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ESTIMATORS FOR LOCAL NON-GAUSSIANITIES

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Abstract

We study the Likelihood function of data given f_{NL} for the so-called local type of non-Gaussianity. In this case the curvature perturbation is a non-linear function, local in real space, of a Gaussian random field. We compute the Cramer-Rao bound for f_{NL} and show that for small values of f_{NL} the 3-point function estimator saturates the bound and is equivalent to calculating the full Likelihood of the data. However, for sufficiently large f_{NL} , the naive 3-point function estimator has a much larger variance than previously thought. In the limit in which the departure from Gaussianity is detected with high confidence, error bars on f_{NL} only decrease as $1/\ln N_{\text{pix}}$ rather than $N_{\text{pix}}^{-1/2}$ as the size of the data set increases. We identify the physical origin of this behavior and explain why it only affects the local type of non-Gaussianity, where the contribution of the first multipoles is always relevant. We find a simple improvement to the 3-point function estimator that makes the square root of its variance decrease as $N_{\text{pix}}^{-1/2}$ even for large f_{NL} , asymptotically approaching the Cramer-Rao bound. We show that using the modified estimator is practically equivalent to computing the full Likelihood of f_{NL} given the data. Thus other statistics of the data, such as the 4-point function and Minkowski functionals, contain no additional information on f_{NL} . In particular, we explicitly show that the recent claims about the relevance of the 4-point function are not correct. By direct inspection of the Likelihood, we show that the data do not contain enough information for any statistic to be able to constrain higher order terms in the relation between the Gaussian field and the curvature perturbation, unless these are orders of magnitude larger than the size suggested by the current limits on f_{NL} .

1 Introduction

In single field slow-roll inflation the level of non-Gaussianity is sharply predicted and very small, less than 10^{-6} [1, 2]. This is quite far from the present experimental sensitivity and probably not attainable with either CMB observations or galaxy redshift surveys. As a result, deviations from a purely Gaussian statistics of density perturbations, if observed, could provide important constraints on models of early cosmology, forcing us to abandon the single-field slow-roll paradigm.

Of course there are many ways in which a signal could be “non-Gaussian”. Given a data set, such as the WMAP maps, there are two possible ways to proceed. One could calculate all kinds of statistics of the data and compare the results with the expectation for a Gaussian field searching for anomalies. This is a fine strategy as long as one adjusts the significance of the result to account for the number of possible deviations that have been explored. There are several anomalies in the WMAP data reported in the literature that have been found in this way (see for example [3–5]). Unfortunately their significance is hard to assess and as a result one is not sure how seriously to take them.

The second approach is to think about the possible physical mechanisms that can lead to non-Gaussianities and search for their particular signatures. In the context of primordial effects one should investigate what types of non-Gaussianity can plausibly be produced in various inflationary models. This approach is further bolstered by the fact that at least at the level of the 3-point function, primordial signals seem to fall into two definite classes. Thus there are only two different signatures one has to look for.

The analysis of inflationary models that go beyond the single field slow-roll class has identified several examples with a relatively high level of non-Gaussianity, within reach of present or forthcoming experiments. For nearly Gaussian fluctuations, the quantity most sensitive to departures from perfect Gaussianity is the 3-point correlation function. In general, each model will give a different correlation between the Newtonian potential modes*:

$$\langle \Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\Phi(\mathbf{k}_3) \rangle = (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) F(k_1, k_2, k_3) . \quad (1)$$

The function F describes the correlation as a function of the triangle shape in momentum space.

The predictions for the function F in different models divide quite sharply into two qualitatively different classes as a consequence of qualitatively different ways of producing correlations among modes [8]. The first possibility is that the source of density perturbations is not the inflaton but a second light scalar field σ . In this case non-Gaussianities are generated by the non-linear relation between the fluctuation $\delta\sigma$ of this field and the final perturbation Φ we

*Even with perfectly Gaussian primordial fluctuations, the observables, *e.g.* the temperature anisotropy, will not be perfectly Gaussian as a consequence of the non-linear relation between primordial perturbations and what we will eventually observe. These effects are usually of order 10^{-5} (see for example [6, 7]) and thus beyond (but not much) present sensitivity. In the following we will disregard these contributions.

observe. This non-linearity is *local* as it acts when the modes are much outside the horizon; schematically we have $\Phi(\mathbf{x}) = g(\mathbf{x}) + f_{\text{NL}}(g^2(\mathbf{x}) - \langle g^2 \rangle) + \dots$, where g is a Gaussian random field. The quadratic piece introduces a 3-point function for Φ of the form

$$F(k_1, k_2, k_3) = f_{\text{NL}} \cdot 2\Delta_{\Phi}^2 \cdot \left(\frac{1}{k_1^3 k_2^3} + \frac{1}{k_1^3 k_3^3} + \frac{1}{k_2^3 k_3^3} \right), \quad (2)$$

where Δ_{Φ} is the power spectrum normalization, $\langle \Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2) \rangle = (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2) \Delta_{\Phi} \cdot k_1^{-3}$, which has been taken as exactly scale invariant. Examples of this mechanism are the curvaton scenario [9] and the variable decay width model [10], which naturally give rise to f_{NL} greater than 10 and 5, respectively. Various subtleties in estimating the size of this type of non-Gaussianity will be the focus of this paper.

The second class of models are single field models with a non-minimal Lagrangian, where the correlation among modes is created by higher derivative operators [11–15]. In this case, the correlation is strong among modes with comparable wavelength and it decays when we take one of k 's to zero keeping the other two fixed. Although different models of this class give a different function F , all these functions are qualitatively very similar. We will call this kind of functions *equilateral* because the signal is maximal for equilateral configurations in Fourier space, whereas for the local form (2) the most relevant configurations are the *squeezed* triangles with one side much smaller than the others. We will not discuss the *equilateral* type of non-Gaussianity in this paper too much. We will just point out that the effects studied in this paper do not apply in that case so that the situation is much simpler.

The strongest constraint on f_{NL} comes from analyzing the 3-point function of the WMAP data set. WMAP is the best available data set because it has the largest number of pixels measured with good signal-to-noise. From the first year data the constraint is [16]:

$$-27 < f_{\text{NL}} < 121 \quad \text{at 95\% C.L.} \quad (3)$$

This constraint is better than that obtained by the WMAP collaboration both using the one year WMAP data [17] and the three year ones [18]. This is so because the WMAP team used a non-optimal estimator which did not adequately treat the effect of anisotropic noise, as already noted in [17, 19]. In [16], we showed that the effect of the anisotropic noise can be substantially reduced with the addition of a linear piece to the estimator. Always in [16] we also constrained the level of the *equilateral* 3-point function. For both types, the departures from Gaussianity still allowed by the data are at the same level.

Given the interest in constraining the level of non-Gaussianity, one may wonder if a statistic other than the 3-point function might extract more information about f_{NL} . There are various contradictory, or at least apparently contradictory, answers to this question in the literature. On the one hand in [20] and [16] it is argued that the 3-point function saturates the Cramer-Rao bound up to terms of order $f_{\text{NL}} A^{1/2}$, where A is the square of the amplitude of curvature perturbations: $A^{1/2} \sim 10^{-5}$. On the other hand calculations of the signal to noise in the 4-point

function by [21] and [22] point to a different conclusion. These papers claim that, even though in the limit of $f_{\text{NL}} \rightarrow 0$ the signal to noise ratio of the 4-point function is negligible, it grows more rapidly with the number of pixels in the data set than for the 3-point function. As a result for values of f_{NL} rather small, say around ~ 50 for an experiment like Planck, the signal to noise in the 4-point function is larger than for the 3-point function and stronger constraints on f_{NL} could be placed by studying the 4-point function. Of course this result is puzzling. One is immediately drawn to the question, what about the 5-point function? And why not the 11-point function? Applying the same arguments as in [21] and [22] would show that the signal to noise ratio becomes larger the higher the n-point function considered. Clearly there is a contradiction.

It is the aim of this paper to clarify this contradiction. We will show that both calculations have missed an interesting subtlety of the local type of non-Gaussianity in the case of scale invariant, or nearly scale invariant, spectrum of primordial perturbations. As a result, the calculation of the noise of various estimators (including the 3-point function) for finite f_{NL} is missing some relevant term. Some of the terms that are naively down by powers of $f_{\text{NL}}A^{1/2}$ are actually much larger, being enhanced by N_{pix} . The growth in the signal to noise for high f_{NL} seen in the above papers is fictitious. We will show that the same subtlety creeps into the calculation of [20] and thus the 3-point function estimator considered there also does not saturate the Cramer-Rao bound for large f_{NL} . We want to stress that even though what was missed was a rather subtle point, it has potentially large consequences on the signal to noise of the estimators previously considered. For example when one is in the regime of large signal to noise, the error bars on f_{NL} from the 3-point function decrease as $1/\ln N_{\text{pix}}$ rather than $N_{\text{pix}}^{-1/2}$. The reader at this point should not panic, we will show that the Cramer-Rao bound in this regime still scales as $N_{\text{pix}}^{-1/2}$ and that it is rather straightforward to extract all of this information from the data either by calculating the full Likelihood or slightly tweaking the 3-point function estimator.

What is the missing subtlety? To understand it, it is best to recall what is the main effect of the local non-Gaussianity: it correlates large and small scales. In the 3-point function, a long wavelength mode modulates the amplitude of all the short wavelengths by the same amount, regardless of the wavelength of the short mode. Furthermore, in [8] we showed that most of the signal in the 3-point function is coming from squeezed triangular configuration in Fourier space. More importantly for this discussion, if one considers the signal to noise as a function of the wavenumber of the long wavelength mode k_L , one gets an equal amount of information from every logarithmic interval in k_L . This in fact is the source of the problem. The smallest k_L in the survey are by definition the ones with the largest cosmic variance as there are the least of them in the survey. As one increases the resolution of the survey the contribution to the signal to noise from the long wavelengths only decreases logarithmically and thus the large cosmic variance of the long modes translates into large variances in the estimators of f_{NL} .

In fact in [16] indications of the importance of the long wavelength modes were emphasizes

in the context of the effect of anisotropic noise on the estimator of the 3-point function. The noise in the map is anisotropic because WMAP spent different amounts of time observing each pixel on the sky. As a result the level of small scale power, for large multipoles where the noise becomes important, varies across the sky. This map of small scale power can randomly align with the particular large scale mode giving a spurious f_{NL} signal. Of course on average this effect is zero as there is no intrinsic correlation between the map of observing time per pixel and the large scale temperature. However for a particular realization, some modes will be correlated (spurious positive f_{NL}) and others anti-correlated (spurious negative f_{NL}). The contribution to the signal from the long wavelength modes will not add exactly to zero, as we have few of them in the survey. The random left over spurious signal effectively increases the variance of the estimator. This effect was noted in the WMAP team analysis [17, 19], where the constraint on f_{NL} got worse as they increased the size of the data set by including more of the small scales. In [16] the estimator was improved by including a linear piece which substantially reduces the effect allowing us to get better constraints. This effect is the reason why the 3-year data analysis by the WMAP team [18] did not appreciably improve the limits.

To clarify the situation we will study the full Likelihood of the data given f_{NL} . We will keep careful track of enhanced terms and thus do a consistent expansion in f_{NL} . We will then calculate the Cramer-Rao bound, extending the results of [20] to non-zero f_{NL} . We will show how the additional terms in the variance of the 3-point function estimator make it become sub-optimal. This can be easily fixed using an improved estimator which asymptotically saturates the Cramer-Rao bound. The use of this estimator is equivalent to the full Likelihood of the data.

The fact that the improved 3-point function estimator is equivalent to the full Likelihood of the data, implies that there is no additional information in the 4-point function. We will also show this explicitly, illustrating how at best the 4-point function is equivalent to the 3-point function. No other statistic such as Minkowski functionals, various wavelet based statistics and other esoteric constructions are worth trying to constrain f_{NL} . *None can be better than the 3-point function.* This is true up to corrections of order $f_{\text{NL}}A^{1/2} \lesssim 10^{-3}$.

The apparent large signal to noise in the 4-point function led to the suggestion [21, 22] that the 4-point function could even be sensitive to higher order terms in the relation between Φ and $\delta\sigma$: $\Phi(\mathbf{x}) = g(\mathbf{x}) + f_{\text{NL}}(g^2(\mathbf{x}) - \langle g^2 \rangle) + f_{\text{NL}}^2\alpha g^3(\mathbf{x}) \dots$. The claim was that the 4-point function could constrain the real parameter α . Of course the third term is a minuscule correction to the first two, even for the largest allowed values of f_{NL} . Thus it is difficult to understand how one could be sensitive to it. Again the missing terms in the variance of the estimator were responsible for the apparent sensitivity to α . Using the full Likelihood we will show that for any realistic experiment there is in fact not enough information about α in the data to constrain it, unless $\alpha > 1/(f_{\text{NL}}A^{1/2}) \gtrsim 10^3$.

Finally we will also show that for a realistic experiment where $\ln N_{\text{pix}}$ is large, the value of

f_{NL} for which improving the naive 3-point function estimator is important is rather large. One should start worrying about it once there is a many σ detection of f_{NL} . As a result our improved estimator will probably be only of academic interest. Our paper mainly provides clarification of various misconceptions in the literature. Given this and to reduce the length of our equations, we will work in the flat sky approximation and neglect the CMB transfer functions. Clearly none of these approximations are material to our results. We also point out that even though we do all our calculations in 2 dimensions, as relevant to the CMB, our conclusions are equally valid for 3 dimensional surveys, such as galaxy surveys or future 21 cm observations.

2 Subtleties of the f_{NL} expansion

The aim of this section is to explicitly show the presence of terms in the variance of the 3-point function estimator of f_{NL} that are enhanced by factors of N_{pix} and thus contribute significantly even though they are naively suppressed by $f_{\text{NL}}A^{1/2}$. We start by introducing our notation and reproducing previous calculations of the variance of the estimator. We then identify the terms that had been previously missed and give a rule of thumb to easily determine when they are important. We will show that in practice the enhanced terms do not correct current upper limits and that they will only become important after a very high signal to noise detection of f_{NL} .

Notation. We work in the flat sky approximation, neglect the transfer function, and assume that the error is dominated by cosmic variance. In this paper we are mostly interested in the scaling properties of the estimators for the non-Gaussianities used for example in [16, 17, 20–24]. These properties are not modified by these approximations[†], while on the other hand their use makes the presence of some physical effects much clearer, as we will later see. Let us briefly set up our conventions. For the Fourier transform we have:

$$\Phi_{\vec{l}} = \frac{\Omega}{N_{\text{pix}}} \sum_{\vec{\theta}} e^{-i\vec{\theta}\cdot\vec{l}} \Phi_{\vec{\theta}}, \quad (4)$$

where Ω and N_{pix} are respectively the angular size and the number of pixels of the sky survey. It is immediate to obtain from this the continuum limit $\Phi_{\vec{l}} = \frac{\Omega}{N_{\text{pix}}} \sum_{\vec{\theta}} e^{-i\vec{\theta}\cdot\vec{l}} \Phi_{\vec{\theta}} \simeq \int d^2\theta e^{-i\vec{\theta}\cdot\vec{l}} \Phi_{\vec{\theta}}$. We also have:

$$\Phi_{\vec{\theta}} = \frac{1}{\Omega} \sum_{\vec{l}} e^{i\vec{\theta}\cdot\vec{l}} \Phi_{\vec{l}} \simeq \int \frac{d^2l}{(2\pi)^2} e^{i\vec{\theta}\cdot\vec{l}} \Phi_{\vec{l}}, \quad (5)$$

and the useful relations:

$$\sum_{\vec{l}} e^{i\vec{l}\cdot(\vec{\theta}_1 - \vec{\theta}_2)} = N_{\text{pix}} \delta_{\vec{\theta}_1, \vec{\theta}_2}, \quad \sum_{\vec{\theta}} e^{-i\vec{\theta}\cdot(\vec{l}_1 - \vec{l}_2)} = N_{\text{pix}} \delta_{\vec{l}_1, \vec{l}_2}. \quad (6)$$

[†]This is confirmed by the fact we are able to recover the same scaling properties found in [20–24] where these approximations were not used.

We are only interested in local non-Gaussianities as the effect we will discuss does not apply to other types. In that case, the observed field $\Phi_{\vec{\theta}}$ is given by a non linear function of a Gaussian field $g_{\vec{\theta}}$ which is local in real space:

$$\Phi_{\vec{\theta}} = f(g_{\vec{\theta}}) = g_{\vec{\theta}} + f_{\text{NL}} \left(g_{\vec{\theta}}^2 - \sigma^2 \right). \quad (7)$$

We will call this field temperature although our results apply to other measures, not just the CMB temperature. In Fourier space the local relation reads:

$$\Phi_{\vec{l}} = (f(g))_{\vec{l}} = g_{\vec{l}} + f_{\text{NL}} \left((g \circ g)_{\vec{l}} - \sigma^2 \Omega \delta_{\vec{l},0} \right), \quad (8)$$

where we have defined $(g \circ g)_{\vec{l}} = \frac{1}{\Omega} \sum_{\vec{k}} g_{\vec{l}-\vec{k}} g_{\vec{k}}$. We will explicitly address later the case of possible higher order corrections in f_{NL} to these definitions. The covariance matrix is defined as

$$\langle g_{\vec{l}_1} g_{\vec{l}_2} \rangle = C_{\vec{l}_1 \vec{l}_2} = C_{l_1} \Omega \delta_{\vec{l}_1, -\vec{l}_2}, \quad (9)$$

where $l^2 = \vec{l} \cdot \vec{l}$, and $C_l = 2\pi A/l^2$, which then implies that

$$\sigma^2 = \langle g_{\vec{\theta}_i} g_{\vec{\theta}_i} \rangle = \frac{2\pi A}{\Omega} \sum_{\vec{l}} \frac{1}{l^2} \simeq \frac{A}{2} \ln N_{\text{pix}}, \quad (10)$$

where in the last passage we have used the continuum limit, and the fact that $N_{\text{pix}} \simeq \Omega l_{\text{max}}^2/(4\pi)$, with l_{max} the maximum of the observed l s. From here on, in order to simplify the notation, we will remove the vector symbol from \vec{l} and $\vec{\theta}$ in all the mathematical expressions when the meaning and the distinction from the modulus $l = |\vec{l}|$ and $\theta = |\vec{\theta}|$ is clear from the context.

Previous results: the missing enhanced terms. In [16, 17] the analysis for the non-Gaussianities of the local kind was performed using a trilinear estimator with signal-to-noise weighting. In the limit of flat sky, unit transfer function, and isotropic noise it reduces to:

$$\mathcal{E} = \frac{1}{N} \sum_l \frac{1}{\Omega C_l} \Phi_l \chi_{-l}, \quad (11)$$

where we have defined the field $\chi_l = (\Phi \circ \Phi)_l - \Omega \sigma^2 \delta_{l,0}$, and where we consider only non degenerate configurations with all the Φ s taken with $l \neq 0$. The normalization

$$N = \sum_l \frac{1}{\Omega C_l} \langle \Phi_l \chi_{-l} \rangle_1 \simeq 8 N_{\text{pix}} \sigma^2, \quad (12)$$

with the subscript ₁ meaning that the expectation value is taken with $f_{\text{NL}} = 1$, has been chosen so that the estimator is unbiased, $\langle \mathcal{E} \rangle = f_{\text{NL}}$ ([‡]). The definition in eq. (7) tells us that the temperature field in the sky Φ_{θ} is to a good approximation a Gaussian field, with a small non-Gaussian correction of order $f_{\text{NL}} \Phi \sim f_{\text{NL}} A^{1/2} \lesssim 10^{-3}$. So one is tempted to expect that higher

[‡]Some of the expressions, like eq. (12) above, when expressed in terms of N_{pix} and σ , will slightly depend on the geometry of the survey which changes the boundary of the domain of integration in Fourier space. Also we will have similar corrections going from flat to full sky. However these effects do not change significantly our results.

order corrections in f_{NL} in the various expressions are suppressed with respect to the leading terms by powers of $f_{\text{NL}}A^{1/2}$ and thus irrelevant. For example, the variance of the estimator in eq. (11) starts with a piece which is of zeroth order in f_{NL} , and which gives:

$$\langle \Delta \mathcal{E}^2 \rangle_{f_{\text{NL}}=0} = \frac{1}{\Omega^2 N^2} \sum_{l'l'} \frac{1}{C_l C_{l'}} \langle \Phi_{l\chi-l} \Phi_{l'\chi-l'} \rangle_{f_{\text{NL}}=0} = \frac{1}{N} = \frac{1}{8N_{\text{pix}} \sigma^2}, \quad (13)$$

where $\Delta \mathcal{E} = \mathcal{E} - \langle \mathcal{E} \rangle$ [§]. One would naively assume that this is the dominant term in the variance, with small corrections of order $f_{\text{NL}}A^{1/2}$, which should contribute at most at order 10^{-3} given the current bound on f_{NL} . This is what was assumed for example in [16, 17, 20], where the estimator (11) was in fact found minimizing the variance at zeroth order in f_{NL} among all trilinear estimators. However, as we will soon see, for the case of local non-Gaussianities, and only for them, there is another parameter which enters into the expansion: N_{pix} . We will see in fact that in certain expressions such as the variance of the estimator above for example, there are terms that although suppressed by powers of $f_{\text{NL}}A^{1/2}$, are enhanced by powers of N_{pix} , and so, depending on the real value of f_{NL} , they might need to be taken into account.

In order to verify that this is actually the case, and to understand the implications of this fact, let us sketch the computation of the variance of the estimator \mathcal{E} keeping higher order terms in f_{NL} . The variance of \mathcal{E} will involve the computation of a 6-point function, which will split in the sum of the product of several different combinations of connected n -point functions, i.e. the product of three 2-point functions, of two 3-point functions, and of a 4-point function and a 2-point function. Concentrating on the last kind of contribution, we will have terms like:

$$\langle \Delta \mathcal{E}^2 \rangle \supset \frac{1}{N^2 \Omega^4} \sum_{l_1 l_2 \tilde{l}_1 \tilde{l}_2} \frac{1}{C_{l_2} C_{\tilde{l}_2}} \langle \Phi_{l_1} \Phi_{\tilde{l}_1} \rangle_c \langle \Phi_{l_2} \Phi_{-l_1-l_2} \Phi_{\tilde{l}_2} \Phi_{-\tilde{l}_1-\tilde{l}_2} \rangle_c, \quad (14)$$

where $\langle \Phi_{l_1} \Phi_{l_2} \dots \Phi_{l_n} \rangle_c$ stays for the connected n -point function. Now, apart from numerical factors, one of the terms in the expansion of the connected 4-point function reads:

$$\langle \Phi_{l_2} \Phi_{-l_1-l_2} \Phi_{\tilde{l}_2} \Phi_{-\tilde{l}_1-\tilde{l}_2} \rangle_c \supset f_{\text{NL}}^2 \Omega^3 \delta_{l_1, -\tilde{l}_1} C_{l_2} C_{l_1} C_{\tilde{l}_2}. \quad (15)$$

Considering the effect of this term in the variance in eq. (14), where we also take the 2-point function at zeroth order in f_{NL} , we obtain:

$$\langle \Delta \mathcal{E}^2 \rangle \supset \frac{f_{\text{NL}}^2}{N^2} \sum_{l_1 l_2 \tilde{l}_2} C_{l_1}^2 \propto \frac{f_{\text{NL}}^2 A N_{\text{pix}}^2}{N^2} \propto \frac{f_{\text{NL}}^2}{\ln^2 N_{\text{pix}}}. \quad (16)$$

Thus this contribution to the relative variance *does not* decrease as $1/(N_{\text{pix}} \ln N_{\text{pix}})$, as one would have naively expected, but there is an enhancement of N_{pix} which make it decrease only as $1/\ln^2 N_{\text{pix}}$.

[§]Notice that the variance of the estimator scales faster than the naive $1/N_{\text{pix}}$ by a factor of $\ln N_{\text{pix}}$. This behaviour is typical of non-Gaussianities of the local kind, where the signal comes from the correlation of the modes of all different scales.

There is a very physical reason for the presence of such enhanced terms for the case of local non-Gaussianities. As we have already said, most of the signal for this kind of non-Gaussianities comes from squeezed configurations, where one of the l s is small and the others are large. More precisely, for the 3-point function, the signal-to-noise from all the squeezed configurations with the smallest l in a given decade is roughly the same for every decade, although there are much fewer modes in a decade of low l . This means that the low l modes are always very important for the estimator \mathcal{E} . Now, the point is that there is an intrinsic variance associated with a configuration with a certain small l , simply because there are very few of those small l s, just $2l + 1$; and this is unaffected by the fact that N_{pix} of the survey increases, because this just increases the l_{max} of the experiment. Therefore the relative variance of the estimator due to these terms decrease only logarithmically with N_{pix} , because this is how the relative importance of the configurations with small l s decreases with N_{pix} . This physical explanation guarantees us that these enhanced terms are not present in the case of equilateral non-Gaussianities, where the importance of the small l s and of the squeezed configurations is marginal.

It is useful to develop a quick thumb rule to understand if a term which is a sum of product of different C_l s is enhanced or not: a term will be enhanced only if at least one C_l is raised to a power larger than one. In fact in this case the summation over the multipoles for this term will be dominated by the lowest l s, so that some l_{min} will appear in the denominator. This makes these terms enhanced by powers of $l_{\text{max}}/l_{\text{min}}$ (see also appendix A).

The relevance of the additional terms. Our discussion shows that the treatment of expressions containing f_{NL} is delicate in the case of local non-Gaussianities. The expansion parameter *is not* just $f_{\text{NL}}A^{1/2}$, but there are terms which can parametrically go as $f_{\text{NL}}A^{1/2}N_{\text{pix}}$, and therefore cannot be neglected. We need to understand the relevance of these terms both for existing limits on f_{NL} as well as their impact on future measurements.

After a careful calculation, we find the following expression for the variance of \mathcal{E} :

$$\langle \Delta \mathcal{E}^2 \rangle = \frac{1}{4AN_{\text{pix}} \ln N_{\text{pix}}} \left(1 + \frac{8f_{\text{NL}}^2 AN_{\text{pix}}}{\pi \ln N_{\text{pix}}} + \dots \right), \quad (17)$$

where \dots represents terms suppressed by powers of $f_{\text{NL}}^2 A$ without any further N_{pix} enhancement. This result shows an important feature of this estimator. Imagine that we have a series of experiments with increasing N_{pix} , and that at some point we detect a non-null f_{NL} . Then, at first the variance will decrease as $1/(N_{\text{pix}} \ln N_{\text{pix}})$, but, after a critical N_{pix} which depends on the actual value of f_{NL} , and which is basically, apart for logarithms, when the signal-to-noise is of order 1, the variance will begin to decrease very slowly as $1/\ln^2 N_{\text{pix}}$, because of the enhanced variance of the term proportional to f_{NL}^2 .

In the analysis performed in [16,17] the variance for a non-zero f_{NL} was assumed to be the same as for $f_{\text{NL}} = 0$, expecting that the f_{NL} corrections would have been small. In the light of the results of this section, we see that this procedure is not always justified. However, for those

analysis, we can verify that the error introduced is very small, as already numerically checked with non-Gaussian Montecarlos in [17, 25]. We can quantify the error in this way: the relative correction to the variance for an f_{NL} at $n\sigma_0$ from the origin, where σ_0 is the variance computed at $f_{\text{NL}} = 0$, is of order $2n^2/(\pi \ln^2 N_{\text{pix}})$. For the WMAP experiment $\ln^2 N_{\text{pix}} \sim 35$, therefore this correction is large for n larger than $\sim 6, 7$. Therefore, if we wish to give a $2 - \sigma$ confidence interval around a certain central value, we see that the enhanced terms will become important for a central value around $4, 5 - \sigma_0$ far from the origin, i.e. in the case of a clear detection of a non zero f_{NL} . In the analysis of [16, 17], the central value of f_{NL} is of the order of only one σ_0 far from the origin. Because of this, the approximation done in [16, 17] of considering for a non-zero f_{NL} the variance at $f_{\text{NL}} = 0$ is numerically justified, with a small error at the percent level, well beneath the error coming from other sources, for example from the uncertainty in the cosmological parameters, which gives an error on the variance of order ten percent [16].

Summarizing, we conclude that the enhanced terms will *not* be important until there is a clear detection of a non-zero f_{NL} . If that happens, they will have to be taken into account. At that point, the variance of the estimator \mathcal{E} will begin to decrease as $1/\ln^2 N_{\text{pix}}$. Given the very slow convergence of \mathcal{E} in this regime, one is led to wonder whether a better estimator exists.

3 Likelihood Calculation

For non-Gaussianities of the local type, it is easy to calculate the full Likelihood for f_{NL} given the data and determine to what extent the data are able to constrain f_{NL} . This is true even in the high signal to noise limit where the previous estimator has an increased variance.

With the full Likelihood it is possible to determine what is the minimum variance that an estimator of f_{NL} can have, the so called Cramer-Rao bound. The bound on the variance is $\langle \partial^2 \mathcal{L} / \partial f_{\text{NL}}^2 \rangle^{-1}$, where \mathcal{L} is minus the logarithm of the Likelihood. In [20] it was proved that the estimator \mathcal{E} of the former section, whose variance scales as $1/(N_{\text{pix}} \ln N_{\text{pix}})$, satisfies this bound at order zero in $f_{\text{NL}} A^{1/2}$. However, we have just learned that this expansion in powers of $f_{\text{NL}} A^{1/2}$ breaks down when f_{NL} is detected because of the presence of enhanced terms. It is therefore worth asking what happens to the Cramer-Rao bound in the same regime, and check if there are enhanced terms also in this case.

By the end of this section, we will see that the Likelihood allows for an expansion in powers of $f_{\text{NL}}^2 A$, without N_{pix} enhancements, and therefore that the Cramer-Rao bound in the presence of a non-null f_{NL} is affected only marginally by terms suppressed by powers of $f_{\text{NL}}^2 A$. This will tell us that the estimator \mathcal{E} of the previous section is just a bad estimator in the large signal to noise regime, and that in principle there can be estimators whose variance in this regime scales as $1/(N_{\text{pix}} \ln N_{\text{pix}})$.

Full Likelihood and Cramer-Rao bound: leading terms. The Likelihood function can be simply obtained inverting eq. (7), and expressing the probability for the Gaussian variables

g as a function of the temperature field Φ :

$$g_\theta = f^{-1}(\Phi_\theta) = \Phi_\theta - \tilde{f}_{\text{NL}} (\Phi_\theta^2 - \sigma^2) + 2\tilde{f}_{\text{NL}}^2 \Phi_\theta (\Phi_\theta^2 - \sigma^2) + \dots \quad (18)$$

The Likelihood will be a function of the parameter \tilde{f}_{NL} , while we keep f_{NL} to denote the true value of the non-Gaussianity parameter. The dots represents higher order terms in \tilde{f}_{NL} coming from the inversion of the function $f(g_\theta)$, which for the moment we neglect. We will come back to them shortly. In Fourier space, expression (18) translates into:

$$g_l = (f^{-1}(\Phi))_l = \Phi_l - \tilde{f}_{\text{NL}} ((\Phi \circ \Phi)_l - \sigma^2 \Omega \delta_{l,0}) + 2\tilde{f}_{\text{NL}}^2 ((\Phi \circ \Phi \circ \Phi)_l - \sigma^2 \Phi_l) + \dots \quad (19)$$

Starting from minus the logarithm of the probability

$$\mathcal{L}_g = \frac{1}{2} \sum_{l_1 l_2} C_{l_1 l_2}^{-1} g_{l_1} g_{l_2} \quad (20)$$

we change variable from g_l to Φ_l taking into account the change in the measure:

$$\mathcal{L} = \frac{1}{2} \sum_{l_1 l_2} C_{l_1 l_2}^{-1} (f^{-1}(\Phi))_{l_1} (f^{-1}(\Phi))_{l_2} - \text{Tr} \ln (\text{J}) \quad (21)$$

where Tr stays for trace in Fourier space, and J is the Jacobian

$$\text{J} = \frac{\partial (f^{-1}(\Phi))_{l_1}}{\partial \Phi_{l_2}} \quad (22)$$

We can now expand to second order in \tilde{f}_{NL} to obtain:

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \sum_l \left(\frac{1}{\Omega C_l} \left(\Phi_l \Phi_{-l} - 2\tilde{f}_{\text{NL}} \chi_l \Phi_{-l} + \tilde{f}_{\text{NL}}^2 (\chi_l \chi_{-l} + 4\Phi_l \eta_{-l}) \right) \right) \\ & + 2\tilde{f}_{\text{NL}} \frac{N_{\text{pix}}}{\Omega} \Phi_{l=0} - 4\tilde{f}_{\text{NL}}^2 \frac{N_{\text{pix}}}{\Omega} \chi_{l=0} - 2\tilde{f}_{\text{NL}}^2 N_{\text{pix}} \sigma^2, \end{aligned} \quad (23)$$

where we have introduced the field $\eta_l = (\chi \circ \Phi)_l$ [¶]. Although the Likelihood contains all the information on the parameter f_{NL} one can derive from an experiment, its computation as a function of \tilde{f}_{NL} can be very challenging in practice. All analysis to date have used an estimator of f_{NL} rather than to calculate the full Likelihood (e.g. [16, 17]).

The statistical properties of \mathcal{L} depend on the underlying true value f_{NL} . To make this explicit we can write

$$\Phi_l = g_l + f_{\text{NL}} ((g \circ g)_l - \Omega \sigma^2 \delta_{l,0}) \quad (24)$$

Plugging back in the expression for the Likelihood, we obtain:

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \sum_l \frac{1}{\Omega C_l} \left(g_l g_{-l} + 2(f_{\text{NL}} - \tilde{f}_{\text{NL}}) (\tilde{\chi}_l g_{-l} - 2f_{\text{NL}} (g_l \tilde{\eta}_{-l} - \langle g_l \tilde{\eta}_{-l} \rangle)) \right) \\ & + \left(f_{\text{NL}} - \tilde{f}_{\text{NL}} \right)^2 \left(\tilde{\chi}_l \tilde{\chi}_{-l} + 2g_l \tilde{\eta}_{-l} + 2(g_l \tilde{\eta}_{-l} - \langle g_l \tilde{\eta}_{-l} \rangle) \right) \\ & + 2\tilde{f}_{\text{NL}} \frac{N_{\text{pix}}}{\Omega} g_{l=0} + \left(2f_{\text{NL}} \tilde{f}_{\text{NL}} - 4\tilde{f}_{\text{NL}}^2 \right) \frac{N_{\text{pix}}}{\Omega} \tilde{\chi}_{l=0} - 2f_{\text{NL}}^2 N_{\text{pix}} \sigma^2, \end{aligned} \quad (25)$$

[¶]Notice that in this section, to keep the formulas as simple as possible, we have assumed that the average of Φ in the patch of the sky survey is observed. As it will become clear later, even if this was not the case, it would not change relevantly the results.

where we have analogously defined $\tilde{\chi}_l = (g \circ g)_l - \Omega \sigma^2 \delta_{l,0}$, and $\tilde{\eta}_l = (\tilde{\chi} \circ g)_l$.

We can use the expression of the Likelihood we have just derived to find the Cramer-Rao bound on the variance of an unbiased estimator for f_{NL} :

$$\left\langle \frac{\partial^2 \mathcal{L}}{\partial \tilde{f}_{\text{NL}}^2} \right\rangle^{-1} = \left(\sum_l \frac{1}{\Omega C_l} \langle \tilde{\chi}_l \tilde{\chi}_{-l} + 2g_l \tilde{\eta}_{-l} \rangle \right)^{-1} = \frac{1}{8N_{\text{pix}} \sigma^2}, \quad (26)$$

where we have used that:

$$\begin{aligned} \sum_l \frac{1}{\Omega C_l} \langle g_l \tilde{\eta}_{-l} \rangle &= \sum_l \frac{1}{\Omega C_l} \langle g_l \frac{1}{\Omega} \sum_{l'} g_{-l-l'} \left(\sum_{l''} g_{l'-l''} g_{l''} - \Omega \sigma^2 \delta_{l',0} \right) \rangle \\ &= \sum_{ll'l''} \frac{2}{\Omega^2 C_l} \Omega C_l \delta_{l,-l'+l''} \Omega C_{l''} \delta_{-l-l',-l''} = 2 \frac{N_{\text{pix}}}{\Omega} \sum_l C_l = 2N_{\text{pix}} \sigma^2, \end{aligned} \quad (27)$$

and analogously:

$$\sum_l \frac{1}{\Omega C_l} \langle \tilde{\chi}_l \tilde{\chi}_{-l} \rangle = 4N_{\text{pix}} \sigma^2. \quad (28)$$

We see from eq. (13) that the estimator of the last section saturates the Cramer-Rao bound for sufficiently small f_{NL} .

The expansion of the Likelihood to second order is consistent. At this point, one may wonder if the higher order terms in f_{NL} might relevantly alter this result with terms that, though suppressed by powers of f_{NL} , are enhanced by factors of N_{pix} , as in the former section for the variance of the estimator \mathcal{E} . It is quite straightforward to check that this is not the case. For example, at quartic level in f_{NL} there are terms like:

$$\begin{aligned} \mathcal{L} \supset f_{\text{NL}}^4 \sum_l \frac{1}{\Omega C_l} (g \circ g \circ g)_l (g \circ g \circ g)_{-l} \\ = f_{\text{NL}}^4 \frac{1}{\Omega^5} \sum_{l l_1 l_2 \tilde{l}_1 \tilde{l}_2} g_{l-l_1-l_2} g_{l_1} g_{l_2} \frac{1}{C_l} g_{-l-\tilde{l}_1-\tilde{l}_2} g_{\tilde{l}_1} g_{\tilde{l}_2}, \end{aligned} \quad (29)$$

whose expectation value contributes to the Cramer-Rao bound in eq. (26) with terms like:

$$f_{\text{NL}}^2 \sum_{l l_1 l_2} \frac{1}{\Omega^2} \frac{C_{l_1} C_{l_2} C_{l-l_1-l_2}}{C_l} = f_{\text{NL}}^2 \sum_{l_1 l_2 l_3} \frac{1}{\Omega^2} \frac{C_{l_1} C_{l_2} C_{l_3}}{C_{l_1+l_2+l_3}} \propto f_{\text{NL}}^2 \sigma^4 N_{\text{pix}}. \quad (30)$$

We recognize this term as not being enhanced also thanks to our thumb rule according to which a term is not enhanced if there are no C_l s raised to a power larger than one. We conclude that for these terms there is no N_{pix} enhancement, and therefore are suppressed by genuine powers of $f_{\text{NL}}^2 A$ with respect to the leading terms in the Cramer-Rao bound in eq. (26). Higher order terms will appear in even powers of f_{NL} , and will give similar contributions of the form:

$$f_{\text{NL}}^{2n-2} \sum_{l_1 l_2 \dots l_{n+1}} \frac{1}{\Omega^n} \frac{C_{l_1} C_{l_2} \dots C_{l_{n+1}}}{C_{l_1+l_2+\dots+l_{n+1}}} \propto f_{\text{NL}}^{2n-2} \sigma^{2n} N_{\text{pix}}, \quad (31)$$

so that we see there is no N_{pix} enhancement for all these terms. At quartic level in f_{NL} there are also terms appearing from the expansion of the Jacobian in the Likelihood, as for example:

$$f_{\text{NL}}^4 \frac{N_{\text{pix}}}{\Omega^4} \sum_{l_1 l_2 l_3} g_{l_1} g_{l_2} g_{l_3} g_{-l_1-l_2-l_3}, \quad (32)$$

whose expectation value gives subleading not-enhanced terms to the Cramer-Rao bound of the form:

$$f_{\text{NL}}^2 \frac{N_{\text{pix}}}{\Omega^2} \sum_{l_1 l_2} C_{l_1} C_{l_2} = f_{\text{NL}}^2 \frac{N_{\text{pix}}}{\Omega^2} \left(\sum_l C_l \right)^2 \propto f_{\text{NL}}^2 \sigma^4 N_{\text{pix}}, \quad (33)$$

and the same conclusion applies unaltered to the higher order terms coming from the expansion of the Jacobian:

$$f_{\text{NL}}^{2n-2} \frac{N_{\text{pix}}}{\Omega^n} \sum_{l_1 l_3 \dots l_{2n-1}} C_{l_1} C_{l_3} \dots C_{l_{2n-1}} = f_{\text{NL}}^{2n-2} \frac{N_{\text{pix}}}{\Omega^n} \left(\sum_l C_l \right)^n \propto f_{\text{NL}}^{2n-2} \sigma^{2n} N_{\text{pix}}. \quad (34)$$

We conclude that these higher order terms, being not enhanced by N_{pix} , do not alter significantly the Cramer-Rao bound in eq. (26).

In order to verify the consistency of the expansion at quadratic order in f_{NL} for the Likelihood as well, we need to check that the higher order terms are irrelevant not only on average, but also on each realization. In appendix A, we show that their variance scales at most as N_{pix}^2 , which makes their contribution to the Likelihood suppressed by powers of $f_{\text{NL}} A^{1/2}$ with respect to the contribution of the quadratic terms. We conclude that the expansion of the Likelihood up to quadratic order is consistent.

Higher order terms in the relation between Φ and g are negligible. Since in this section we have been very careful in keeping track of higher order terms in f_{NL} , it is useful to comment on the possibility that additional contributions come from the presence of higher order terms in the relation between Φ and the underlying Gaussian field of eq. (7):

$$\Phi_\theta = g_\theta + f_{\text{NL}}(g_\theta^2 - \sigma^2) + \alpha f_{\text{NL}}^2 g_\theta (g_\theta^2 - \sigma^2) + \dots \quad (35)$$

with α an unknown real parameter. Physically we expect these corrections to be there with α of order one. We need the Likelihood at second order in f_{NL} , so that one might worry that our results now depend on α . This would be very strange as physically these third order terms in the expansion above are a very small correction to the, already small, second order terms. The data should not be sensitive at all to α . One can check that this is indeed what happens. We leave the details of the algebra to appendix B. We will find that, although α enters into the Likelihood at order f_{NL}^2 , terms containing α cancel on each realization up to terms suppressed by $1/N_{\text{pix}}$. Therefore there is no sensitivity to α and it can safely be set to zero.

In summary, we have written the Likelihood for f_{NL} up to second order in f_{NL} , proving that this expansion is consistent, with the higher order terms only giving negligible contributions

suppressed by powers of $f_{\text{NL}}A^{1/2}$. This has also allowed us to verify that higher order terms in f_{NL} in the Likelihood do not give rise to enhanced contributions to the Cramer-Rao bound. Thus we conclude that the result in eq. (26) is only corrected by terms of order $f_{\text{NL}}A^{1/2}$, without N_{pix} enhancement. After what observed in the previous section, this was not *a priori* guaranteed.

4 No need to worry: simple estimators can saturate the Cramer-Rao bound

Now that we have determined the Cramer-Rao bound, in this section we look for a new estimator which continues to have a variance close to the bound even in the high signal-to-noise regime. We do this to further understand the origin of the enhanced terms and point out how a simple change in the estimator based on our intuitive understanding can make the estimator saturate the bound.

We will start from the original estimator \mathcal{E} in sec. 2, and we will explicitly show the way in which the enhanced terms cause the slow convergence of the estimator in the large signal-to-noise regime. After this it will become easy to guess a new estimator that, apart from small corrections, saturates the Cramer-Rao bound when the enhanced terms are large.

Explicit origin of the increased variance. Let us therefore start from the estimator \mathcal{E} in eq. (11) and express it in terms of Gaussian variables as:

$$\mathcal{E} = \frac{1}{N\Omega} \sum_l \frac{1}{C_l} (g_l \tilde{\chi}_{-l} + f_{\text{NL}} (\tilde{\chi}_l \tilde{\chi}_{-l} + 2g_l \tilde{\eta}_{-l})) \equiv \mathcal{E}_0 + f_{\text{NL}} \mathcal{E}_1 + \dots \quad (36)$$

where $+\dots$ represent higher order terms in f_{NL} , and where we have defined:

$$\mathcal{E}_0 = \frac{1}{N\Omega} \sum_l \frac{1}{C_l} g_l \tilde{\chi}_{-l}, \quad (37)$$

$$\mathcal{E}_1 = \frac{1}{N\Omega} \sum_l \frac{1}{C_l} (\tilde{\chi}_l \tilde{\chi}_{-l} + 2g_l \tilde{\eta}_{-l}). \quad (38)$$

Notice that $\langle \mathcal{E}_0 \mathcal{E}_1 \rangle = 0$, $\langle \mathcal{E}_0 \rangle = 0$, and $\langle \mathcal{E}_1 \rangle = 1$. Therefore the variance of the estimator can be written as:

$$\langle \Delta \mathcal{E}^2 \rangle = \langle \mathcal{E}_0^2 \rangle + f_{\text{NL}}^2 \langle \Delta \mathcal{E}_1^2 \rangle. \quad (39)$$

As we discussed, the variance at zeroth order in f_{NL} is

$$\langle \Delta \mathcal{E}^2 \rangle_{f_{\text{NL}}=0} = \langle \mathcal{E}_0^2 \rangle = \frac{1}{8N_{\text{pix}}\sigma^2} \quad (40)$$

which saturates the Cramer-Rao bound in (26). However, as we noted in sec. 2, the variance of \mathcal{E}_1 behaves like:

$$\langle \Delta \mathcal{E}_1^2 \rangle \sim \frac{2}{\pi \ln^2 N_{\text{pix}}}, \quad (41)$$

decreasing only logarithmically. Therefore it is going to dominate the variance of the estimator for $N_{\text{pix}}/\ln N_{\text{pix}} \gtrsim 1/(f_{\text{NL}}^2 A)$. Apart for the logarithm, this is when the signal-to-noise becomes of order 1.

The enhanced variance of \mathcal{E}_1 could have been anticipated. This term contains, at leading order in f_{NL} , all the signal of non-Gaussianity. We know that for the local kind of non-Gaussianity the signal comes from squeezed configurations with two of the three l s very large, and one of the l s very small. The contribution to \mathcal{E}_1 of squeezed configurations with the smallest of the l s within a certain decade is roughly independent of the decade, although there are very few low l multipoles. *The value of \mathcal{E}_1 on a given realization depends quite strongly on the particular value of the few lowest multipoles*; this explains why it converges to its average $\langle \mathcal{E}_1 \rangle = 1$ very slowly ^{||}. Progressively the contribution of the first decades of modes becomes negligible, so that the dependence on the particular value of the lowest multipoles goes away. However this happens only logarithmically in N_{pix} as this is the way in which the contribution from the lowest multipoles decays.

Improved Estimator. Now that we understand better the problem of the estimator \mathcal{E} , it is easy to find an improved estimator for the large signal-to-noise regime. We can think about the large variance of \mathcal{E} as coming from a “wrong normalization”. Although the estimator is clearly unbiased its value strongly depends on the amplitude of the low l modes, so that if on a particular realization we have a small amplitude in the first multipoles, the value of the estimator will be small and viceversa. This effect cancels on average (that is why the estimator is unbiased) but it is the source of the large variance. Anyway this effect can clearly be corrected as we surely know the amplitude of the low l modes in each particular realization: we just have to divide by a “realization dependent” normalization. We define a new estimator $\tilde{\mathcal{E}}$

$$\begin{aligned} \tilde{\mathcal{E}} &= \frac{N\Omega}{\sum_l \frac{1}{\tilde{c}_l} (\chi_l \chi_{-l} + 2\Phi_l \eta_{-l})} \mathcal{E} = \frac{\sum_l \frac{1}{\tilde{c}_l} \Phi_l \chi_{-l}}{\sum_l \frac{1}{\tilde{c}_l} (\chi_l \chi_{-l} + 2\Phi_l \eta_{-l})} = \\ &= f_{\text{NL}} + \frac{\sum_l \frac{1}{\tilde{c}_l} \tilde{g}_l \chi_{-l}}{\sum_l \frac{1}{\tilde{c}_l} (\tilde{\chi}_l \tilde{\chi}_{-l} + 2g_l \tilde{\eta}_{-l})} + \dots = f_{\text{NL}} + \frac{\mathcal{E}_0}{\mathcal{E}_1} + \dots \end{aligned} \quad (42)$$

where in the second line we have expressed everything in terms of Gaussian variables. Neglected terms are suppressed with respect to the ones we kept by genuine powers of $f_{\text{NL}} A^{1/2}$. Neglecting these terms, the new estimator $\tilde{\mathcal{E}}$ is unbiased: $\langle \tilde{\mathcal{E}} \rangle = f_{\text{NL}}$, as $\mathcal{E}_0/\mathcal{E}_1$ is an odd function of the Gaussian variables g and it has thus zero average.

We can now verify that the new estimator converges to the Cramer-Rao bound. We can write $\mathcal{E}_1 = 1 + \delta\mathcal{E}_1$, where $\delta\mathcal{E}_1$ is of the order of $\langle \delta\mathcal{E}_1^2 \rangle^{1/2} \sim 2^{1/2}/(\pi^{1/2} \ln N_{\text{pix}})$. For large N_{pix} we

^{||}As shown in appendix A, terms in \mathcal{E} of higher order in f_{NL} , even including a possible contribution from terms proportional to α , contribute to the variance of the estimator with terms which are not enhanced by powers of N_{pix} more than what \mathcal{E}_1 already is. Therefore they are suppressed with respect to the contribution of \mathcal{E}_1 by a genuine power of $f_{\text{NL}} A^{1/2}$.

can thus expand the denominator

$$\tilde{\mathcal{E}} \simeq f_{\text{NL}} + \mathcal{E}_0 - \mathcal{E}_0 \delta \mathcal{E}_1 . \quad (43)$$

The variance introduced by the third piece scales like $1/(\text{N}_{\text{pix}} \ln^3 \text{N}_{\text{pix}})$, more rapidly than the Cramer-Rao bound $\propto 1/(\text{N}_{\text{pix}} \ln \text{N}_{\text{pix}})$. After a while we are therefore left with the variance of \mathcal{E}_0 that, as we know, satisfies the Cramer-Rao bound. It is worth noticing that already at the level of the WMAP experiment $\ln^2 \text{N}_{\text{pix}} \simeq 35$, so the deviation of this estimator from the Cramer-Rao bound is already rather small. The important point is that this good behavior of $\tilde{\mathcal{E}}$ is not spoiled when we enter in the large signal-to-noise regime.

The improved normalization only depends on the large scales. Our understanding of the enhanced variance of the estimator \mathcal{E} relies on the fact that always a significant fraction of the signal is coming from low l modes with a great intrinsic variance. If this is true, it better be that the solution to this problem depends strongly on the low l s. Here we therefore verify that \mathcal{E}_1 can be written to good approximation in terms of just the first few modes. In appendix B we have shown that

$$S_2 = \sum_l \frac{1}{\Omega C_l} g_l \tilde{\eta}_{-l} \quad (44)$$

is fully correlated with the quantity

$$S_1 = \frac{\text{N}_{\text{pix}}}{\Omega^2} \sum_l g_l g_{-l} , \quad (45)$$

up to corrections $\mathcal{O}(1/\text{N}_{\text{pix}})$, so that on each realization $\Delta S_2 = 3\Delta S_1$ with very good accuracy. In the same fashion one can prove that also the quantity

$$S_3 = \sum_l \frac{1}{\Omega C_l} \tilde{\chi}_l \tilde{\chi}_{-l} \quad (46)$$

is fully correlated to S_1 (up to corrections $\mathcal{O}(1/\text{N}_{\text{pix}})$) and that $\Delta S_3 = 4\Delta S_1$. This implies that also \mathcal{E}_1 is fully correlated with S_1 and therefore, on each realization, we can write:

$$\mathcal{E}_1 = \langle \mathcal{E}_1 \rangle + \frac{10\Delta S_1}{N} . \quad (47)$$

Now, the important point is that in order to compute the quantity ΔS_1 on a given realization one needs, to good approximation, only the first few modes. This can be seen from the computation of the variance of S_1 in Appendix B:

$$\langle \Delta S_1^2 \rangle = 2 \frac{\text{N}_{\text{pix}}^2}{\Omega^2} \sum_l C_l^2 \simeq \frac{2\pi A^2 \text{N}_{\text{pix}}^2}{\Omega} \left(\frac{1}{l_{\text{min}}^2} - \frac{1}{l_{\text{max}}^2} \right) \quad (48)$$

which shows that the contribution to the variance of the high l s is completely irrelevant. We therefore conclude that the value of \mathcal{E}_1 on each realization can be determined just by looking at the first few modes, in agreement with our intuition.

This last remark has also relevant consequences from the computational point of view. In fact it seems at first sight very hard to use the new estimator in the analysis of CMB data, as it contains 4-point functions and one has to deal with the complications of the spherical geometry and of the transfer function. On the other hand the dependence on only the first few modes makes the modification computationally quite light.

Relation to the full Likelihood calculation. The Likelihood (23) contains all the information on the parameter f_{NL} . To reconstruct it from the data we just need the coefficients of the terms linear and quadratic in \tilde{f}_{NL} . *These two combinations of the data are sufficient statistics for f_{NL} .* Notice that given our discussion above it is not so complicated to analyze the full Likelihood function: we just need the same kind of terms entering in the estimator $\tilde{\mathcal{E}}$ above. A natural question is whether one can get better constraints on f_{NL} using the full Likelihood function instead of an estimator.

First of all it is straightforward to check that the maximum Likelihood estimator, which can be easily derived from eq. (23), has the same good properties of our improved estimator discussed above. It is unbiased up to corrections $\mathcal{O}(1/N_{\text{pix}}^{1/2})$ and it asymptotically saturates the Cramer-Rao bound up to corrections decaying as $1/\ln^2 N_{\text{pix}}$.

The Cramer-Rao bound, being the average value of the second derivative of the log-Likelihood, gives the average value of error bars that one gets. The difference in using the full Likelihood is that the curvature of it changes realization by realization, as it is given by the \tilde{f}_{NL}^2 term in eq. (23). Usually this distinction between the curvature of the Likelihood in a particular realization and its average value is irrelevant, as the difference scales like $1/N_{\text{pix}}$. This is not true in our case. The variance of the curvature of the Likelihood function only scales as $1/\ln^2 N_{\text{pix}}$. This is again intuitive: given the strong dependence on the lowest multipoles, a realization with an excess of power in the low ls compared with the average will be more constraining than one with suppressed power on large scales. In the first case in fact it is easier to see the non-Gaussian correlation between the low ls and the short scale power. This difference is anyway not that large: for real experiments that have a chance of detecting f_{NL} , $1/\ln N_{\text{pix}}$ is rather small.

We reach an important conclusion. *The use of our improved estimator, or equivalently the maximum Likelihood one, is equivalent to the full Likelihood of the data up to small corrections suppressed by $1/\ln N_{\text{pix}}$.* This closes the door to any additional attempt to improve the limits on f_{NL} .

5 Comments on estimating f_{NL} using the 4-point function

It has been proposed in [21] and more recently in [22] that an estimator for f_{NL} based on the 4-point function has a variance which decreases as $1/N_{\text{pix}}^2$, so that it might be better than a 3-point function estimator for large enough f_{NL} . Of course our analysis in the previous sections shows that the Cramer-Rao bound scales as $1/(N_{\text{pix}} \ln N_{\text{pix}})$, so that no estimator can do better

than this. Moreover we proved that a slight modification of the 3-point function estimator makes the new estimator $\tilde{\mathcal{E}}$ approach asymptotically the Cramer-Rao bound. Rather than stop here and rely on the above “theorems” we want to show explicitly in this section what goes wrong in the naive calculation of the variance of the 4-point function estimator. We will see that the $1/N_{\text{pix}}^2$ scaling of the variance does not hold once the signal-to-noise is larger than one and the “enhanced” terms are taken into account. We will also show explicitly that there is no additional information about f_{NL} in the 4-point function that is not already captured by the 3-point function.

The variance of the proposed 4-pt estimator has enhanced terms. Let us begin proving that the variance of the estimator introduced in [21, 22] does not scale as $1/N_{\text{pix}}^2$. The proposed estimator, analogously to the estimator \mathcal{E} in the 3-point function case, is the linear combination of 4-point correlators which maximizes the signal-to-noise in the limit $f_{\text{NL}} \rightarrow 0$:

$$\mathcal{E}_4 = \frac{1}{N_4} \sum_{l_1 l_2 l_3} \frac{\langle \Phi_{l_1} \Phi_{l_2} \Phi_{l_3} \Phi_{l_4} \rangle_{c,1}}{C_{l_1} C_{l_2} C_{l_3} C_{l_4}} \Phi_{l_1} \Phi_{l_2} \Phi_{l_3} \Phi_{l_4}, \quad (49)$$

where N_4 is a normalization constant which makes the estimator unbiased $\langle \mathcal{E}_4 \rangle = f_{\text{NL}}^2$. The sum is restricted to momentum conserving, $l_4 = -l_1 - l_2 - l_3$, non-degenerate quadrilaterals. The subscript $_1$ in $\langle \Phi_{l_1} \dots \Phi_{l_4} \rangle_{c,1}$ means that the connected 4-point function is evaluated with $f_{\text{NL}} = 1$. The variance of this estimator is:

$$\begin{aligned} \langle \Delta \mathcal{E}_4^2 \rangle = & \quad (50) \\ & \frac{\sum_{l_1 l_2 l_3} \sum_{\tilde{l}_1 \tilde{l}_2 \tilde{l}_3} \frac{\langle \Phi_{l_1} \Phi_{l_2} \Phi_{l_3} \Phi_{l_4} \rangle_{c,1}}{C_{l_1} C_{l_2} C_{l_3} C_{l_4}} \frac{\langle \Phi_{\tilde{l}_1} \Phi_{\tilde{l}_2} \Phi_{\tilde{l}_3} \Phi_{\tilde{l}_4} \rangle_{c,1}}{C_{\tilde{l}_1} C_{\tilde{l}_2} C_{\tilde{l}_3} C_{\tilde{l}_4}} \langle \Phi_{l_1} \Phi_{l_2} \Phi_{l_3} \Phi_{l_4} \Phi_{\tilde{l}_1} \Phi_{\tilde{l}_2} \Phi_{\tilde{l}_3} \Phi_{\tilde{l}_4} \rangle}{\left(\sum_{l_1 l_2 l_3} \frac{\langle \Phi_{l_1} \Phi_{l_2} \Phi_{l_3} \Phi_{l_4} \rangle_{c,1}^2}{C_{l_1} C_{l_2} C_{l_3} C_{l_4}} \right)^2} - f_{\text{NL}}^4. \end{aligned}$$

Here we are interested in the scaling with N_{pix} of the different terms, therefore we do not keep track of the various combinatorial and numerical factors, and also of possible logarithmic corrections. Using the fact that the connected 4-point function behaves, at leading order in f_{NL} , like:

$$\langle \Phi_{l_1} \Phi_{l_2} \Phi_{l_3} \Phi_{l_4} \rangle_c \sim f_{\text{NL}}^2 \cdot C_{l_1} C_{l_1+l_2} C_{l_4} + \text{symm}, \quad (51)$$

we find that the denominator of the first term contains terms that behave like:

$$\sum_{l_1 l_2 l_3} \frac{\langle \Phi_{l_1} \Phi_{l_2} \Phi_{l_3} \Phi_{l_4} \rangle_{c,1}^2}{C_{l_1} C_{l_2} C_{l_3} C_{l_4}} \sim \sum_{l_1 l_2 l_3} \frac{C_{l_1}^2 C_{l_1+l_2}^2 C_{l_4}^2}{C_{l_1} C_{l_2} C_{l_3} C_{l_4}} \simeq \sum_{l_1 l_2 l_3} \frac{C_{l_1} C_{l_1+l_2} C_{l_4}}{C_{l_2} C_{l_3}}. \quad (52)$$

We already met these kind of summations and we know they are dominated by $l_1 + l_2 \sim l_{\text{min}}$ and enhanced by a factor of N_{pix} with respect to the naive scaling.

$$\sum_{l_1 l_2 l_3} \frac{\langle \Phi_{l_1} \Phi_{l_2} \Phi_{l_3} \Phi_{l_4} \rangle_{c,1}^2}{C_{l_1} C_{l_2} C_{l_3} C_{l_4}} \sim N_{\text{pix}}^2 \sum_l C_l^2 \propto N_{\text{pix}}^2 A^2. \quad (53)$$

For the numerator we have to compute the 8-point function, which in general will be the sum of the product of four 2-point function, of two connected 3-point functions and one 2-point function, of two connected 4-point functions, and so on. At zeroth order in f_{NL} , we have only 2-point functions and we obtain something which scales as the square root of the denominator. Therefore, if ones stops at this level, one finds that the variance decreases as $1/N_{\text{pix}}^2$. This is the result obtained in [21] and [22].

However, as we learned in sec. 2, there are other terms in the numerator which, though suppressed by powers of $f_{\text{NL}}^2 A$, are enhanced by powers of N_{pix} . It turns out that the product of two 4-point functions, of the 5-point with the 3-point one, and of the 6-point with the 2-point one, all give rise to enhanced terms with the same scaling. For example a term with two connected 4-point functions is

$$\begin{aligned} & \sum_{l_1 l_2 l_3 \tilde{l}_1 \tilde{l}_2 \tilde{l}_3} \frac{\langle \Phi_{l_1} \Phi_{l_2} \Phi_{l_3} \Phi_{l_4} \rangle_{c,1}}{C_{l_1} C_{l_2} C_{l_3} C_{l_4}} \frac{\langle \Phi_{\tilde{l}_1} \Phi_{\tilde{l}_2} \Phi_{\tilde{l}_3} \Phi_{\tilde{l}_4} \rangle_{c,1}}{C_{\tilde{l}_1} C_{\tilde{l}_2} C_{\tilde{l}_3} C_{\tilde{l}_4}} \langle \Phi_{l_1} \Phi_{l_2} \Phi_{\tilde{l}_3} \Phi_{\tilde{l}_4} \rangle_c \langle \Phi_{\tilde{l}_1} \Phi_{\tilde{l}_2} \Phi_{l_3} \Phi_{l_4} \rangle_c \\ & \simeq f_{\text{NL}}^4 \sum_{l_1 l_2 l_3 \tilde{l}_1 \tilde{l}_2 \tilde{l}_3} \frac{C_{l_1} C_{l_4} C_{l_1+l_2}^2}{C_{l_2} C_{l_3}} \frac{C_{\tilde{l}_1} C_{\tilde{l}_4} C_{\tilde{l}_1+\tilde{l}_2}^2}{C_{\tilde{l}_2} C_{\tilde{l}_3}} + \dots \end{aligned} \quad (54)$$

where $+\dots$ represents terms of higher order in f_{NL} . The sums will be dominated by the region with $l_1 + l_2 \sim l_{\text{min}}$ and $\tilde{l}_1 + \tilde{l}_2 \sim l_{\text{min}}$, so that we obtain

$$f_{\text{NL}}^4 N_{\text{pix}}^4 \left(\sum_l C_l^2 \right)^2 \propto f_{\text{NL}}^4 A^4 N_{\text{pix}}^4 \quad (55)$$

as we wanted to show. Putting together the behavior of the different terms we get the scaling of the variance of the proposed estimator up to logarithmic corrections

$$\langle \Delta \mathcal{E}_4^2 \rangle \sim \frac{1 + f_{\text{NL}}^4 A^2 N_{\text{pix}}^2}{A^2 N_{\text{pix}}^2}, \quad (56)$$

where we see the importance of enhanced terms at numerator. For comparison, it is useful to write the same schematic relation for the analogous 3-point function estimator \mathcal{E} we discussed in sec. 2:

$$\langle \Delta \mathcal{E}^2 \rangle \sim \frac{1 + f_{\text{NL}}^2 A N_{\text{pix}}}{A N_{\text{pix}}}. \quad (57)$$

We notice that the enhanced terms become parametrically important for both the estimators when $f_{\text{NL}}^2 A N_{\text{pix}} \sim 1$ which is, apart for logarithms, the regime when the signal-to-noise is of order one.

Is there additional information in the 4-point function? In the limit in which the enhanced terms are negligible, we see that the variance of \mathcal{E}_4 is of the order of the square of the variance of \mathcal{E} , which means that \mathcal{E}_4 will give parametrically the same limits on f_{NL} in this regime. However, in the previous sections, we have shown that in the same limit the 3-point function estimator \mathcal{E} saturates the Cramer-Rao bound; therefore in this regime there is nothing

which could be added by the use of the 4-point function. This agrees with the numerical result of [22], where it is shown that in this regime the limit on f_{NL} obtained from the 4-point function estimator is always slightly worse than the one obtained using the 3-point function. Given the Cramer-Rao bound, we can even say something more: in this regime no improvement can be achieved from combining the two estimators.

On the other hand when the enhanced terms become important we see that the variance of both estimators does not decrease anymore (apart for logarithmic terms). In particular there is no $1/N_{\text{pix}}^2$ scaling for the 4-point function. As it was shown in sec. 4, in this regime one must consider “fractional” estimators, which are not just polynomial in the data.

Explicit relation between the 3-point and 4-point estimators. It is worth pointing out an explicit relationship between the 4-point and 3-point function estimators, to show that there is really nothing new in the 4-point estimator \mathcal{E}_4 , which is not already taken into account using \mathcal{E} .

Let us remind once again that, for local non-Gaussianities, the signal-to-noise of the 3-point function is concentrated on squeezed configurations, where one of the three l s is small, and the other two are large and almost opposite. The 3-point function estimator \mathcal{E} is basically doing a weighted sum of the signal contained in all the configurations, where the weight is the signal-to-noise ratio. Therefore, the estimator \mathcal{E} can be well approximated by a sum over just the squeezed configurations:

$$\mathcal{E} \propto \sum_L \Phi_L \sum_l \frac{\Phi_{-L-l}\Phi_l}{C_l} \equiv \sum_L \Phi_L \mathcal{K}_{-L} \quad (58)$$

where L is a large scale multipole and l is a small scale one, and \mathcal{K}_L is defined as:

$$\mathcal{K}_L = \sum_l \frac{\Phi_{-L-l}\Phi_l}{C_l}. \quad (59)$$

In the presence of a non-zero f_{NL} , \mathcal{K}_L will contain, when expressed in terms of Gaussian variables a contribution

$$\mathcal{K}_L \sim 2f_{\text{NL}} g_L. \quad (60)$$

The estimator \mathcal{E} correlates this contribution with the long-wave mode Φ_{-L} .

Analogously in the case of the estimator \mathcal{E}_4 , as we have seen, the signal comes from squeezed configurations, where the four vectors l s are approximately opposite in pairs. Therefore, the estimator \mathcal{E}_4 can be written to a good approximation as a sum over just these configurations:

$$\mathcal{E}_4 \propto \sum_L C_L \mathcal{K}_L \mathcal{K}_{-L}. \quad (61)$$

Again in the presence of a non-zero f_{NL} we are correlating the non-Gaussian contribution inside each of the \mathcal{K} 's, giving an average signal $\propto f_{\text{NL}}^2$.

From this it should be clear that the two estimators are clearly not independent and that the 4-point one is less efficient because the non-Gaussian contribution must come out of both the \mathcal{K} 's, while it is more efficient to directly correlate \mathcal{K}_L with the mode Φ_{-L} as in eq. (58).

6 Summary

It is perhaps unfortunate that our paper is filled with so many equations, the message however is simple. The analysis of the local type of non-Gaussianity for scale invariant perturbations is somewhat more subtle than one might have guessed: a naive $f_{\text{NL}}A^{1/2}$ expansion is not always appropriate. The physical origin of the effect is clear: long wavelength modes modulate the amplitude of the short wavelengths and the amplitude of this modulation produced by long wavelengths of every decade in scale is the same. Cosmic variance severely affects this long wavelengths and because their relative information contribution only decreases logarithmically with the number of pixels, one ends up with large variances for the naive f_{NL} estimators.

The basic point is that when one calculates the normalization of the f_{NL} estimator one uses the average level of large scale fluctuations as opposed to the power in the individual realization one happens to have. As a result, in the limit of large signal to noise, this relatively large uncertainty in the normalization of the estimator severely enhances its variance. Fortunately one knows the amplitude of the modes in a given realization by direct measurement so it is almost trivial to fix the problem by choosing a normalization that depends on the particular realization.

Writing down the full Likelihood one can explicitly calculate how well one should in principle be able to constrain f_{NL} and explicitly check how the effect mentioned above comes in. One can show that a simple modification of the naive estimator recovers all the information that the data contain and that in fact using that estimator is basically equivalent to calculating the full Likelihood, up to corrections $\mathcal{O}(1/\ln N_{\text{pix}})$. As a result, one is also convinced that other statistics such as the 4-point function, Minkowski functionals, wavelets, etc can at best extract as much information on f_{NL} as the 3-point function. In any event they would not contribute *additional* information on f_{NL} , so once the 3-point function is measured there is nothing else to be done.

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Appendices

A Proof that the enhancement is at most of order N_{pix}

In this appendix we want to prove that the variance of the sums that appear in the Likelihood and in the estimators scales at most as N_{pix}^2 , i.e. that the possible enhancement with respect to the naive scaling is at most of order N_{pix} . In order to do this, following the discussion in sec. 2, where we explained the thumb rule for discovering enhanced terms, it is enough to show that in the expression in Fourier space of the variance there is at most one C_l raised at most to the power of two. This corresponds to an enhancement of one factor of N_{pix} , while further enhancements would require either one C_l raised to a power larger than two, or more than one C_l squared.

A good rearrangement of the various terms is obtained if we start in real space, where the terms we are interested in can be written in the general form:

$$\sum_{\theta_1\theta_2} C_{\theta_1\theta_2}^{-1} g_{\theta_1}^M g_{\theta_2}^N, \quad (62)$$

where M and N are two positive integers, and $C_{\theta_i\theta_j}$ is the covariance matrix in real space. This is related to the one in Fourier space by the following relation:

$$C_{\theta_i\theta_j} = \frac{1}{\Omega} \sum_l C_l e^{il\cdot(\theta_i-\theta_j)}. \quad (63)$$

Analogously, C^{-1} can be expressed in real space as:

$$C_{\theta_i\theta_j}^{-1} = \frac{\Omega}{N_{\text{pix}}^2} \sum_l \frac{1}{C_l} e^{il\cdot(\theta_i-\theta_j)}. \quad (64)$$

Let us compute the variance of the general term in eq. (62). This will be the sum of terms of the form:

$$\sum_{\theta_1\theta_2\theta_3\theta_4} C_{\theta_1\theta_2}^{-1} C_{\theta_3\theta_4}^{-1} (C_{\theta_1\theta_2})^\alpha (C_{\theta_1\theta_3})^\beta (C_{\theta_1\theta_4})^\gamma (C_{\theta_2\theta_3})^\delta (C_{\theta_2\theta_4})^\Sigma (C_{\theta_3\theta_4})^\rho, \quad (65)$$

with the positive integers $\alpha, \beta, \gamma, \delta, \Sigma, \rho$ constrained to satisfy $\alpha + \beta + \gamma + \delta + \Sigma + \rho = (N + M)/2$.

We can now express each of the C s and C^{-1} s in Fourier space with the relations (63) and (64). After this, the summation over the angles $\theta_1, \theta_2, \theta_3$ and θ_4 becomes trivial, each of these giving a Kronecker delta. In particular, the summations over θ_1 and θ_2 give the constraints:

$$l_1 + l_1^\alpha + \dots + l_\alpha^\alpha + l_1^\beta + \dots + l_\beta^\beta + l_1^\gamma + \dots + l_\gamma^\gamma = 0, \quad (66)$$

$$-l_1 - l_1^\alpha - \dots - l_\alpha^\alpha + l_1^\delta + \dots + l_\delta^\delta + l_1^\Sigma + \dots + l_\Sigma^\Sigma = 0 \quad (67)$$

where l_1 is the l associated to the Fourier transform of $C_{\theta_1\theta_2}^{-1}$, l_i^α , with $i = 1, \dots, \alpha$, are the l s associated to the Fourier transform of $(C_{\theta_1\theta_2})^\alpha$, and analogously for the other C s. These two constraints can be usefully rewritten as:

$$l_1 = -l_1^\alpha - \dots - l_\alpha^\alpha - l_1^\beta - \dots - l_\beta^\beta - l_1^\gamma - \dots - l_\gamma^\gamma, \quad (68)$$

$$l_1^\beta + \dots + l_\beta^\beta + l_1^\gamma + \dots + l_\gamma^\gamma + l_1^\delta + \dots + l_\delta^\delta + l_1^\Sigma + \dots + l_\Sigma^\Sigma = 0 . \quad (69)$$

From the summation over θ_3 and θ_4 , we obtain two analogous constraints that can be written as:

$$l_2 = l_1^\beta + \dots + l_\beta^\beta + l_1^\delta + \dots + l_\delta^\delta - l_1^\rho - \dots - l_\rho^\rho , \quad (70)$$

$$l_1^\beta + \dots + l_\beta^\beta + l_1^\gamma + \dots + l_\gamma^\gamma + l_1^\delta + \dots + l_\delta^\delta + l_1^\Sigma + \dots + l_\Sigma^\Sigma = 0 , \quad (71)$$

where l_2 is the l associated to the Fourier transform of $C_{\theta_3\theta_4}^{-1}$. We see that the second of these constraints is equivalent to the one in eq. (69). The presence of a redundant constraint is a manifestation of the fact that the term we started with in eq. (62) was rotationally invariant.

After the summation over the angles, we are left with summations only over the l s:

$$\sum_{l_1 l_2} \sum_{l_1^\alpha \dots l_\alpha^\alpha} \dots \sum_{l_1^\rho \dots l_\rho^\rho} \frac{(C_{l_1^\alpha} \dots C_{l_\alpha^\alpha}) \dots (C_{l_1^\rho} \dots C_{l_\rho^\rho})}{C_{l_1} C_{l_2}} , \quad (72)$$

subject to the three independent constraints we have found. Notice that the N_{pix} factors in (64) exactly cancel with the four ones from the sums over θ_i . Now, the first and third constraints can be used to eliminate the summations over l_1 and l_2 , leaving us with:

$$\sum_{l_1^\alpha \dots l_\alpha^\alpha} \dots \sum_{l_1^\rho \dots l_\rho^\rho} \frac{(C_{l_1^\alpha} \dots C_{l_\alpha^\alpha}) \dots (C_{l_1^\rho} \dots C_{l_\rho^\rho})}{C_{l_1} C_{l_2}} , \quad (73)$$

with l_1 and l_2 given by eq. (68) and (70). The only remaining constraint is eq. (69). After applying it, the variance is in the form such that we can quickly apply our thumb rule, and understand its level of enhancement. Naively given that the number of C_l s is $(N + M)/2 - 2$ and the number of summations is $(N + M)/2 - 1$ we get a behavior $\sim N_{\text{pix}}$. But it can happen that the last constraint makes two C_l s at numerator equal. In this case the sum goes as N_{pix}^2 . No further enhancement is possible.

B Higher order corrections in the definition of local non-Gaussianities

In Fourier space eq. (35) reads:

$$\Phi_l = g_l + f_{\text{NL}} \tilde{\chi}_l + \alpha f_{\text{NL}}^2 \tilde{\eta}_l . \quad (74)$$

In this appendix we want to prove that, although α enters in the Likelihood at order f_{NL}^2 , this does not imply that data are sensitive to this parameter, unless it is huge compared to the naive estimate $\alpha \sim \mathcal{O}(1)$. The new term gives the following contribution to the Likelihood at order f_{NL}^2 :

$$\mathcal{L}_\alpha = \alpha f_{\text{NL}}^2 \left(\sum_l -\frac{1}{\Omega C_l} (g_l \tilde{\eta}_{-l} - \langle g_l \tilde{\eta}_{-l} \rangle) + 3 \frac{N_{\text{pix}}}{\Omega} \tilde{\chi}_{l=0} \right) , \quad (75)$$

where we have neglected terms which are independent of \tilde{f}_{NL} . We notice that both terms above have zero average. However this is not enough to prove that there is no relevant dependence on α because both terms have enhanced variance, so that they converge to zero very slowly. Their importance with respect to the other terms in the Likelihood decreases as $1/\ln^2 N_{\text{pix}}$. What we are now going to prove is that, although both terms have large variance, they are strongly correlated and their contributions in eq. (75) cancel up to terms suppressed by $1/N_{\text{pix}}$. Therefore the dependence on α is extremely small as expected on physical grounds.

Defining:

$$S_1 = \frac{N_{\text{pix}}}{\Omega^2} \sum_l g_l g_{-l}, \quad S_2 = \sum_l \frac{1}{\Omega C_l} g_l \tilde{\eta}_{-l}, \quad (76)$$

we can write eq. (75) as:

$$\mathcal{L}_\alpha = \alpha \tilde{f}_{\text{NL}}^2 (-\Delta S_2 + 3\Delta S_1). \quad (77)$$

We can now compute the correlation functions of S_1 and S_2 :

$$\langle S_1 \rangle = \frac{N_{\text{pix}}}{\Omega^2} \sum_l \langle g_l g_{-l} \rangle = \frac{N_{\text{pix}}}{\Omega} \sum_l C_l, \quad (78)$$

$$\langle \Delta S_1^2 \rangle = \frac{N_{\text{pix}}^2}{\Omega^4} \sum_{l\bar{l}} \langle g_l g_{-l} g_{\bar{l}} g_{-\bar{l}} \rangle - \langle S_1 \rangle^2 = 2 \frac{N_{\text{pix}}^2}{\Omega^2} \sum_l C_l^2, \quad (79)$$

$$\langle S_2 \rangle = 2 \frac{N_{\text{pix}}}{\Omega} \sum_l C_l, \quad (80)$$

as computed before, and:

$$\begin{aligned} \langle \Delta S_2^2 \rangle &= \sum_{\tilde{l}\bar{l}} \frac{1}{\Omega C_l} \frac{1}{\Omega C_{\bar{l}}} \langle g_l \tilde{\eta}_{-l} g_{\bar{l}} \tilde{\eta}_{-\bar{l}} \rangle - \langle S_2 \rangle^2 \\ &= \sum_{\tilde{l}\bar{l}} \frac{1}{\Omega^2 C_l C_{\bar{l}}} \langle g_l \left(\frac{1}{\Omega} \sum_{\nu} g_{-l-\nu} \left(\frac{1}{\Omega} \sum_{\nu'} g_{\nu'-\nu} g_{\nu''} - \Omega \sigma^2 \delta_{\nu,0} \right) \right) \right. \\ &\quad \left. \times g_{\bar{l}} \left(\frac{1}{\Omega} \sum_{\tilde{\nu}} g_{-\bar{l}-\tilde{\nu}} \left(\frac{1}{\Omega} \sum_{\tilde{\nu}'} g_{\tilde{\nu}'-\tilde{\nu}''} g_{\tilde{\nu}''} - \Omega \sigma^2 \delta_{\tilde{\nu},0} \right) \right) \right\rangle - \langle S_2 \rangle^2. \end{aligned} \quad (81)$$

The summation is dominated by those terms which come from the contraction of the g_l in each S_2 with one of the three g_l s contained inside the $\tilde{\eta}_{-l}$ from the same S_2 term. These are enhanced by a factor of N_{pix} with respect to the other contributions. Neglecting subleading terms we obtain:

$$\langle \Delta S_2^2 \rangle = 18 \frac{N_{\text{pix}}^2}{\Omega^2} \sum_l C_l^2. \quad (82)$$

Finally for $\langle \Delta S_1 \Delta S_2 \rangle = \langle S_1 S_2 \rangle - \langle S_1 \rangle \langle S_2 \rangle$, we obtain:

$$\langle \Delta S_1 \Delta S_2 \rangle = 6 \frac{N_{\text{pix}}^2}{\Omega^2} \sum_l C_l^2, \quad (83)$$

again keeping only leading terms. We see that S_1 and S_2 are fully correlated, up to correction $\mathcal{O}(1/N_{\text{pix}})$:

$$\frac{\langle \Delta S_1 \Delta S_2 \rangle}{(\langle \Delta S_1^2 \rangle \langle \Delta S_2^2 \rangle)^{1/2}} = \frac{6 \frac{N_{\text{pix}}^2}{\Omega^2} \sum_l C_l^2}{\left(2 \frac{N_{\text{pix}}^2}{\Omega^2} \sum_l C_l^2 \times 18 \frac{N_{\text{pix}}^2}{\Omega^2} \sum_l C_l^2 \right)^{1/2}} = 1 . \quad (84)$$

On each realization we have:

$$\Delta S_2 = \left(\frac{\langle \Delta S_2^2 \rangle}{\langle \Delta S_1^2 \rangle} \right)^{1/2} \Delta S_1 = 3 \Delta S_1 , \quad (85)$$

and therefore $\mathcal{L}_\alpha = 0$ up terms suppressed by $1/N_{\text{pix}}$.

We conclude that terms which depend on α are negligible in the Likelihood.

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