

United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

**THEORY OF EIGENVALUES FOR PERIODIC
NONSTATIONARY MARKOV PROCESSES:
THE KOLMOGOROW OPERATOR AND ITS APPLICATIONS**

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July 2005

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Abstract

We present an eigenvalue theory to study the stochastic dynamics of nonstationary time-periodic Markov processes. The analysis is carried out by solving an integral operator of the Fredholm type, i.e. considering complex-valued functions fulfilling the Kolmogorow compatibility condition. We show that the asymptotic behavior of the stochastic process is characterized by the smaller time-scale associated to the spectrum of the Kolmogorow operator. The presence of time-periodic elements in the evolution equation for the semigroup is considered to apply a Floquet analysis, then the first non-trivial Kolmogorow's eigenvalue is interpreted from a physical point of view. This nontrivial characteristic time-scale strongly depends on the interplay between the stochastic behavior of the process and the time-periodic structure of the Fokker-Planck equation for continuous processes, or the periodically modulated Master Equation for discrete Markov processes. We present pedagogical examples in a finite dimensional vector space to calculate the Kolmogorow characteristic time-scale for discrete Markov processes.

I. GENERAL STATEMENTS

It is well known that from a given stochastic prescription^{1,2} (Stratonovich, Ito, etc.) any stochastic differential equation (SDE) with a delta-correlated Gaussian noise drives to a well defined Markov Process³. A continuous Markov Process is completely characterized by its Fokker-Planck Operator (FPO), which can immediately be written from the corresponding SDE. If some parameter of the SDE is time-dependent, the stochastic process will not be stationary. Particularly if such a dependence is time-periodic the stochastic process is called a Periodic Nonstationary Markov Process (PNMP)^{1,4}. Now we want to discuss a method for solving a Fokker-Planck dynamics with time-periodic drift or diffusion matrix. Let the Fokker-Planck equation be:

$$\begin{aligned} \partial_t P(q, t) &= \left[-\frac{\partial}{\partial q_\nu} K^\nu(q, t) + \frac{\epsilon}{2} \frac{\partial^2}{\partial q_\nu \partial q_\mu} Q^{\nu\mu}(q, t) \right] P(q, t) \\ &= \mathcal{L}_{FP}(q, \partial_q, t) P(q, t). \end{aligned} \tag{1.1}$$

Here q stands for the set of variables (q_1, \dots, q_n) and summation over the double appearing indices ν, μ is understood. The drift $K^\nu(q, t)$ and diffusion matrix $Q^{\nu\mu}(q, t)$ are supposed to be time-periodic with time-discrete translation invariance $t \rightarrow t + T$, i.e.

$$K^\nu(q, t + T) = K^\nu(q, t) \tag{1.2}$$

$$Q^{\nu\mu}(q, t + T) = Q^{\nu\mu}(q, t), \tag{1.3}$$

ϵ is the parameter which measures the noise strength. The propagator (conditional probability density) of the Fokker-Planck dynamics $P(q, t|q_0, t_0)$ is a solution of (1.1) with the initial condition

$$P(q, t_0|q_0, t_0) = \delta(q - q_0).$$

The propagator is nonnegative for any q and q_0 and satisfies normalization to one. If K^ν and $Q^{\nu\mu}$ are time-independent the Fokker-Planck dynamics can be mapped into an eigenvalue problem, then the propagator can be expanded into a bi-ortonormal set of eigenfunctions of the FPO²⁻⁵. The need of the adjoint eigenfunctions is due to the fact that in general the FPO is not Hermitian nor normal. In the restricted case of Detailed Balance the problem can be mapped into a self-adjoint negative semi-definite eigenvalue problem, which shows the existence of a complete set of eigenfunctions with negative (or zero) eigenvalues, but for general FPOs not even the existence of a complete set of eigenfunctions can be proved. We are going to show that for the PNMP the dynamics of the system can still be studied as an eigenvalue problem, but the kind of operator to be solved is an integral one. We will show that the task is reduced to the eigenvalue analysis of a Fredholm equation⁶ with a nonsymmetric kernel. In the following sections we will give some applications of the eigenvalue theory, i.e. we deduce some connections between eigenvalues, eigenfunctions and quantities which characterize the dynamics and mixing of the system, like correlation functions, the Lyapunov function, the spectrum and the generalized switching time between attractors.

II. THE KOLMOGOROW OPERATOR

Every solution $f(q, t)$ of the Fokker-Planck equation (1.1) satisfies the Kolmogorow compatibility condition

$$f(q, t) = \int P(q, t|q', t') f(q', t') dq', \quad (2.1)$$

for all $t' \leq t$.

Definition: *The Kolmogorow Operator is given by ($t_2 \geq t_1$):*

$$\mathcal{U}(t_2, t_1) : f(q) \rightarrow \int P(q, t_2|q', t_1) f(q') dq',$$

i.e. the evolution of every solution of the Fokker-Planck equation is obtained by the application of the Kolmogorow operator:

$$f(q, t_2) = \mathcal{U}(t_2, t_1) f(q, t_1).$$

This is once again the Kolmogorow compatibility condition.

Proposition: *The Kolmogorow operator satisfies the semigroup laws:*

$$\mathcal{U}(t_1, t_1) = id \quad (2.2)$$

$$\mathcal{U}(t_3, t_1) = \mathcal{U}(t_3, t_2) \mathcal{U}(t_2, t_1). \quad (2.3)$$

If the FPO is time-periodic (see 1.2-1.3), the Kolmogorow operator has the periodicity:

$$\mathcal{U}(t_2 + T, t_1 + T) = \mathcal{U}(t_2, t_1). \quad (2.4)$$

Property (2.2) follows from the initial condition for the propagator, property (2.3) from the Chapman-Kolmogorow equation, which is valid for every Markov process. From (1.1) to (1.3) it is easy to see that the propagator has the periodicity

$$P(q, t + T|q_0, t_0 + T) = P(q, t|q_0, t_0),$$

from which property (2.4) follows. Due to the fact that the propagator generally is not symmetric under the transformation $q \leftrightarrow q_0$, the Kolmogorow operator in general is not self-adjoint. Its adjoint is given by:

$$\mathcal{U}(t_2, t_1)^+ : \phi(q) \rightarrow \int \phi(q') P(q', t_2|q, t_1) dq'.$$

Proposition: *The adjoint Kolmogorow operator satisfies:*

$$\begin{aligned} \mathcal{U}(t_1, t_1)^+ &= id \\ \mathcal{U}(t_3, t_1)^+ &= \mathcal{U}(t_2, t_1)^+ \mathcal{U}(t_3, t_2)^+, \end{aligned} \quad (2.5)$$

and if the FPO is time-periodic (1.2-1.3):

$$\mathcal{U}(t_2 + T, t_1 + T)^+ = \mathcal{U}(t_2, t_1)^+.$$

These properties follow immediately from the corresponding properties of the Kolmogorow operator.

If $\phi(q, t)$ is a solution of the Fokker-Planck backwards equation

$$\begin{aligned} \partial_t \phi(q, t) &= \left[K^\nu(q, t) \frac{\partial}{\partial q_\nu} + \frac{\epsilon}{2} Q^{\nu\mu}(q, t) \frac{\partial^2}{\partial q_\nu \partial q_\mu} \right] \phi(q, t) \\ &= \mathcal{L}_{FP}(q, \partial_q, t)^+ \phi(q, t), \end{aligned} \quad (2.6)$$

then its evolution backwards in time is obtained by the application of the adjoint Kolmogorow operator:

$$\phi(q, t_1) = \mathcal{U}(t_2, t_1)^+ \phi(q, t_2). \quad (2.7)$$

We will call this equation the *adjoint Kolmogorow compatibility* condition.

III. EVOLUTION IN ONE PERIOD OF TIME

Now we consider the space of all complex-valued functions with finite norm satisfying the Kolmogorow compatibility condition. In particular, we are interested in the eigenvalue problem of $\mathcal{U}(t + T, t)$. Since the Kolmogorow operator in general is not self-adjoint, we are looking for a complete bi-ortonormal set of eigenfunctions of $\mathcal{U}(t + T, t)$ and its adjoint $\mathcal{U}(t + T, t)^+$:

$$\mathcal{U}(t + T, t) f_i(q, t) = k_i f_i(q, t) \quad (3.1)$$

$$\mathcal{U}(t + T, t)^+ \phi_i(q, t + T) = k_i \phi_i(q, t + T) \quad (3.2)$$

$$\{\phi_i, f_j\} = \int \phi_i(q, t + T) f_j(q, t) dq = \delta_{ij} \quad (3.3)$$

Using the definitions and properties of the previous section, the next Lemma follows immediately.

Lemma: *Let $f(q, t)$ satisfy the Kolmogorow compatibility condition (2.1) and $\phi(q, t)$ satisfy the adjoint Kolmogorow compatibility condition (2.7), then we have:*

(a) *If $f(q, t_0)$ is an eigenfunction of $\mathcal{U}(t_0 + T, t_0)$ with eigenvalue k then $f(q, t)$ is an eigenfunction of $\mathcal{U}(t + T, t)$ with the same eigenvalue k for all t .*

If $\phi(q, t_0 + T)$ is an eigenfunction of $\mathcal{U}(t_0 + T, t_0)^+$ with eigenvalue k then $\phi(q, t + T)$ is an eigenfunction of $\mathcal{U}(t + T, t)^+$ with the same eigenvalue k for all t .

(b) *The eigenfunctions $f_i(q, t)$ and $\phi_i(q, t)$ have the Floquet structure*

$$\begin{aligned} f_i(q, t) &= e^{-\lambda_i t} g_i(q, t) \\ \phi_i(q, t) &= e^{\lambda_i t} \gamma_i(q, t) \end{aligned} \quad (3.4)$$

where the functions $g_i(q, t)$ and $\gamma_i(q, t)$ are periodic in t

$$\begin{aligned} g_i(q, t + T) &= g_i(q, t) \\ \gamma_i(q, t + T) &= \gamma_i(q, t) \end{aligned} \tag{3.5}$$

and λ_i must be chosen in such a way that the eigenvalue k_i has the form

$$k_i = e^{-\lambda_i T}$$

(c) The integral $\int \phi(q, t + T)f(q, t)dq$ does not depend on t , i.e. the scalar product $\{\phi, f\}$ in (3.3) is well defined.

Proof: (a) Since $\mathcal{U}(t + T, t)$ is periodic in t it is enough to show the proof for $t_0 + T > t > t_0$:

$$\begin{aligned} \mathcal{U}(t + T, t)f(q, t) &= \mathcal{U}(t + T, t)\mathcal{U}(t, t_0)f(q, t_0) \\ &= \mathcal{U}(t + T, t_0)f(q, t_0) \\ &= \mathcal{U}(t + T, t_0 + T)\mathcal{U}(t_0 + T, t_0)f(q, t_0) \\ &= \mathcal{U}(t + T, t_0 + T)kf(q, t_0) \\ &= k\mathcal{U}(t, t_0)f(q, t_0) \\ &= kf(q, t) \end{aligned}$$

The proof for $\phi(q, t)$ can be stated analogously.

(b) Let $k_i = e^{-\lambda_i T}$ then $g_i(q, t)$ is periodic in t :

$$\begin{aligned} g_i(q, t + T) &= e^{\lambda_i(t+T)} f_i(q, t + T) \\ &= e^{\lambda_i(t+T)} \mathcal{U}(t + T, t) f_i(q, t) \\ &= e^{\lambda_i(t+T)} k_i f_i(q, t) \\ &= e^{\lambda_i t} f_i(q, t) \\ &= g_i(q, t) \end{aligned}$$

The proof for $\phi_i(q, t)$ is again completely analogue.

(c) Let $t_2 > t_1$:

$$\begin{aligned} \int \phi(q, t_1 + T)f(q, t_1)dq &= \int (\mathcal{U}(t_2 + T, t_1 + T)^+ \phi(q, t_2 + T)) f(q, t_1)dq \\ &= \int (\mathcal{U}(t_2, t_1)^+ \phi(q, t_2 + T)) f(q, t_1)dq \\ &= \int \phi(q, t_2) (\mathcal{U}(t_2, t_1) f(q, t_1)) dq \\ &= \int \phi(q, t_2) f(q, t_2) dq \end{aligned}$$

Up to now this Lemma was in principle only a conclusion from the time periodicity of our problem (i.e., Floquet theorem⁷). If we further take into account that our equations describe probability distributions of Markov processes, we can make the following conclusions:

(d) *There always exists an eigenvalue $k_0 = 1$ ($\lambda_0 = 0$) with a constant adjoint eigenfunction $\phi_0(q, t) = \gamma_0(q, t) = 1$.*

(e) *Eigenfunctions for other eigenvalues have zero integral:*

$$\int f_i(q, t) dq = \int g_i(q, t) dq = 0 \quad \text{for } k_i \neq 1$$

(f) *If the drift and diffusion matrix are not singular, the eigenvalue $k_0 = 1$ is not degenerate, and its eigenfunction is the Asymptotic Time-Periodic Distribution (ATPD) $f_0(q, t) = g_0(q, t) = P_{as}(q, t)$.*

(g) *All other eigenvalues have a modulus smaller than 1:*

$$|k_i| < 1 \quad \text{i.e. real part } \lambda_i > 0 \quad \text{for } i = 1, 2, \dots$$

Proof: (d) Since the propagator is a normalized probability density we have $\mathcal{U}(t+T, t)1 = \int P(q', t+T|q, t) dq' = 1$.

(e) Since the Fokker-Planck dynamics conserves the integral we have $\int f_i(q, t+T) dq = \int f_i(q, t) dq = e^{-\lambda_i t} \int g_i(q, t) dq$ but the periodicity of $g_i(q, t)$ gives $\int f_i(q, t+T) dq = e^{-\lambda_i(t+T)} \int g_i(q, t+T) dq = e^{\lambda_i(t+T)} \int g_i(q, t) dq$. Both are only possible if either $e^{\lambda_i T} (= k_i) = 1$ or $\int g_i(q, t) dq = 0$ and therefore $\int f_i(q, t) dq = 0$.

(f) Under these conditions⁸ the system approaches a unique ATPD $P_{as}(q, t)$ for $t \rightarrow \infty$. The eigenfunctions with eigenvalue 1 are precisely the time-periodic functions satisfying (3.1). But $P_{as}(q, t)$ is the only such function (besides scalar multiples).

(g) Since every solution of the Fokker-Planck dynamics approaches the ATPD $P_{as}(q, t)$ for $t \rightarrow \infty$ all other eigenfunctions must vanish for $t \rightarrow \infty$, so $|k_i|$ must be smaller than 1. This proof follows from the existence of the Lyapunov function for PNMP⁸, but part (g) of the Lemma can also be proved without using the uniqueness of the ATPD. Consider (3.2), thus from the definition of the adjoint Kolmogorow operator it follows that

$$k_i \phi_i(q, t+T) = \int \phi_i(q', t+T) P(q', t+T|q, t) dq'. \quad (3.6)$$

Now we use that the propagator is nonnegative for any q and q' and satisfies normalization to one, and denote q_m if q is such that $|\phi_i(q, t+T)| = \max$. Then from (3.6)

$$\begin{aligned} |k_i| |\phi_i(q_m, t+T)| &= \left| \int \phi_i(q', t+T) P(q', t+T|q_m, t) dq' \right| \\ &\leq \int |\phi_i(q', t+T)| P(q', t+T|q_m, t) dq' \\ &\leq \int |\phi_i(q_m, t+T)| P(q', t+T|q_m, t) dq' \\ &\leq |\phi_i(q_m, t+T)|, \end{aligned}$$

so therefore $|k_i| \leq 1$.

For the rest of the article we will order the eigenvalues with decreasing modulus $1 = k_0 > |k_1| > |k_2| > \dots$, i.e. λ_i with increasing real part.

Up to now nothing is said about the completeness of the eigenfunctions system. Actually this cannot be proved in general. But Lemma (a) and (b) show that from the existence of a complete set of eigenfunctions for some fixed time t_0 follows the existence for all times t . For further conclusions we assume that such a complete set of eigenfunctions exists, so that the functions $f_i(q, t)$ and $\phi_i(q, t)$ satisfy:

$$\begin{aligned} \{\phi_i(q, t+T), f_j(q, t)\} &= \delta_{ij} \\ \sum_{i=0}^{\infty} \phi_i(q', t+T) f_i(q, t) &= \delta(q' - q) \end{aligned} \quad (3.7)$$

Now we can expand every function $h(q, t)$ which satisfies the Kolmogorow compatibility condition (2.1) in a series of eigenfunctions:

$$h(q, t) = \sum_{i=0}^{\infty} A_i f_i(q, t) = \sum_{i=1}^{\infty} A_i e^{-\lambda_i t} g_i(q, t)$$

where the coefficients A_i can be obtained from:

$$A_i = \{\phi_i, h\} = \int \phi_i(q, t+T) h(q, t) dq$$

In particular the propagator can be written as

$$P(q, t|q_0, t_0) = \sum_{i=0}^{\infty} A_i(q_0, t_0) e^{-\lambda_i(t-t_0)} g_i(q, t) \quad (3.8)$$

where the coefficients $A_i(q_0, t_0)$ are periodic in t_0 . This can easily be seen from the periodicity of $P(q, t|q_0, t_0)$ and $g_i(q, t)$.

IV. PERIODIC DETAILED BALANCE

Proving the existence of a complete set of eigenfunctions of the Kolmogorow operator is still an open question. The problems arise because from one hand the differential representation of the Kolmogorow operator involves a time ordered exponential²⁻⁵, which is difficult to handle

$$\mathcal{U}(t_2, t_1) = \vec{T} \exp \left(\int_{t_1}^{t_2} \mathcal{L}_{FP}(q, \partial_q, \tau) d\tau \right),$$

on the other hand in its integral representation the kernel (i.e. the propagator) is in general not symmetric. For time-independent Markov processes a symmetrization of the FPO is possible under the condition of detailed balance with even variables under time inversion:

$$P(x, t|y, 0) P_{st}(y) = P(y, t|x, 0) P_{st}(x)$$

($P_{st}(x)$ is the stationary solution of such a Markov process). This fact drives to a self-adjoint Fokker-Planck and Kolmogorow operator, which guarantees the completeness of the set of eigenfunctions^{2,3,5}. Furthermore under this conditions the symmetrized FPO is negative semi-definite, so that its eigenvalues

are negative real numbers, i.e. the eigenvalues of the Kolmogorow operator are real numbers between 0 and 1. In our case we have no continuous time translation symmetry but a discrete symmetry under the translation $t \rightarrow t + T$. A symmetrization of the Kolmogorow operator is possible under a similar condition as the detailed balance which is compatible with our discrete time translation symmetry and which we will call periodic detailed balance.

Definition: *The periodic detailed balance (PDB) is held if*

$$P(x, t + T|y, t)P_{as}(y, t) = P(y, t + T|x, t)P_{as}(x, t) \quad \text{for all } x, y, t$$

Proposition: *If the periodic detailed balance is fulfilled, the Kolmogorow operator $\mathcal{U}(t + T, t)$ is self-adjoint under the scalar product*

$$\{\eta, \xi\} = \int \eta(x)\xi(x)/P_{as}(x, t)dx$$

Proof:

$$\begin{aligned} \{\eta, \mathcal{U}(t + T, t)\xi\} &= \int \int \eta(x)P(x, t + T|y, t)\xi(y)/P_{as}(x, t)dx dy \\ &= \int \int P(y, t + T|x, t)\eta(x)\xi(y)/P_{as}(y, t)dx dy \\ &= \{\mathcal{U}(t + T, t)\eta, \xi\} \end{aligned}$$

Corollary: *If PDB is fulfilled there exists a complete set of eigenfunctions of the Kolmogorow operator $\mathcal{U}(t + T, t)$.*

V. STRONG MIXING

Strong mixing was originally introduced as one of several conditions that a stochastic process must satisfy in order that the central limit theorem is applicable. This question does not arise here, since our process is in general non-Gaussian. We are more involved in the strong mixing condition as a form of asymptotic independence.

Definition: *Let $q(t)$ be a realization of a stochastic process. The correlation function is defined by*

$$\langle\langle q(t)q(t') \rangle\rangle = \langle q(t)q(t') \rangle - \langle q(t) \rangle \langle q(t') \rangle$$

where the simple bracket represents the ensemble average.

Particularly the correlation function is calculated for the long time limit, i.e. by using for the 1-time probability distribution the ATPD $P_{as}(q, t)$. Then the asymptotic 2-time second moment is given by:

$$\langle q(t)q(t') \rangle_{as} = \int \int qq'P(q', t'|q, t)P_{as}(q, t)dq dq'$$

where we have $t' \geq t$ without a loss of generality. We can consider the correlation function as a function of the variables t and $\tau = t' - t$. By using the representation of the propagator in the set of eigenfunctions of the Kolmogorow operator we get:

$$\langle\langle q(t + \tau)q(t) \rangle\rangle_{as} = \sum_{i=1}^{\infty} e^{-\lambda_i \tau} B_i(t, \tau) \quad (5.1)$$

where the functions

$$B_i(t, \tau) = \int \int q q' A_i(q, t) g_i(q', t + \tau) g_0(q, t) dq dq', \quad i = 1, 2, 3 \dots$$

are periodic functions in t and τ . Therefore the asymptotic correlation function is an oscillatory decreasing function of τ for every fixed time t , which goes to zero for $\tau \rightarrow \infty$. This proves the following

Corollary: *Every periodic nonstationary Markov process characterized by non-singular drift and diffusion matrix is strong mixing.*

A. Correlation function for large τ

If we arrange the eigenvalues of the Kolmogorow operator in the order $1 = k_0 > |k_1| > |k_2| > \dots$ and retain only the slowest decreasing summand in the expansion (5.1) of the asymptotic correlation function, we get:

$$\langle\langle q(t + \tau)q(t) \rangle\rangle_{as} \approx e^{-\lambda_1 \tau} B_1(t, \tau)$$

Then after each period of time in τ the asymptotic correlation function falls by a factor $k_1 = e^{-\lambda_1 T}$, i.e. the first eigenvalue of the Kolmogorow operator smaller than 1 characterizes the slope of the decay of $\langle\langle q(t + \tau)q(t) \rangle\rangle_{as}$ as a function of τ .

B. The Lyapunov function

Another interesting object, which also gives dynamical information of a stochastic process, is the Lyapunov function. Traditionally this function was introduced in order to analyze the decay of the initial preparation of the system.

Definition: *Let the system be prepared in one point q_0 at some certain time t_0 . The Lyapunov function is defined by:*

$$\mathcal{H}(t) = \int P(q, t | q_0, t_0) \ln \frac{P(q, t | q_0, t_0)}{P_{as}(q, t)} dq$$

Note that for every q_0 , $\mathcal{H}(t)$ is a non-negative decreasing function and that the approach of the system from the preparation point q_0 towards the ATPD is characterized by the decreasing of $\mathcal{H}(t)$ ⁸. If we again use the expansion of the propagator in eigenfunctions of the Kolmogorow operator and take only the slowest decreasing part we get the following

Corollary: *In the long-time regime the Lyapunov function $\mathcal{H}(t)$ is an oscillatory decreasing function, which falls by a factor k_1^2 in each period of time.*

Note that in the asymptotic regime the decay of the correlation function (as a function of τ) is slower than the decay of the Lyapunov function.

It is possible to interpret the time-scale λ_1^{-1} that appears in the eigenvalue k_1 as a generalization of the Kramers escape time for periodic nonstationary Markov processes. An important question for stationary processes, conveniently answered in terms of the FPO framework, is: what is the time required for the passage of a prepared initial state to the final stationary state. Historically this problem was first studied by Kramers¹¹, and many mathematicians¹² and physicists¹³ have developed different perturbation approaches to tackle this fundamental problem for bistable situations with clearly separated times scales. In the present section we were concerned with a similar question but for a nonstationary periodically modulated Markov process. But because the system may reach periodically a situation where the stabilities and instabilities may not be well pronounced, the analysis is even more complex than in the Kramers problem due to the disparate time-scales that the system may have⁹. Therefore the hope to find an analytical expression characterizing the passage times is even more unlikely, so in the next section we introduce discrete Markov processes in order to find some analytical answers to the characterization of λ_1^{-1} .

VI. APPLICATIONS TO FINITE DIMENSIONAL VECTOR SPACES

A continuous nonlinear stochastic Markovian process can rarely be solved analytically, and solving the integral equation related to the Kolmogorow eigenvalue problem is even more complex. If we can arrange the eigenvalues of the Kolmogorow operator $\mathcal{U}(t+T, t)$ in the form $1 = k_0 > |k_1| > |k_2| > \dots$, we have shown that the asymptotic behavior of the periodic nonstationary process is controlled by k_1 . The important point to remark here is that by finding the first nontrivial eigenvalue of Kolmogorow's operator, we get the time-scale λ_1^{-1} which controls the asymptotic stochastic relaxation of the system, see section V. In fact, in previous works⁹ we have studied numerically a noise induced order-disorder transition by employing the mentioned integral operator $\mathcal{U}(t+T, t)$. Here we will extend our presentation concerning the Floquet analysis of *nonstationary periodically modulated* processes to the case when the Markov process is discrete. In this situation the evolution equation of the process is controlled by a Master Equation (ME) rather than by a Fokker-Planck one, so in order to solve $\mathcal{U}(t+T, t)$ we only have to deal with a matrix eigenvalue problem.

The ME of an arbitrary discrete Markov process is characterized by a Master Hamiltonian matrix \mathbf{H} of finite dimension $N \times N$ (we exclude here the analysis of a system whose range is denumerably infinite). This matrix \mathbf{H} is real and in general non-symmetric, and it must fulfill the fundamental conditions:

$$\begin{aligned}\mathbf{H}_{jl} &\geq 0, \quad j \neq l \\ \mathbf{H}_{jj} &= - \sum_{l(\neq j)} \mathbf{H}_{lj}.\end{aligned}$$

Therefore for a *nonstationary* Markov process, any result will have to be based on the *two* properties: (1) $\mathbf{H}_{jl}(t) \geq 0$, $j \neq l$, (2) for each l we have $\sum_j \mathbf{H}_{jl}(t) = 0$. This means that in general the instantaneous matrix $\mathbf{H}(t)$ cannot be diagonalized. Consider now the situation when the process is nonstationary but *periodically modulated*. In this case we have the additional temporal symmetry: $\mathbf{H}_{jl}(t) = \mathbf{H}_{jl}(t+T)$, thus we can apply our Floquet analysis of section III. In this case the Kolmogorow integral problem reduces to an eigenvector analysis. This fact can be seen by noting that the Kolmogorow operator is

$$\mathcal{U}(t_2, t_1) = \vec{T} \exp \left(\int_{t_1}^{t_2} \mathbf{H}(\tau) d\tau \right) \equiv \mathbf{P}(t_2 | t_1), \quad (6.1)$$

where $\mathbf{P}(t_2 | t_1)$ is the matrix Green's function of the ME $\dot{\mathbf{P}} = \mathbf{H}(t) \cdot \mathbf{P}$. Note the similitude with the propagator introduced in section II when we deal with a continuous Markov process. From this comment it is trivial to see that the Kolmogorow eigenvalue problem, see (3.1-3.3), reduces to a simple eigenvector problem of dimension N . We remark that in order to fully characterize the temporal behavior of an arbitrary discrete N -level PNMP, the Kolmogorow operator technique turns out to be a suitable and fundamental approach to tackle that sort of nonstationary problems.

A. Master Equation toy model

A toy discrete stochastic process which, in the present context, can analytically be solved is the so called *dichotomic process*^{2-4,10} originally introduced by Kubo and Anderson. As we pointed out before, in opposition to continuous Markov processes, the dichotomic process has two discrete levels and the evolution equation that governs its propagator is the ME. Then the Kolmogorow operator approach can immediately be applied.

Here we will work out a nonstationary dichotomic process for the case when the ME has a discrete symmetry under the time translation $t \rightarrow t + T$. In the present case the Kolmogorow operator is a 2×2 matrix (6.1), and the eigenvalues problem (3.1-3.3) reads

$$\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \cdot \begin{pmatrix} f_i(1, t) \\ f_i(2, t) \end{pmatrix} = k_i \begin{pmatrix} f_i(1, t) \\ f_i(2, t) \end{pmatrix} \quad (6.2)$$

$$\begin{pmatrix} \phi_i(1, t+T) & \phi_i(2, t+T) \end{pmatrix} \cdot \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = k_i \begin{pmatrix} \phi_i(1, t+T) & \phi_i(2, t+T) \end{pmatrix} \quad (6.3)$$

$$\left\{ \vec{\phi}_i, \vec{f}_j \right\} \equiv \begin{pmatrix} \phi_i(1, t+T) & \phi_i(2, t+T) \end{pmatrix} \cdot \begin{pmatrix} f_j(1, t) \\ f_j(2, t) \end{pmatrix} = \delta_{ij}, \quad \forall i, j = 0, 1. \quad (6.4)$$

As we commented in the Lemma, the elements of the propagator are evaluated in one-period of time

$$P_{\alpha\beta} \equiv P_{\alpha\beta}(t+T | t), \quad \forall \alpha, \beta = 1, 2,$$

and they are periodic functions of t fulfilling the normalization conditions

$$P_{11} + P_{21} = 1, \quad P_{12} + P_{22} = 1.$$

As expected, the one-time ATP probability (vector) of the discrete system $P(\alpha, t), \forall \alpha, \beta = 1, 2$, is related with the propagator (matrix) in the long-time limit.

The eigenvalue problem (6.2-6.4) can immediately be solved, so we get for the eigenvalue $k_0 = 1$ the right and left eigenvectors

$$\vec{f}_0 \equiv f_0(1, t) \begin{pmatrix} 1 \\ \frac{1-P_{11}}{1-P_{22}} \end{pmatrix}, \quad \vec{\phi}_0 \equiv C \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

the function $f_0(1, t)$ and the constant C can be determined from normalization of the ATP probability and the scalar-product condition $\{\vec{\phi}_0, \vec{f}_0\} = 1$.

From the eigenvalue k_1 we get

$$\vec{f}_1 \equiv f_1(1, t) \begin{pmatrix} 1 \\ \frac{P_{11}-1}{P_{22}-k_1} \end{pmatrix}, \quad \vec{\phi}_1 \equiv \phi_1(1, t+T) \begin{pmatrix} 1 \\ \frac{P_{11}-k_1}{P_{11}-1} \end{pmatrix},$$

and from orthonormality, $\{\vec{\phi}_0, \vec{f}_1\} = \{\vec{\phi}_1, \vec{f}_0\} = 0$, we obtain

$$P_{11} + P_{22} = 1 + k_1, \tag{6.5}$$

which is nothing more than the trace of the Kolmogorow operator, see (6.1). From the scalar-product $\{\vec{\phi}_1, \vec{f}_1\} = 1$ we get the condition

$$f_1(1, t)\phi_1(1, t+T) = \left(1 + \frac{P_{11}-k_1}{P_{22}-k_1}\right)^{-1}.$$

Using the Floquet structure (see Lemma, part [b]) we know that $f_1(1, t) = e^{-\lambda_1 t} \Theta(t)$ and $\phi_1(1, t+T) = e^{+\lambda_1(t+T)} \Xi(t)$, where $\Theta(t)$ and $\Xi(t)$ are periodic functions of time. It is simple to see that they fulfill

$$\Theta(t)\Xi(t) = k_1 \left(\frac{P_{22}-k_1}{1-k_1}\right) = k_1 \left(\frac{P_{11}-1}{k_1-1}\right). \tag{6.6}$$

So we arrive to the set of vectors

$$\begin{aligned} \vec{f}_0 &= \frac{1}{1-k_1} \begin{pmatrix} 1-P_{22} \\ 1-P_{11} \end{pmatrix}; \quad \vec{\phi}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \vec{f}_1 &\equiv e^{-\lambda_1 t} \begin{pmatrix} \Theta(t) \\ -\Theta(t) \end{pmatrix}; \quad \vec{\phi}_1 = e^{+\lambda_1(t+T)} \begin{pmatrix} \Xi(t) \\ \frac{P_{11}-k_1}{P_{11}-1}\Xi(t) \end{pmatrix}. \end{aligned} \tag{6.7}$$

The calculation of the functions $\Theta(t)$ and $\Xi(t)$ can be done by imposing the condition that the following equality holds for any pair of times $\{t, t_0\}$ provided $t \geq t_0$.

$$\begin{aligned} \mathbf{P}(t | t_0) &= \sum_j \vec{f}_j(t) \cdot \vec{\phi}_j(t_0 + T)^\top \\ &= \frac{1}{1-k_1} \begin{pmatrix} 1-P_{22}(t+T | t) & 1-P_{22}(t+T | t) \\ 1-P_{11}(t+T | t) & 1-P_{11}(t+T | t) \end{pmatrix} + \\ &\quad \frac{e^{\lambda_1(t_0-t+T)} \Theta(t)\Xi(t_0)}{P_{11}(t_0+T | t_0)-1} \begin{pmatrix} P_{11}(t_0+T | t_0)-1 & P_{11}(t_0+T | t_0)-k_1 \\ -P_{11}(t_0+T | t_0)+1 & -P_{11}(t_0+T | t_0)+k_1 \end{pmatrix}. \end{aligned} \tag{6.8}$$

Note that (6.6) follows from (6.8) considering that $\mathbf{P}(t_0 | t_0) = \mathbf{I}$, i.e., the identity matrix. Defining $\Gamma(t, t_0) \equiv Tr[\mathbf{P}(t | t_0)] - 1$ we can write the equations

$$\begin{aligned}\Gamma(t, t_0)e^{-\lambda_1(t_0-t)} &= \frac{\Theta(t)}{\Theta(t_0)} \\ \frac{P_{11}(t+T|t) - 1}{k_1 - 1} \frac{k_1}{\Theta(t)} &= \Xi(t),\end{aligned}\tag{6.9}$$

from where both functions $\Theta(t)$ and $\Xi(t)$ can be calculated; see (6.14) and (6.16) for two particular cases.

In the present 2×2 model there are only two important time-scales, T and λ_1^{-1} , interestingly the only function that remains to be calculated is the trace of the propagator, which in fact depends on the temporal structure of $\mathbf{H}(t)$. On the other hand, the decay of the slowest decreasing (antisymmetric) eigenvector \vec{f}_1 is dominated by the time-scale λ_1^{-1} .

Note that in the continuous case the eigenfunction $f_1(q, t)$ has to be antisymmetric with only one zero, because such a function can only decay when there is a current across the origin⁹. This situation is analogous to the Kramers metastable problem for stationary Markov processes, mathematically the FPO is parabolic and its propagator can be written as an eigenfunction expansion, with the eigenvalues appearing in exponential decay factors associated with each eigenfunction. Thus it turns out that the lowest non-trivial eigenvalue of the FPO is just the inverse of the mean first passage time, in the large barrier limit³. But this simple interpretation for the Kramers problem, in general does not appear in periodic nonstationary Markov processes.

The two-point correlation function, (5.1), in the present case reads

$$\langle\langle q(t+\tau)q(t) \rangle\rangle_{as} = e^{-\lambda_1\tau} B_1(t, \tau),\tag{6.10}$$

where

$$B_1(t, \tau) = \frac{1}{k_1} \sum_{\alpha\beta} q_\alpha q_\beta g_1(q_\alpha, t+\tau) \gamma_1(q_\beta, t) g_0(q_\beta, t).$$

Here q_α is the arbitrary value that the dichotomic process can take, the functions $g_i(q_\alpha, t)$ and $\gamma_i(q_\beta, t)$ can be read, using Lemma part [b], from of the biorthogonal set (6.7), then we get

$$B_1(t, \tau) = \frac{\Theta(t+\tau)\Xi(t)}{k_1} \left(\frac{1-P_{22}}{1-k_1} \right) (q_1 - q_2)^2\tag{6.11}$$

In order to have an explicit expression for the nontrivial eigenvalue k_1 we need to specify the time-structure of the one-period of time propagator $\mathbf{P}(t+T|t)$. In general this matrix is obtained from the solution of the ME

$$\frac{d\mathbf{P}}{dt} = \mathbf{H} \cdot \mathbf{P},\tag{6.12}$$

where \mathbf{H} is given in terms of the transition probability rate $W_{\alpha\beta}$

$$\mathbf{H} = \begin{pmatrix} -W_{21} & W_{12} \\ W_{21} & -W_{12} \end{pmatrix}.\tag{6.13}$$

As a particular asymmetric model we assume here that the transition probability rate W_{12} is periodic in time with the structure $W_{12} = \mathcal{A} + \mathcal{B} \cos \omega t$, with $\mathcal{A} - \mathcal{B} \geq 0$; and that the reverse transition rate is constant $W_{21} = \mathcal{C} \geq 0$. Solving the system (6.12) it is possible to write

$$\mathbf{P}(t | t_0) \equiv \begin{pmatrix} P_{11}(t | t_0) & P_{12}(t | t_0) \\ P_{21}(t | t_0) & P_{22}(t | t_0) \end{pmatrix},$$

with

$$\begin{aligned} P_{11}(t | t_0) &= \Gamma(t, t_0) + \int_{t_0}^t (\mathcal{A} + \mathcal{B} \cos \omega t') \Gamma(t, t') dt' \\ P_{22}(t | t_0) &= \Gamma(t, t_0) + \int_{t_0}^t \mathcal{C} \Gamma(t, t') dt' \\ P_{21}(t | t_0) &= 1 - P_{11}(t | t_0) \\ P_{12}(t | t_0) &= 1 - P_{22}(t | t_0), \end{aligned}$$

where $\Gamma(t, t')$ is

$$\Gamma(t, t') = \exp \left[-(\mathcal{A} + \mathcal{C})(t - t') + \frac{\mathcal{B}}{\omega} (\sin \omega t - \sin \omega t') \right]. \quad (6.14)$$

This expression *completely* solves the Kolmogorow problem we have posed before. As a matter of fact, it can be seen that the following expression holds

$$1 + \Gamma(t, t') = P_{11}(t | t') + P_{22}(t | t').$$

Therefore, by taking in the former expression $t' = t - T$, and comparing with Eq. (6.5), the eigenvalue k_1 can be written in the form:

$$k_1 = P_{11}(t + T | t) + P_{22}(t + T | t) - 1 = \exp[-(\mathcal{A} + \mathcal{C})T]. \quad (6.15)$$

Note that for the present model, due to the symmetry of the periodic modulation in W_{12} , the Kolmogorow time-scale is independent of \mathcal{B} and it is just characterized by the value: $\lambda_1^{-1} = (\mathcal{A} + \mathcal{C})^{-1}$. Nevertheless, as we have already mentioned, in general the time-scale λ_1^{-1} strongly depends on the temporal structure of the matrix $\mathbf{H}(t)$. For example, if the time-dependent W_{12} transition rate were of the Arrhenius type, the eigenvalue k_1 would not be independent of the amplitude of the periodic modulation.

1. Periodically modulated Arrhenius model

Consider an asymmetric Arrhenius' type model, so we now assume that the transition probability rate W_{12} is periodic in time with the temporal structure $W_{12} = \mathcal{A} \exp(b \cos \omega t)$, and as before $W_{21} = \mathcal{C}$ is constant in time. Solving the system (6.12) for this model it is possible to write

$$\Gamma(t, t') = \exp \left[-\mathcal{C}(t - t') - \mathcal{A} \int_{t'}^t \exp(b \cos \omega s) ds \right]. \quad (6.16)$$

Therefore, Kolmogorow's time-scale $\lambda_1^{-1} = -T / \log |k_1|$ reads

$$\lambda_1^{-1} = [\mathcal{C} + \mathcal{A} I_0(b)]^{-1}, \quad (6.17)$$

where $I_0(b)$ is the Hyperbolic Bessel function¹⁶; in the case $b = 0$ there is no periodic modulation and we reobtain the static or Kramers-like time scale. In analogy with the splitting problem³, note that in the static case the rate λ_1 is just given in terms of the Kramers scape rates \mathcal{C} and \mathcal{A} .

The nonadiabatic formula (6.17) is therefore an useful starting point to study the relaxation of our stochastic model in the presence of an external periodic modulation. It is interesting to introduce here a perturbation in the amplitude of modulation b , in order to compare Kolmogorow's time-scale λ_1^{-1} versus the static time-scale $\tau_s = [\mathcal{C} + \mathcal{A}]^{-1}$. Using the expansion of the Bessel function, from (6.17) we get

$$\lambda_1^{-1} = \tau_s \left(1 - \mathcal{A} \tau_s \left(\frac{b^2}{4} + \dots \right) \right), \quad (6.18)$$

it is worthwhile pointing here that (6.17) is independent of T , showing therefore that (6.18) is not a perturbative expression in the period of the modulation. Thus the comparison of λ_1^{-1} against T will give information concerning the lost of correlation during the time-scale of the external periodic modulation.

We see from the propagator expansion (3.8), using (6.7) and (6.8) for a finite dimensional vector space, that by increasing the amplitude of modulation b the antisymmetric Floquet eigenvector $\vec{g}_1(t) = (\Theta(t), -\Theta(t))$ dies out faster to reach to the ATP probability (positive vector $\vec{g}_0(t)$). In a similar context, Kolmogorow's eigenvalue k_1 gets smaller for increasing b , thus predicting a faster strong mixing of the two-point correlation function (6.10).

We have proved in section V that in general the relaxation of the *correlation* and of the *Lyapunov* function are, in the asymptotic limit, controlled by the Kolmogorow time-scale $\sim \lambda_1^{-1}$. Therefore in order to study the asymptotic relaxation of any PNMP, the important task is to determine the first non-trivial eigenvalue k_1 . More complex objects can similarly be studied in terms of the biorthogonal set of eigenvectors of $\mathcal{U}(t+T, t)$; for example, this is the case of the stochastic resonance phenomenon.

2. Stochastic resonance in a 2×2 model

The stochastic resonance is a name *coined* in order to characterize the situation when the addition of noise into a periodically modulated nonlinear system leads to an amplification of the signal-to-noise spectrum, i.e., this is a cooperative result showing the interplay between deterministic and random dynamics in a time modulated system¹⁴. The signature of this phenomena can easily be seen from the dynamics of a ME, as it is in the present 2×2 model. Consider the exact expression for the correlation function (6.10). The power spectral density $S(\omega, t)$ is the Fourier transform of the correlation function with respect to the variable τ . Here the time t corresponds to the time at which one begins to take data, and since the phase of the signal with respect to this time is usually not known (in decoherent systems), one can assume that t itself is a random variable distributed uniformly over one period of the signal. Then the spectral density of interest is the average over the random phase variable:

$$\begin{aligned} S(\omega) &= \frac{1}{T} \int_0^T \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega\tau} \langle\langle q(t+\tau)q(t) \rangle\rangle_{as} d\tau \right] dt \\ &= \frac{1}{T} \int_0^T \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega\tau} e^{-\lambda_1\tau} B_1(t, \tau) d\tau \right] dt. \end{aligned}$$

As we mention before, from (6.11) it is simple to see that $B_1(t, \tau)$ is a periodic function of τ , in fact given in terms of $\Theta(t + \tau)$. This τ -dependence implies the destruction of the Lorentzian shape for the

spectra, a fact that ultimately leads to the stochastic resonance phenomenon shown in the signal-to-noise spectrum.

For a general time modulated Markov problem, in order to study the amplification of the signal-to-noise spectrum the important task—in the *asymptotic* analysis—is to consider just the associated key function $B_1(t, \tau)$ in the corresponding expansion (5.1).

B. Diagonalizable models

If the N -level Master Hamiltonian can be diagonalized at any *instant* t , we could calculate the spectrum of $\mathcal{U}(t+T, t)$ by the following device. Consider the auxiliary time-parametric eigenvalue problem

$$\mathbf{H}(t') |n(t')\rangle = E_n(t') |n(t')\rangle,$$

where $E_n(t)$, $n = 0, 1, 2 \dots$ are auxiliary quasienergies. Assuming the existence a complete *bracket* set $|n(t')\rangle$, i.e., $\sum_n |n(t)\rangle\langle n(t)| = \mathbf{I}$ (the identity matrix) it is possible to show that

$$\text{Tr} [\mathbf{P}(t | t')] = \sum_n \exp \int_{t'}^t E_n(s) ds. \quad (6.19)$$

From this expression we can calculate recursively any Kolmogorow's eigenvalue. Using the notation $\Gamma(t+T, t) = \text{Tr} [\mathbf{P}(t+T | t)] - 1$, in principle any eigenvalue $k_n = e^{-\lambda_n T}$ can be calculated by noting that

$$\lim_{T \rightarrow \infty} \frac{\frac{d}{dT} \Gamma(t+T, t)}{\Gamma(t+T, t)} \rightarrow -\lambda_1,$$

and for λ_2 we have

$$\lim_{T \rightarrow \infty} \frac{\frac{d}{dT} [\Gamma(t+T, t) - k_1]}{\Gamma(t+T, t) - k_1} \rightarrow -\lambda_2,$$

and so forth. We see that the important task is to determine the trace of matrix Green's function $\mathbf{P}(t+T | t)$.

We note that in general a time-parametric Master Hamiltonian matrix $\mathbf{H}(t)$ cannot be diagonalized; on the other hand only if the condition of PDB is fulfilled the diagonalization of $\mathcal{U}(t+T, t)$ could be assured, see section IV.

In order to get more insight into the structure of the Kolmogorow spectrum let us now consider a particular nonsymmetric 3×3 model. Assume that the Master Hamiltonian has the particular structure

$$\mathbf{H} = \begin{pmatrix} -(\beta + \gamma) & \alpha & \alpha \\ \beta & -(\alpha + \gamma) & \beta \\ \gamma & \gamma & -(\alpha + \beta) \end{pmatrix}, \quad (6.20)$$

where α, β, γ are positive possible time-periodic functions. The physical interpretation of this $\mathbf{H}(t)$ can easily be done by using a level diagrammatic representation, see figure 1. Solving the system (6.12) for this model it is possible to write a closed equation for the trace of the propagator, then we get

$$\begin{aligned} \Gamma(t, t') &= \text{Tr} [\mathbf{P}(t | t')] - 1 \\ &= 2 \exp \left[- \int_{t'}^t (\alpha(s) + \beta(s) + \gamma(s)) ds \right]. \end{aligned} \quad (6.21)$$

We see that even when there is a degenerated quasienergy for (6.20), i.e., $E_0(t) = 0$, $E_{1,2}(t) = -\alpha - \beta - \gamma$, due to the fact that this $\mathbf{H}(t)$ can be diagonalized at any *instant* t , expression (6.21) is in accordance with (6.19). From (6.21) it is possible to see that Kolmogorow's time-scale λ_1^{-1} is given, for this particular case, in terms of one period of time area of the transition rates.

C. About the Suzuki-Trotter approach

If $\mathbf{H}(t)$ cannot be diagonalized at any instant t , nothing more can be told concerning the possibility of finding analytically the non-trivial eigenvalue k_1 of the Kolmogorow operator. Therefore in order to end the analysis of the spectrum of $\mathcal{U}(t+T, t)$, we give here a numerical approach that can easily be implemented in a finite dimensional vector space.

Consider the formal expression (6.1) to represent the Kolmogorow operator. Following the theory of time-ordered exponential, any ordered exponential operator can be expressed by an ordinary exponential operator in terms of the super-operator \mathbf{T} as¹⁵

$$\vec{T} \exp \left(\int_t^{t+T} \mathbf{H}(\tau) d\tau \right) = \exp [T (\mathbf{H}(t) + \mathbf{T})],$$

where the super-operator \mathbf{T} is defined by its action over any operator (differentiable or not) $\mathbf{A}(t)$ and $\mathbf{B}(t)$:

$$\mathbf{A}(t) e^{T\mathbf{T}} \mathbf{B}(t) = \mathbf{A}(t+T) \mathbf{B}(t).$$

Thus, using Trotter's formulae it is possible to prove that

$$\mathcal{U}(t+T, t) = \lim_{n \rightarrow \infty} e^{\frac{T}{n} \mathbf{H}(t+T)} \dots e^{\frac{T}{n} \mathbf{H}(t+\frac{2T}{n})} e^{\frac{T}{n} \mathbf{H}(t+\frac{T}{n})}. \quad (6.22)$$

From this expression for $\mathcal{U}(t+T, t)$ the characteristic polynomial can be calculated as a perturbation in the small parameter T/n .

In order to study how fast is the convergence of this approach, let us tackle an interesting 3×3 model. This situation physically may correspond to the case when a periodically modulated external pumping is acting on some discrete *level 3* to produce a current to the *level 1*, while the rest of transition rates W_{ij} are time independent. We can, for example, consider

$$\mathbf{H}(t) = \begin{pmatrix} -1 & 1 & \alpha(t) \\ 0.5 & -2 & 0 \\ 0.5 & 1 & -\alpha(t) \end{pmatrix}. \quad (6.23)$$

Clearly this is an out of equilibrium model with a diode between states *2* and *3*. Now we assume an Arrhenius-like form for the time-periodic transition rate $\alpha(t) = \mathcal{A} e^{b \cos \omega t}$. It is simple to see that if at time t' , $\alpha(t') = 3$ or 1 , the matrix $\mathbf{H}(t')$ cannot be diagonalized. Thus in order to calculate the eigenvalue k_1 we apply the Suzuki-Trotter approach to estimate the characteristic polynomial of $\mathcal{U}(t+T, t)$

$$\mathcal{P}(k) = \det |\mathbf{P}(t+T | t) - k \mathbf{1}|.$$

By studying the eigenvalues $k_{1,2}$ we found that the convergence of (6.22) is well defined, and good results are obtained just for $n \sim 10$. In addition the trivial root $k_0 = 1$ is always present for any n as was expected by normalization of the propagator. In figure 2 we show the calculation of the time-scale λ_1^{-1} for $\mathcal{A} = 1$ and $b = 1, 2$ as a function of n (in the present case $k_{1,2}$ are two conjugated eigenvalues, and the saturation value of $\lambda_1^{-1} = T/\ln |k_1|^{-1}$ are 0.4628 and 0.3718 for $b = 1, 2$ respectively). We have checked that (6.22) provides a straightforward method to calculate the first nontrivial time-scale λ_1^{-1} of periodically modulated discrete Markov processes. As expected, comparing the present 3×3 case with the similar 2-level system of section VI.A.1, by increasing the amplitude of the periodic modulation b , the time-scale λ_1^{-1} decreases predicting a faster relaxation of the associated antisymmetric eigenvector $\vec{g}_1(t)$.

D. Final discussions

In the present paper we have introduced the Kolmogorow eigenvalue approach to study the stochastic dynamics of continuous or discrete Periodic Nonstationary Markov Process, by exploring its Floquet structure. This theory is encoded in the Lemma of section III, reducing the analysis to an eigenvalue problem of a Fredholm equation with a nonsymmetrical kernel. In general we have proved that the asymptotic relaxation of periodically modulated Markov process is governed by the time-scale λ_1^{-1} which is characterized by the real part of the first nontrivial eigenvalue k_1 of the Kolmogorow operator.

In section VI.A we have discussed pedagogical examples showing the interplay of the combined effect of fluctuations and external time-periodic modulations. Taking for example an Arrhenius-like model for the time-modulation of the transition rates, we have shown that if the amplitude of periodic modulation is small, there is a range of values of the physical parameters where the time-scale λ_1^{-1} is of the order of the Kramers time. But in general the mechanism leading to diffusion is nontrivial and the calculation of λ_1^{-1} is of great value to understand the relaxation and mixing of periodically modulated Markov processes. We have shown that Kolmogorow's spectrum analysis is also of interest in the study of the stochastic resonance¹⁷, and in nonequilibrium (order-disorder) phase transitions by considering the relaxation of the asymptotic two-time correlation function⁹.

For a finite dimensional vector space, if the Kolmogorow operator cannot be diagonalized we could only expect a Jordan form for $\mathcal{U}(t + T, t)$. Nevertheless finding the first non-trivial eigenvalue k_1 gives a very important piece of information concerning the asymptotic relaxation of any discrete Periodic Nonstationary Markov Process. As a matter of fact we have also presented the Suzuki-Trotter approach to calculate the characteristic polynomial in order to get the spectrum of the Kolmogorow operator. To end the section concerning finite dimensional vector spaces, we worked out an example showing that a Suzuki-Trotter numerical approach is a very suitable algorithm to tackle the calculation of Kolmogorow time-scale for discrete Periodic Nonstationary Markov processes.

Acknowledgments. M.O.C. thanks early discussions with Dr. A. Becker. This work has partially been done within the framework of the Associateship Scheme of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy .

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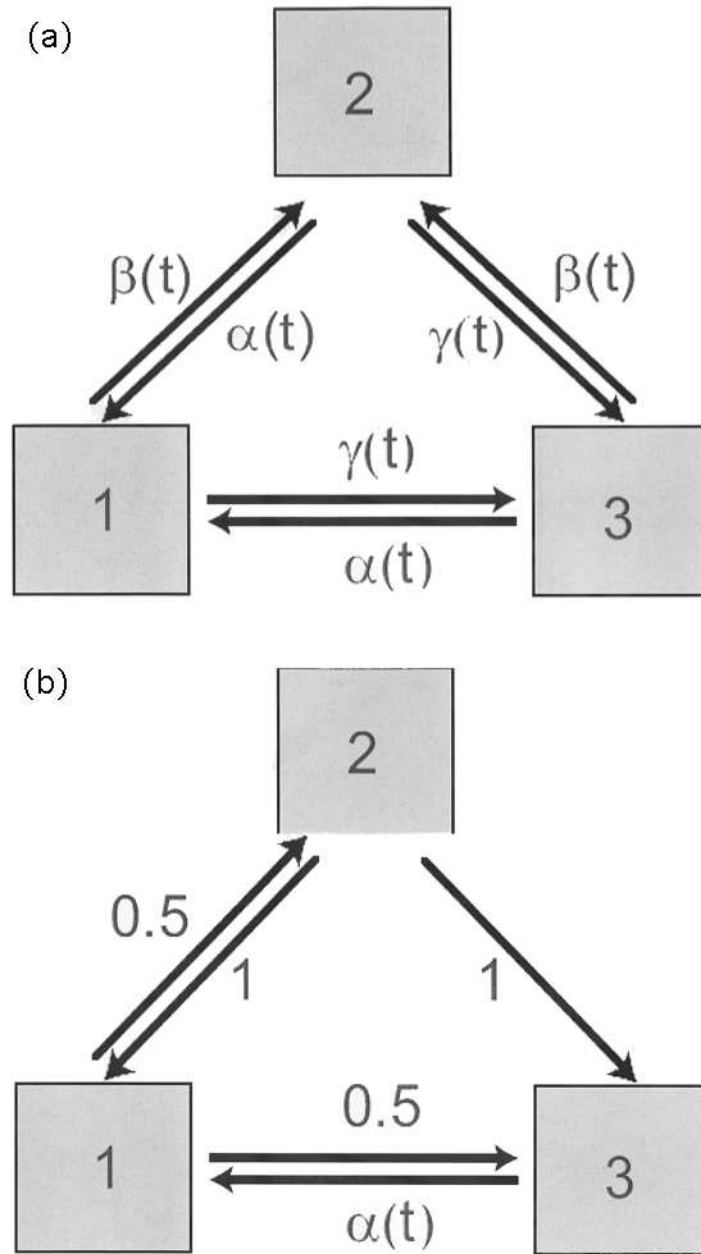


FIG. 1. (a) Schematic drawing representing the transitions rates in a 3-level stochastic process characterized by the Master Hamiltonian (6.20). Note that in principle the transition rates can be time-periodically modulated. (b) Schematic drawing representing the transitions rates characterized by the Master Hamiltonian (6.23).

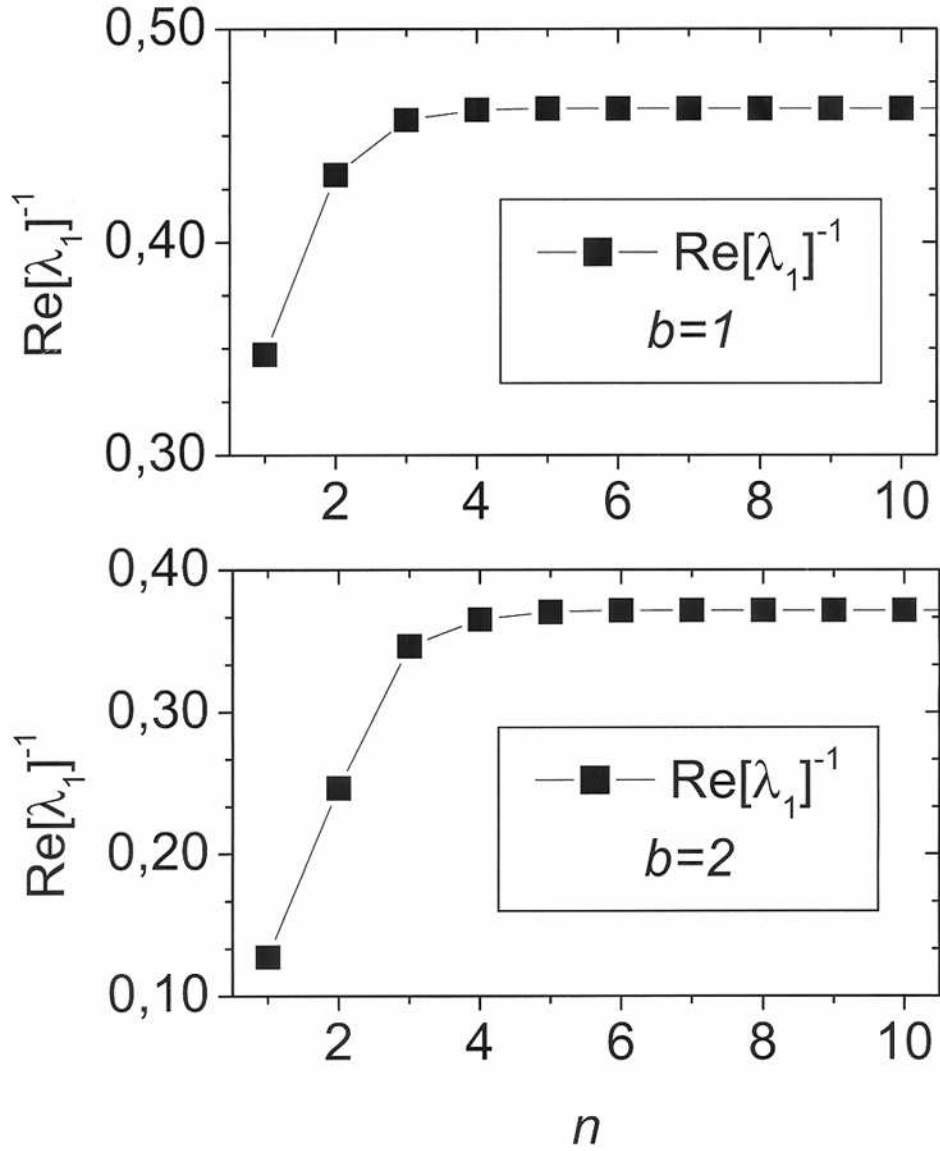


FIG. 2. Plot of the Kolmogorow time-scale $\lambda_1^{-1} = T/\ln|k_1|^{-1}$ using (6.22) for $\mathcal{A} = 1, T = 1$ and $b = 1, 2$, as a function of the iteration number n . Suzuki-Trotter's discretization time integral is T/n .