrt{u}$ are defined for positive and negative longitudinal refractive indexes, respectively. The non-principal branches have no physical meaning according to Landau prescription for choosing contour and should not be taken into account. Thus, it is necessary to choose the right sign of square root to get the proper analytical function in the original integral forms (1) and (2) of Shkarofsky functions.

From expression (8), it is possible to deduce also the following symmetry properties of the principal branch of function $\mathcal{F}_q(z,a)$ in complex plane:

$\mathcal{F}_q(z^*,a) = \mathcal{F}_q(z,a)^* + 2\pi i f(a-z,a)$, where $f(u,a) = e^{-a-u}(u/a)^{(q-1)/2} I_{q-1}\left[2(au)^{1/2}\right]$.if $\operatorname{Re} z < a$

$\mathcal{F}_q(z^*,a)=\mathcal{F}_q(z,a)^*$ if $\operatorname{Re} z > a$.

From these symmetry properties it follows that it is enough to define Shkarofsky function $\mathcal{F}_q(z,a)$ in three quadrants: in the two quadrants that satisfy $\operatorname{Re} z < a$ and in any of the quadrants in which it holds that $\operatorname{Re} z > a$.

To calculate the principal branch of function $\mathcal{F}_q(z,a)$ one may use the expression (8). The first integral in (8) is divergent on real axis if $\operatorname{Re} z < a$ and must be understood as the principal part in the sense of Cauchy. One may use the nonsingular form (4) to calculate it, since the function under the integral is odd with respect to the point $z = a$, at the real axis and near it. Obviously, outside the real axis, where the first integral in (8) converges, the principal value of this integral coincides with the integral itself. The second integral in (8) is also divergent on real axis if $\operatorname{Re} z < a$ but it can be rewritten in the form

$$-\frac{i}{\operatorname{Im} z}\int_0^{\infty} du \frac{f(u,a)\operatorname{Im}^2 z}{(u+\operatorname{Re} z-a)^2+\operatorname{Im} z^2}.$$

In this case we have a singularity outside the integral and, hence, the integral may be calculated for any value near the real axis.

The real and imaginary parts of Shkarofsky function $\mathcal{F}_{5/2}(z,a)$ are presented in Figures 1 and 2 respectively, for perpendicular propagation ($a=0$), i.e., the Dnestrovski function $F_{5/2}(z)$ for complex frequencies, calculated using the integral forms (4) and (8). For comparison, $\mathcal{F}_{5/2}(z,a)$ is shown in Figure 3 and 4 for a moderate value of a($a=1.5$, corresponding to T=1keV and $N_{\|}$=0.1). For large values of the parameter a, $\mathcal{F}_{5/2}(z,a)$ is close to the non-relativistic PDF whose diagram is presented in [7].

4. Conclusions.

In this work, the calculation of weakly relativistic PDFs, i. e. Shakarofsky functions $\mathcal{F}_q(z,a)$, for complex argument is shown. The calculation starts presenting a new way to estimate those functions for real argument z and for half-integer index $q \geq 3/2$, on the basis of the nonsingular form of the main value of Stieltjes type integral. The former method allows one to continue analytically Shkarofsky functions (given for both signs of the square root) for any real value of the parameter a from the real axis into the whole Riemann surfaces and to study the structure of those surfaces.

It has been shown that the Riemann surfaces for every Shkarofsky function, defined by integrals (1) and (2), are two double-valued functions defined on the whole complex plane with pricked out point a and differ each other by the sign.

The analytical continuation of Shkarofsky function from real axis into complex region, and the prescription of Landau to pass the pole, allows one to extract two single-valued analytical principal branches from the two double-valued functions, defined on the complex plane with a cut along the ray $\mathrm{Re}\, z = a$ $(\mathrm{Im}\, z < 0)$. Thus, in the weakly relativistic case, the sign of the square root should be properly chosen in the original integrals (1) and (2), defining Shkarofsky functions except on the contour of integration. Therefore, sign "+" before $\sqrt{u}$ in form (2) corresponds to positive longitudinal refractive index and the sign "-" to negative one.

This recipe can be used to calculate singular integrals for non-Maxwellian distribution functions and other integral transformations of Stieltjes type.

Acknowledgements

This work has been partially supported by the Spanish Ministerio de Educación y Ciencia under Project ENE2004-06957.

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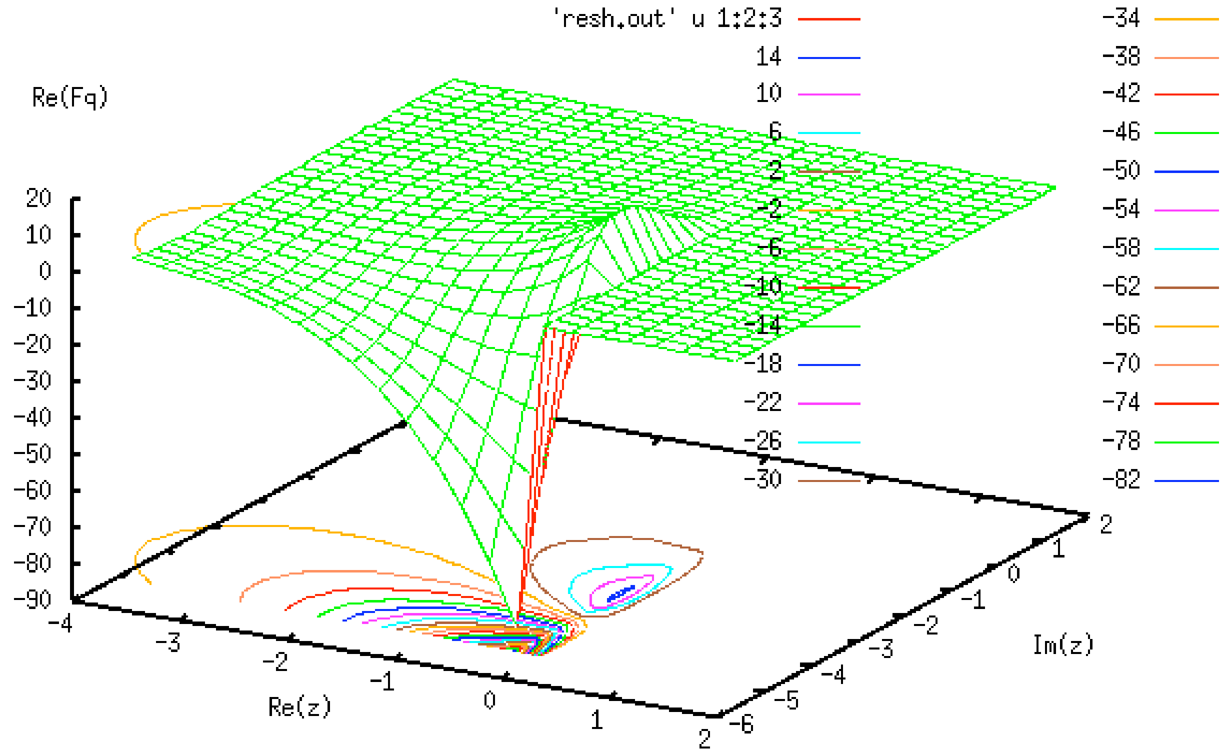

Figure 1: Real part of Shkarovsky function of index 5/2, $\mathcal{F}_{5/2}(z,a)$, for complex frequency and perpendicular propagation.

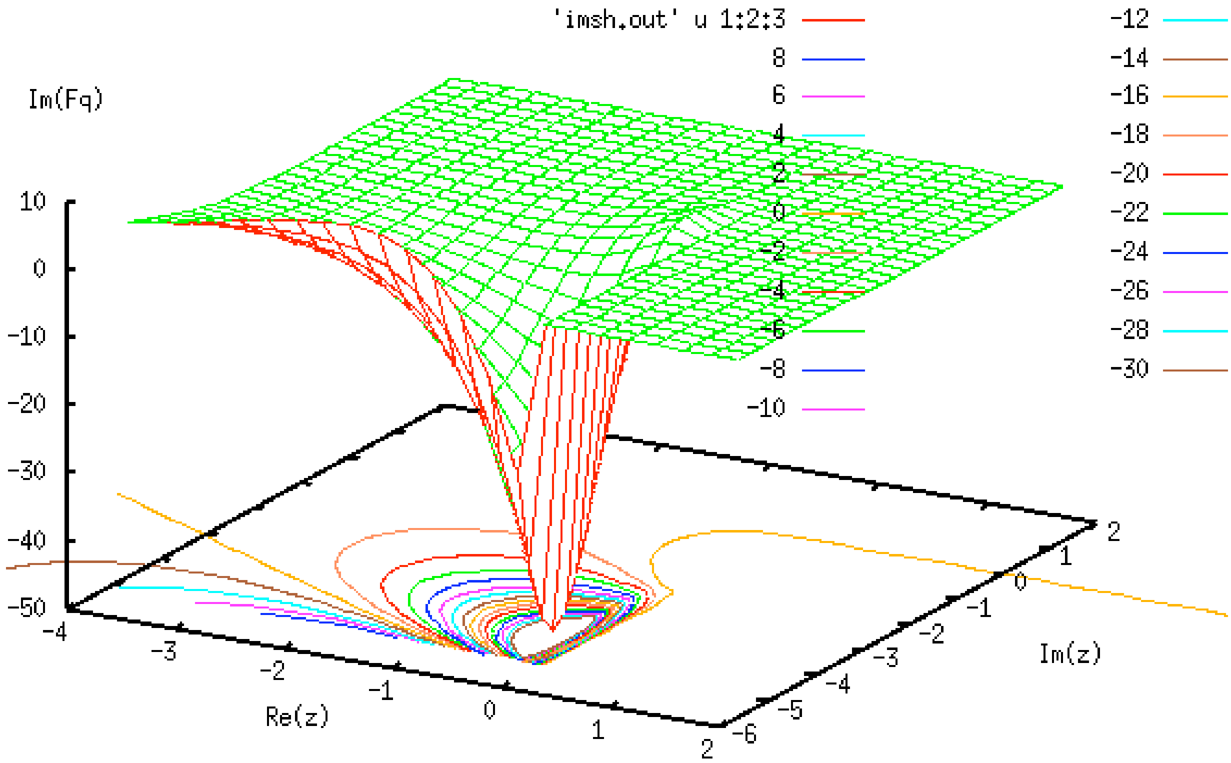

Figure 2: Imaginary part of $\mathcal{F}_{5/2}(z,a)$ for complex frequency and perpendicular propagation.

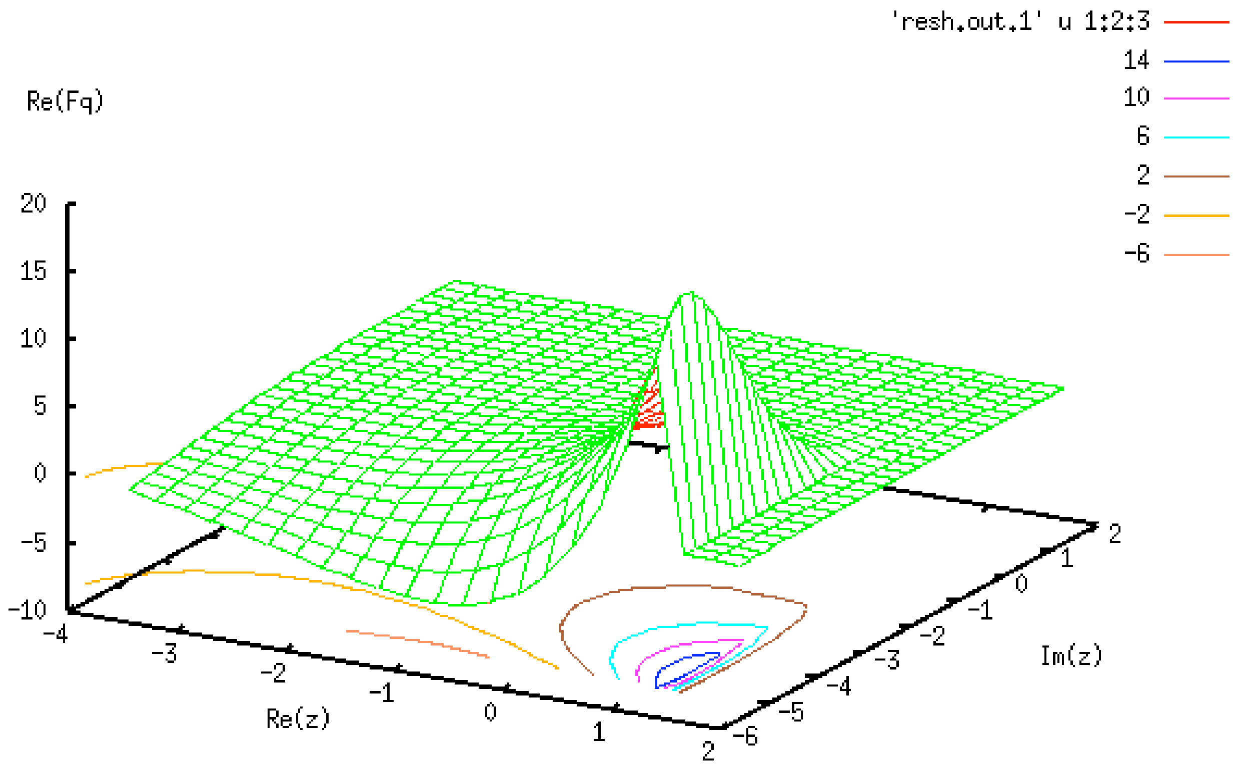

Figure 3: The same as in Figure 1, but for $N_{||}$=0.1 (Te=1 keV).

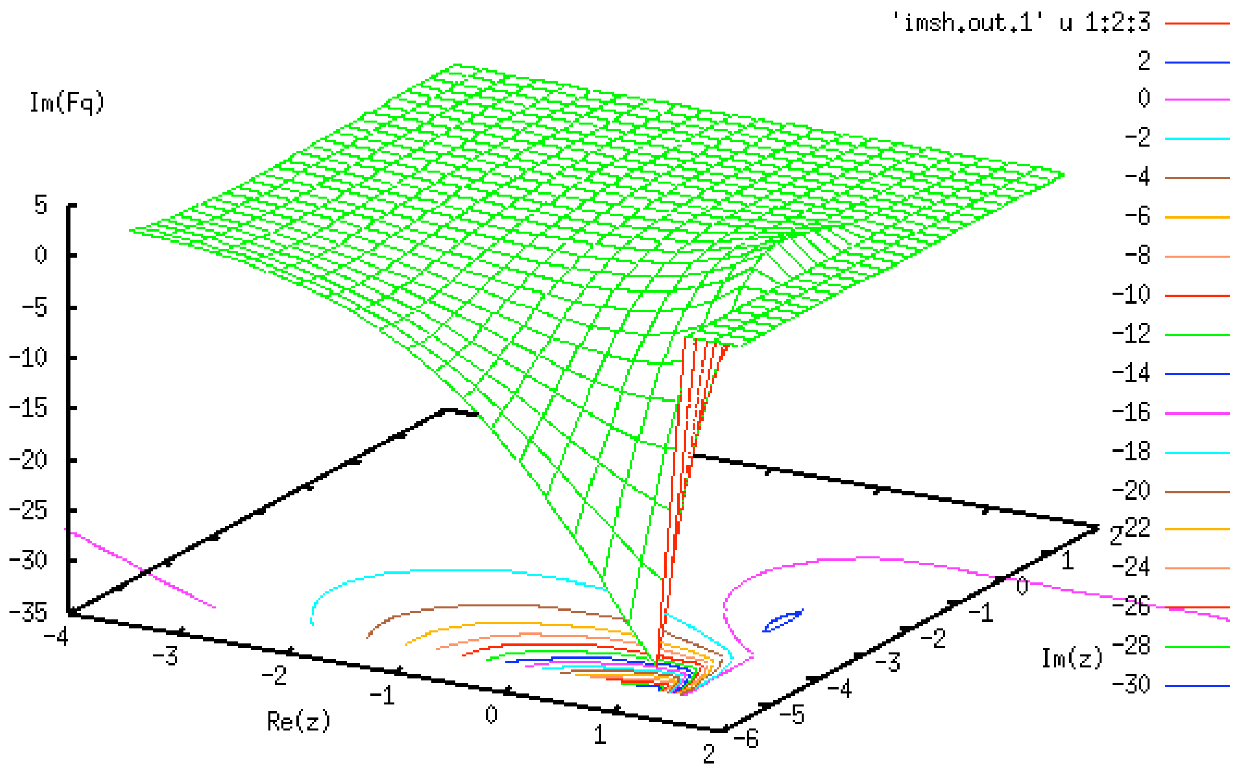

Figure 4: The same as in Figure 2, but for $N_{||}$=0.1 (Te=1 keV).