Recent Results on Rest Frames for Radiating Isolated Systems

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Abstract

The gravitational waves, that are expected to be observed with the new generation of detectors, are supposed to be generated by systems involving very compact objects during processes like the coalescence of binary systems.

First order estimation of the gravitational radiation emitted by those systems can be carried out from the quadrupole formula. A more precise description of the gravitational radiation requires improved formulae which would probably take into account further multipole structure of the sources.

This conduces immediately to the issue of which is the appropriate frame of reference that should be used to define the multipole structure of the system. Even the notion of angular momentum is dependent on the frame that one is using to make the calculation.

For example, if one uses an inappropriate frame, which is moving and is not round, to describe the asymptotics of Schwarzschild spacetime, one could find a nonzero angular momentum and a complicated multipole structure.

Instead, when one wishes to describe the gravitational radiation, one is interested in determining the intrinsic structure of the sources, and therefore intends to avoid information associated to an inappropriate choice of frame.

In a few words, one is interested in using a frame that captures the idea of being at rest with the sources, which is around the sources and that is centered in relation to the central distribution. All these properties are required to be satisfied at all times; which is not a trivial requirement for systems in which the back reaction due to gravitational radiation are important.

The appropriate reference frame at future null infinity is constructed using the so-called *nice sections* that we introduced some time ago.

In this occasion we would like to report on the latest results of our program; which allow us to provide a notion of center of mass frame, which is used to define intrinsic angular momentum, and can also be used to describe the multipolar structure of the sources of gravitational waves.

1 Introduction

At present the observational evidence for the existence of gravitational radiation can be explained in the terms of the quadrupole radiation formula[4],

$$F_Q = \frac{1}{5}\sum_{i,j=1}^{3}\left(\frac{d^3Q_{ij}}{dt^3}\right)^2 \quad . \tag{1}$$

In the future we expect to have access to observation that would require another formula which will take into account further structure of the sources; namely angular momentum and multipole moments.

$$F = F(\texttt{multipole structure}). \tag{2}$$

This conduces us to the question of the reference frame in which these quantities are defined.

In the absence of gravitational radiation, things are easy; since the asymptotic structure of a stationary isolated system mimics that of linearized gravity. This means that it is possible to single out a rest frame suitable for calculating the intrinsic angular momentum[5] and multipole structure of the source.

In linearized gravity the geometrical structure is characterized by the symmetries of Minkowski spacetime; namely the Poincaré group; whose generators are the four translations and the six Lorentz rotations:

$$k_\alpha = \frac{\partial}{\partial x^\alpha} \qquad , \qquad k_{\alpha\beta} = x_\alpha\frac{\partial}{\partial x^\beta} - x_\beta\frac{\partial}{\partial x^\alpha},$$

with $\alpha, \beta = 0, 1, 2, 3$.

Given an energy momentum tensor one can define global quantities associated with the generators of the Poincaré group. In this way one has

$$P_\alpha = \int_V T_{ab}\, k^b_\alpha \, dV^a \quad , \quad J_{\alpha\beta} = \int_V T_{ab}\, k^b_{\alpha\beta}\, dV^a,$$

where V is a spacelike hypersurface and it is observed that the total momentum is defined in terms of the generators of the translations k_α, and the angular momentum is defined in terms of the generators of the Lorentz rotations.

Due to the divergent free nature of the energy momentum tensor, these global quantities are conserved; i.e., they are independent of the particular spacelike hypersurface V; and therefore depend only on the boundary of V, the 2-surface S.

In order to define a rest frame system, one can transform to a reference system in which the spacelike components of the vector P^α are zero. This fixes the Lorentz freedom of the Poincaré group, leaving a four dimensional space of translations.

2 Isolated radiating systems

Things are much more complicated when one considers a gravitating isolated system which emits gravitational radiation. These systems are modeled by asymptotically flat spacetimes. In this case the asymptotic geometric structure at future

null infinity (normally called *scri* and denoted by $\mathcal{I}$) is characterized by the infinite dimensional Bondi-Metzner-Sachs (BMS) group. In this group one can still identify a Lorentz freedom associated to six asymptotic symmetries, and an infinite dimensional freedom of translations; the so-called supertranslations.

The topology of future null infinity is $\mathbb{R} \times S^2$, whose natural coordinates are the Bondi coordinates; which are labeled by $(u, \zeta, \bar{\zeta})$, where $(\zeta, \bar{\zeta})$ are complex stereographic coordinates of the sphere S^2 and u takes values in $\mathbb{R}$. Given an arbitrary function $\gamma(\zeta, \bar{\zeta})$ of the sphere to real numbers, we say that $u = \gamma(\zeta, \bar{\zeta})$ defines a section of $\mathcal{I}$. The BMS group is defined by the following coordinate transformation:

$$\tilde{u} = K\left(u - \gamma(\zeta, \bar{\zeta})\right), \qquad \tilde{\zeta} = \frac{a\zeta + b}{c\bar{\zeta} + d}, \tag{3}$$

with

$$ad - bc = 1, \qquad K = \frac{1 + \zeta\bar{\zeta}}{|a\zeta + b|^2 + |c\zeta + d|^2}$$

where a, b, c, d are complex constants and γ is a real function on the sphere.

The generators of the BMS group can be expressed in terms of the Bondi coordinate system; in particular the supertranslations are

$$k_{lm} = Y_{lm}(\zeta, \bar{\zeta})\frac{\partial}{\partial u}, \tag{4}$$

where $l = 0, 1, ..., \infty$, $m = -l, -l+1, ..., l$ and Y_{lm} are the spherical harmonics.

In an analogous way, as it was done in linearized gravity, one can define the components of the corresponding supermomentum[6], in terms of a Bondi system, by the expression

$$P_{lm} = -\frac{1}{\sqrt{4\pi}} \int_{\Sigma} Y_{lm}(\zeta, \bar{\zeta})\ \Psi(u, \zeta, \bar{\zeta})\ d\Sigma^2, \tag{5}$$

where Σ is a section of future null infinity and

$$\Psi = \Psi_2 + \sigma\dot{\bar{\sigma}} + \eth^2\bar{\sigma}; \tag{6}$$

with Ψ_2 and σ being the leading order asymptotic behavior of the second Weyl tensor component and the Bondi shear respectively, and where we are using the Geroch-Held-Penrose (GHP) notation[2] for the eth operator of the unit sphere; and the dot means $\partial/\partial u$.

The total Bondi energy-momentum vector at a given section of null infinity can be expressed by

$$(P^{\mathtt{a}}) = \left(P_{00}, -\frac{1}{\sqrt{6}}(P_{11} - P_{1,-1}), \frac{i}{\sqrt{6}}(P_{11} + P_{1,-1}), \frac{1}{\sqrt{3}}P_{10}\right), \tag{7}$$

where $\mathtt{a} = \mathtt{0}, \mathtt{1}, \mathtt{2}, \mathtt{3}$.

In order to define a rest frame system one could make a Lorentz rotation so that in the new frame the Bondi momentum is timelike. But in general the other spacelike components of the supermomentum will still be different from zero.

There is a one to one relation between asymptotic Minkowskian frames and Bondi systems at future null infinity. And, there is also a one to one relation between a Bondi system and a section together with a timelike direction.

Since we have the natural timelike direction at future null infinity given by the Bondi momentum, the question can be phrased in the following way: can one find sections at future null infinity such that all the spacelike components of the supermomentum are zero?

If we can find them they would be the natural choice for defining rest frames in which it will be *meaningful* to define the multipole structure of the sources.

These set of sections are called *nice sections*[6] and are determined by the equation

$$\eth^2\bar{\eth}^2\gamma = \Psi(\gamma,\zeta,\bar{\zeta}) + K^3(\gamma,\zeta,\bar{\zeta})M(\gamma)\,, \tag{8}$$

where the Bondi mass is given by $M = P^{\mathtt{a}}P_{\mathtt{a}}$, the conformal factor K can be related to the Bondi momentum by

$$K = \frac{M}{P^{\mathtt{a}}l_{\mathtt{a}}}, \tag{9}$$

$P^{\mathtt{a}}$ is evaluated at the section $u = \gamma$; which is calculated through the integral (7) and $l^{\mathtt{a}}$ is given by

$$(l^{\mathtt{a}}) = \left(1, \frac{\zeta+\bar{\zeta}}{1+\zeta\bar{\zeta}}, \frac{\zeta-\bar{\zeta}}{i(1+\zeta\bar{\zeta})}, \frac{\zeta\bar{\zeta}-1}{1+\zeta\bar{\zeta}}\right). \tag{10}$$

Equation (8) is the nice section equation and it is interesting to know whether it has solutions for radiating isolated systems.

When these sections were initially introduced[6] we proved local existence.

Later we proved global existence[9], with techniques that used the implicit function theorem. The result was that for small radiation data there exists a global 4-parameter family of nice sections at future null infinity. Due to the nature of the techniques employed, the notion of "small radiation" has a topological meaning.

Our last result[1] on the proof of existence is based on the fixed point theorem, which allows us to have a physically realistic condition on the gravitational radiation.

Let us define a real function $x = x(\zeta,\bar{\zeta})$ on the sphere to be a translation if $\eth^2\bar{\eth}^2 x = 0$. Note that this implies that x has an expansion in spherical harmonics with $l = 0, 1$. An arbitrary regular function γ can be decomposed in the form

$$\gamma = x + y,$$

where y has an expansion with $l \geq 2$. Given an arbitrary x, equation (8) is an equation for y.

Our new result is the following Theorem[1]:

Theorem 1 If Ψ is a smooth function on scri, the total energy P^0 is bounded by the constant E_0, the total mass M is bounded from below by $M \geq M_0$, $M_0 > 0$; and the gravitational energy density flux $|\dot{\sigma}|^2 \leq \lambda$; where the constant λ satisfies

$$\lambda < \frac{\sqrt{27}}{4}(1 + (2C_K)^4)^{-1},$$

and C_K is given by

$$C_K = \frac{E_0}{M_0} + \sqrt{\frac{E_0^2}{M_0^2} - 1};$$

then

(i) For every translation x there exists a solution y of equation (8), and y is a smooth function on the sphere.

(ii) The solutions $\gamma = x + y(x)$ are continuous in the 4-parameter translation x, and if x_1 and x_2 are two translations such that the difference, $x_2 - x_1$, corresponds to a future directed timelike vector, then

$$\gamma(x_1, \zeta, \bar{\zeta}) < \gamma(x_2, \zeta, \bar{\zeta}).$$

Since the mass M is a decreasing function of u, the constant M_0 is the final rest mass as $u \to \infty$. Note that while M_0 is Lorentz invariant P^0 is not.

It is important to stress that all the hypotheses of the theorem are in terms of physical quantities; namely the total mass M, the energy P^0 and the gravitational energy density flux $|\dot{\sigma}|^2$. Theorem 1 essentially says that solutions of equation (8) exist and have the expected physical properties, when the gravitational radiation of the spacetime is finite but not too high.

The property stated in part *(ii)* of the theorem can be visualized in figure 1.

To give an idea of the physical relevance of the conditions let us compare with the case of the head-on collision of two black holes[8] [7]. In figure 2 one can find the quadrupole radiation energy for the head on black hole collision, the bounds required in our theorem and the area theorem[3] bounds.

3 Intrinsic angular momentum and center of mass

Having the rest frames defined at future null infinity one can concentrate in defining the physical quantities of the isolated system as for example the intrinsic angular momentum.

We can improve on a previous [5] definition of angular momentum at future null infinity based on charge integrals of the Riemann tensor on a section Σ of the form:

$$Q_\Sigma = \int_\Sigma C \tag{11}$$

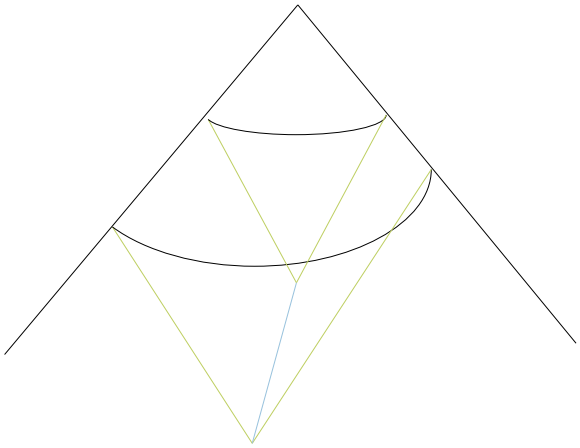

Figure 1: Associated to a time translation in the space of parameters there is a time supertranslation in the space of solutions. In Minkowski spacetime, a translation in the space of parameters corresponds to a physical translation in the interior of the spacetime.

where the two-form C_{ab} is given in terms of the Riemann tensor by

$$C_{ab} \equiv R^{*}_{ab}{}^{cd}\, w_{cd}, \tag{12}$$

and where the two-form w_{ab} is to be determined. Following [5] it is convenient to express this form in terms of its spinorial components and to require on scri

$$-\nabla_{A}{}^{B'}\, w^{AB} + \text{c.c.} = v^{BB'} \tag{13}$$

and

$$\nabla_{E'(E}\, w_{FG)} = 0; \tag{14}$$

where `c.c.` means complex conjugate and the vector $v^{BB'}$ is a generator of asymptotic Lorentz rotations.

Expressing w^{AB} in terms of its components

$$w^{AB} = w_0\, \hat{\imath}^A \hat{\imath}^B - w_1 \left(\hat{o}^A \hat{\imath}^B + \hat{\imath}^A \hat{o}^B\right) + w_2\, \hat{o}^A \hat{o}^B \tag{15}$$

the solutions of these equations for stationary spacetimes are given by

$$w_2 = -\frac{1}{3} v_{\bar{m}}, \tag{16}$$

$$w_1 = w_1^{00}(\zeta, \bar{\zeta}) + \frac{1}{6}\, u\, \eth v_{\bar{m}}, \tag{17}$$

$$w_0 = w_0^{00} + u \left(-2 w_1^{00} + \frac{2}{3} \sigma v_{\bar{m}}\right) - \frac{1}{6}\, u^2\, \eth^2 v_{\bar{m}} \tag{18}$$

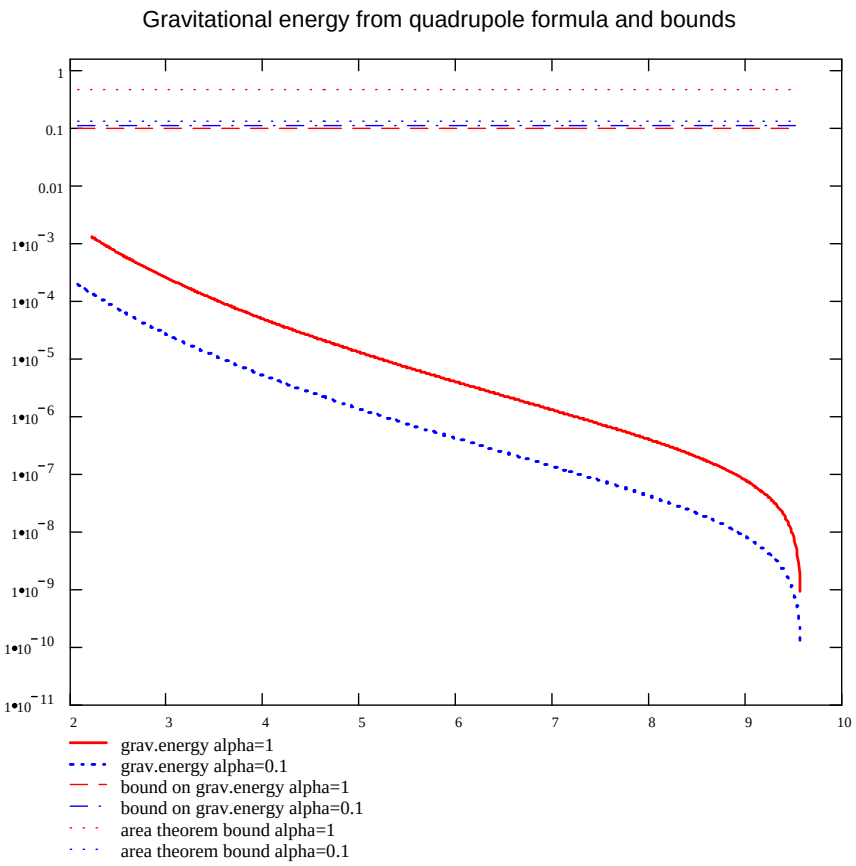

Figure 2: The first two curves from below, show the quadrupole radiation in the head on black hole collision with equal mass (alpha = 1 case) and when one of the masses is 10% of the companion (alpha = 0.1 case). The upper two lines show the bounds coming from the area theorem for these systems. Right below the area theorem bounds appear the conditions required in our theorem; it can be seen that they are almost the same as the area theorem bounds, and therefore the conditions are very reasonable.

where $v_{\bar{m}} = v_{AB'}\,\hat{\imath}^A\hat{o}^{B'}$ and w_1^{00} and w_0^{00} are spin weight 0 and 1 functions respectively that solve the equations

$$\eth^2 w_1^{00} = \frac{1}{3}\eth\sigma\;\bar{\eth}\bar{a} + \frac{1}{2}\sigma\;\eth\bar{\eth}\bar{a} = -\eth\sigma\;w_2 - \frac{3}{2}\sigma\;\eth w_2 \tag{19}$$

and

$$\eth w_0^{00} = -2\sigma w_1^{00}. \tag{20}$$

In this way one obtains a two-form with functional dependence

$$w_{AB}^0 = w_{AB}^0\left(\sigma^0(\zeta,\bar{\zeta}), v_{\bar{m}}; u, \zeta, \bar{\zeta}\right). \tag{21}$$

When the spacetime contains gravitational radiation the shear σ depends on the time u and the equations (13) and (14) in general have no solution. However, given a nice section Σ and an asymptotic symmetry v^a we define the two-form w_{ab} on Σ by the assignment

$$w_{AB} = w_{AB}^0\left(\sigma(u = u_0, \zeta, \bar{\zeta}), v_m; u, \zeta, \bar{\zeta}\right). \tag{22}$$

With this definition the charge integral of the Riemann tensor can be expressed by

$$Q_\Sigma(w) = \int_\Sigma C = 4\int_\Sigma \Big(-w_2\left(\Psi_1^0 + 2\sigma\eth\bar{\sigma} + \eth\left(\sigma\bar{\sigma}\right)\right) + 2w_1\left(\Psi_2^0 + \sigma\dot{\bar{\sigma}} + \eth^2\bar{\sigma}\right)\Big)\, d\Sigma^2 + \text{c.c.}. \tag{23}$$

This equation is reminiscent of the relation in Minkowski spacetime of the angular momentum, the intrinsic angular momentum and the momentum, namely

$$J^{\alpha\beta} = S^{\alpha\beta} + R^\alpha\, P^\beta - P^\alpha\, R^\beta, \tag{24}$$

in which $J^{\alpha\beta}$ is the total angular momentum, $S^{\alpha\beta}$ is the intrinsic angular momentum, P^α is the total momentum and R^β is the translation dependence of the angular momentum. The condition

$$J^{\alpha\beta} = S^{\alpha\beta}, \tag{25}$$

eliminates three degrees of freedom in the choice of R^α, leaving only the possibility of parallel translation to P^α.

This same condition is used in equation (23) to eliminate three degrees of freedom contained in the definition of the nice sections Σ and therefore in Q_Σ. Noting that one can express

$$v_m = \eth a, \tag{26}$$

we request

$$Q_\Sigma(a) = 0 \qquad \text{for all} \quad a = \bar{a}. \tag{27}$$

This is the appropriate condition that leaves a one-dimensional family of nice sections Σ that can legitimately be called *center of mass frames*.

Using these frames $\Sigma_{\mathtt{cm}}$, the intrinsic angular momentum is defined through

$$S(w) = Q_{\Sigma_{\mathtt{cm}}}(w). \tag{28}$$

Our definition of angular momentum has the appropriate behavior in the presence of gravitational radiation. To see this, imagine that there is a spacetime in which one can distinguish three stages; starting with a stationary regime, passing through a radiating stage and ending in a stationary regime (see Fig. 3). By construction it is clear that our definition gives the intrinsic angular momentum in all the stages. In particular in the first and third, it agrees with the accepted notions of angular momentum.

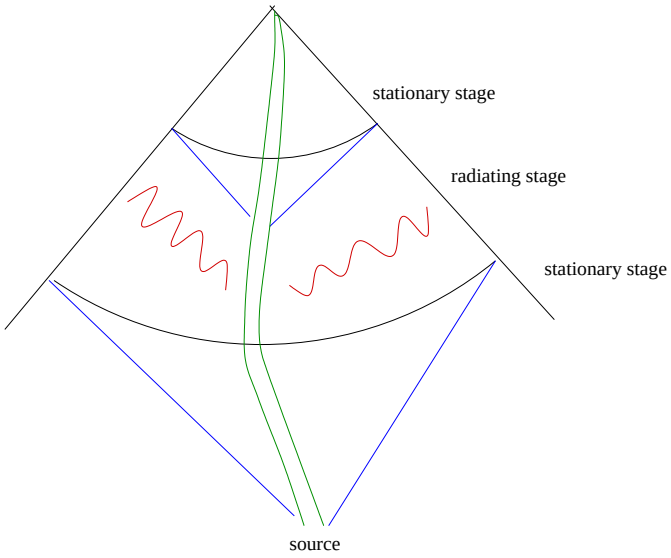

Figure 3: Behavior of the notion of intrinsic angular momentum in a spacetime with three stages; beginning with a stationary stage, continuing with a radiating stage and ending with a stationary stage.

It is important to remark that to construct the center of mass frames one needs the notion of angular momentum and that in order to determine the intrinsic angular momentum one uses the construction of the center of mass frames. That is, both concepts are fixed in the same procedure.

As a summary let us say that given an observer (a gravitational wave detector) at the point p of future null infinity; there is a unique rest frame characterized by a

nice section Σ and the Bondi momentum $P^{\mathrm{a}}(\Sigma)$ which is the center of mass frame of the isolated system at this retarded time; therefore this is the indicated frame to define the multipole moments and to describe their dynamical evolution.

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