AE-4

A slowing-down problem

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AKTIEBOLAGET ATOMENERGI

STOCKHOLM · SWEDEN · 1958

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Summary:

An infinitely long circular cylinder of radius a is surrounded by an infinite moderator. Both media are non-capturing. The cylinder emits neutrons of age zero with a constant source density of S. We assume that the ratios of the slowing-down powers

$\frac{\xi_1 \Sigma_{s1}}{\xi_2 \Sigma_{s2}}$ and of the diffusion constants $\frac{D_1}{D_2}$ are independent of the neutron energy.

The slowing-down density is calculated for two cases, a) when the slowing-down power of the cylinder medium is very small, and b) when the cylinder medium is identical with the moderator. The ratios of the slowing-down density at the age τ and the source density in the two cases are called ψ_V and ψ_M respectively. ψ_M and ψ_V are functions of $y = \frac{a^2}{4\tau}$. We find that

$$\psi_V(y) = \frac{2}{\pi^2} \int_0^\infty \frac{\exp\left(-\frac{u^2}{4y}\right)}{J_1^2(u) + Y_1^2(u)} \frac{du}{u}$$

$$\psi_M(y) = \frac{1}{2}\left[1 - e^{-2y} I_0(2y)\right]$$

These two functions are tabulated for y = 0 (0.01) 0.25.

ψ_V and ψ_M are used for calculating the resonance escape probability in report AEF-71.

Completion of manuscript October 1957
Revised June 1958
Printed June 1958

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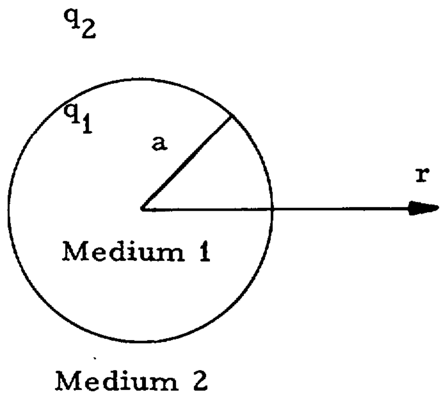
1. Introduction

In order to calculate p as described in AEF-71 (1) it is necessary to know the slowing-down density at the surface of a fuel element solely for neutrons that have been emitted by the same element. We therefore consider the following two-medium problem.

A long circular cylinder with radius a, diffusion coefficient D_1, and slowing-down power $\xi_1 \Sigma_{s1}$ is situated in an infinite homogeneous medium for which the diffusion coefficient is D_2 and slowing-down power $\xi_2 \Sigma_{s2}$. It is assumed that the ratio of the diffusion coefficients, $D_1:D_2$, is constant at all relevant energies, and likewise for $\xi_1 \Sigma_{s1} : \xi_2 \Sigma_{s2}$, though the two constants of proportionality are different. The cylinder emitting neutrons has a constant source density S. It is required to find the slowing-down density, especially at the surface of the cylinder. Two special cases are considered: that where $\xi_1 \Sigma_{s1} << \frac{1}{a}$, and that where $\xi_1 \Sigma_{s1} = \xi_2 \Sigma_{s2}$, $D_1 = D_2$.

The problem is first treated by Laplace transformation of the age equations, whereupon the special conditions are introduced into the solution. The second special case, that with the same medium throughout, is then treated in two further ways: by Hankel transformation, and by applying the slowing-down kernel for a line source.

2. Two-medium problem treated by Laplace transformation



The slowing-down equations are:

$$\frac{\partial q_1}{\partial \tau_1} = \nabla^2 q_1 + S\,\delta(\tau_1) \qquad 2.1$$

$$\frac{\partial q_2}{\partial \tau_2} = \nabla^2 q_2 \qquad 2.2$$

Fig. 1

The boundary condition at r = a is

$$\left(\frac{q_1}{\xi_1 \Sigma_{s1}}\right)_a = \left(\frac{q_2}{\xi_2 \Sigma_{s2}}\right)_a \qquad 2.3$$

$$\left(\frac{D_1}{\xi_1 \Sigma_{s1}} \frac{\partial q_1}{\partial r}\right)_a = \left(\frac{D_2}{\xi_2 \Sigma_{s2}} \frac{\partial q_2}{\partial r}\right)_a \qquad 2.4$$

The ages τ_1 and τ_2 are defined by

$$\tau_1 = \int_0^u \frac{D_1(u^1)}{\xi_1 \Sigma_{s1}(u^1)}\, du^1 \qquad 2.5$$

$$\tau_2 = \int_0^u \frac{D_2(u^1)}{\xi_1 \Sigma_{s1}(u^1)}\, du^1 \qquad 2.6$$

where

$$u = \ln \frac{E_0}{E} \qquad 2.7$$

It is now assumed that

$$\frac{D_2(u)}{\xi_2 \Sigma_{s2}(u)} = \alpha \cdot \frac{D_1(u)}{\xi_1 \Sigma_{s1}(u)} \qquad 2.8$$

$$\xi_1 \Sigma_{s1} (u) = \beta \cdot \xi_2 \Sigma_{s2} (u) \qquad 2.9$$

from which we get

$$\tau_2 = \alpha \tau_1 \qquad 2.10$$

$$\frac{\partial}{\partial \tau_1} = \alpha \frac{\partial}{\partial \tau_2} \qquad 2.11$$

Eliminating τ_1, and replacing τ_2 by τ in eqs. 2.1, 2.2, 2.3, and 2.4,

$$\alpha \frac{\partial q_1}{\partial \tau} = \nabla^2 q_1 + S\, \delta \left(\frac{\tau}{\alpha}\right) \qquad 2.12$$

$$\frac{\partial q_2}{\partial \tau} = \nabla^2 q_2 \qquad 2.13$$

$$(q_1)_a = \beta (q_2)_a \qquad 2.14$$

$$\left(\frac{\partial q_1}{\partial r}\right)_a = \alpha \left(\frac{\partial q_2}{\partial r}\right)_a \qquad 2.15$$

Laplace transformation of 2.12 and 2.13 gives

$$\int_0^\infty q(r, \tau)\, e^{-p\tau}\, d\tau = \bar{q}\,(r, p) \qquad 2.16$$

$$\int_0^\infty \frac{\partial q}{\partial \tau} e^{-p\tau} d\tau = \overset{\infty}{\underset{0}{I}} q e^{-p\tau} + p \cdot \int_0^\infty q e^{-p\tau} d\tau = p \cdot \bar{q} \qquad 2.17$$

$$\int_0^\infty S \delta \left(\frac{\tau}{\alpha}\right) e^{-p\tau} d\tau = \alpha S \qquad 2.18$$

2.12 and 2.13 become

$$\nabla^2 \bar{q}_1 - \alpha p \bar{q}_1 = -\alpha S \qquad 2.19$$

$$\nabla^2 \bar{q}_2 - p \bar{q}_2 = 0 \qquad 2.20$$

with boundary conditions analogous to 2.14 and 2.15. The physically possible solutions of 2.19 and 2.20 are

$$\bar{q}_1 = A \cdot I_0 (\sqrt{\alpha p} \cdot r) + \frac{S}{p} \qquad 2.21$$

$$\bar{q}_2 = B \cdot K_0 (\sqrt{p} \cdot r) \qquad 2.22$$

The constants A and B may be determined from 2.14 and 2.15:

$$A = -\frac{S}{p} \frac{\sqrt{\alpha} K_1 (\sqrt{p} \cdot a)}{N} \qquad 2.23$$

$$B = \frac{S}{p} \frac{I_1 (\sqrt{\alpha p}\ a)}{N} \qquad 2.24$$

where

$$N = \sqrt{\alpha} K_1 (\sqrt{p}\ a) I_0 (\sqrt{\alpha p}\ a) + \beta K_0 (\sqrt{p}\ a) I_1 (\sqrt{\alpha p}\ a) \qquad 2.25$$

The inverse transformation yields

$$q_1 = S - \frac{S}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{p\tau} \frac{\sqrt{\alpha}\, K_1(\sqrt{p}\ a)\, I_0(\sqrt{\alpha p}\ r)}{N} \frac{dp}{p} \qquad 2.26$$

$$q_2 = \frac{S}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{p\tau} \frac{I_1(\sqrt{\alpha p}\ a)\, K_0(\sqrt{p}\ r)}{N} \frac{dp}{p} \qquad 2.27$$

where c is to be chosen so that all the zeros of N lie to the left of x = c in the complex plane.

3. Case I. $\xi_1 \Sigma_{s1} << 1/a$; Laplace transformation

In this special case

$$\alpha << 1 \qquad 3.1$$

$$\beta << 1 \qquad 3.2$$

At the same time

$$\frac{\beta}{\alpha} = \frac{D_1}{D_2} \qquad 3.3$$

$I_0(\sqrt{\alpha p}\ a)$ and $I_1(\sqrt{\alpha p}\ a)$ in 2.25 may be expanded in series:

$$N = \sqrt{\alpha}\left(1 + \frac{\alpha p a^2}{4} + \frac{\alpha^2 p^2 a^4}{64} + \dots\right) K_1(\sqrt{p}\ a) + \beta \frac{\sqrt{\alpha p}\ a}{2}\left(1 + \frac{\alpha p a^2}{8} + \dots\right) K_0(\sqrt{pa}) =$$

$$= \sqrt{\alpha}\left\{ K_1(\sqrt{p}\ a) + \alpha\left[\frac{pa^2}{4} K_1(\sqrt{p}\ a) + \frac{D_1}{D_2}\frac{\sqrt{p}\ a}{2} K_0(\sqrt{p}\ a)\right] + \dots \right\} \qquad 3.4$$

$$N \sim \sqrt{\alpha}\, K_1(\sqrt{p}\ a)\left[1 + \alpha\frac{pa^2}{4} + \frac{D_1}{D_2}\,\alpha\,\frac{\sqrt{p}\ a}{2}\,\frac{K_0(\sqrt{p}\ a)}{K_1(\sqrt{p}\ a)}\right] \qquad 3.5$$

For small α and β the fraction in the integrand of eq. 2.26 may be written

$$(1 + \alpha \frac{p r^2}{4})(1 - \alpha \frac{p a^2}{4} - \alpha \frac{D_1}{D_2} \frac{\sqrt{p}\, a}{2} \frac{K_0(\sqrt{p}\, a)}{K_1(\sqrt{p}\, a)}) \sim$$

$$\sim 1 - \frac{\alpha p}{4}(a^2 - r^2) - \frac{\alpha}{2} \frac{D_1}{D_2} \sqrt{p}\, a \frac{K_0(\sqrt{p}\, a)}{K_1(\sqrt{p}\, a)} \qquad 3.6$$

$$q_1 = S\left\{1 - \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{p\tau} \left[1 - \frac{\alpha p}{4}(a^2 - r^2) - \frac{\alpha}{2} \frac{D_1}{D_2} \sqrt{p}\, a \frac{K_0(\sqrt{p}\, a)}{K_1(\sqrt{p}\, a)}\right] \frac{dp}{p}\right\} =$$

$$= S\,\alpha\left\{\frac{a^2 - r^2}{4}\, \delta(\tau) + \frac{a}{2} \frac{D_1}{D_2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{p\tau} \frac{K_0(\sqrt{p}\, a)}{K_1(\sqrt{p}\, a)} \frac{dp}{\sqrt{p}}\right\} \qquad 3.7$$

$$q_2 = \frac{a S}{2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{p\tau} \frac{K_0(\sqrt{p}\, r)}{K_1(\sqrt{p}\, a)} \frac{dp}{\sqrt{p}} \qquad 3.8$$

Let us now confine our attention to q_2, with $r = a$.

MacDonald (2) has shown that $K_n(z)$ has no zeros with $|\arg z| \leq \frac{\pi}{2}$

Hence $K_1(\sqrt{p}\, a) \cdot \sqrt{p}$ has no zeros. The path of integration may be chosen as in fig. 2, since the integrand decreases sufficiently rapidly as $|p|$ becomes large. The integral may be divided into three parts as shown in the figure.

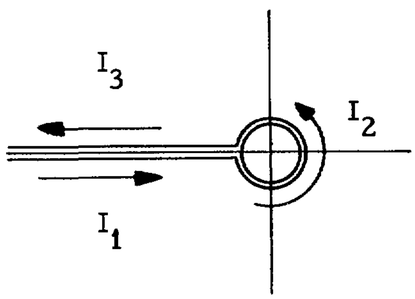


Fig. 2

$$I_1 = \int_{-\infty}^{-\epsilon} e^{p\tau} \frac{K_0(\sqrt{p}\,a)}{K_1(\sqrt{p}\,a)} \frac{dp}{\sqrt{p}} \qquad 3.9$$

$$I_2 = \int_{-\pi}^{+\pi} e^{\epsilon\tau e^{i\varphi}} \frac{K_0(\sqrt{\epsilon}\, e^{\frac{i\varphi}{2}} a)}{K_1(\sqrt{\epsilon}\, e^{\frac{i\varphi}{2}} a)} \cdot i\sqrt{\epsilon}\, e^{\frac{i\varphi}{2}} d\varphi \qquad 3.10$$

$$I_3 = \int_{-\epsilon}^{-\infty} e^{p\tau} \frac{K_0(\sqrt{p}\,a)}{K_1(\sqrt{p}\,a)} \frac{dp}{p} \qquad 3.11$$

The integral I_2 is of the order of magnitude of ϵ. The following substitutions may be made in I_1 and I_3 respectively:

$$p = -u^2 \quad \sqrt{p} = -iu$$
$$p = -u^2 \quad \sqrt{p} = iu \qquad 3.12$$

$$I_1 = \int_{\infty}^{\sqrt{\epsilon}} e^{-\tau u^2} \frac{K_0(-i\,au)}{K_1(-i\,au)} \frac{2}{i}\, du = 2i \int_{\sqrt{\epsilon}}^{\infty} e^{-\tau u^2} \frac{K_0(-i\,au)}{K_1(-i\,au)}\, du \qquad 3.13$$

From reference (3) we have

$$K_0(-i\,au) = \frac{\pi i}{2} H_0^{(1)}(au) = \frac{\pi i}{2}\left[J_0(au) + i \cdot Y_0(au)\right] \qquad 3.14$$

$$K_1(-i\,au) = -\frac{\pi}{2} H_1^{(1)}(au) = -\frac{\pi}{2}\left[J_1(au) + i\, Y_1(au)\right] \qquad 3.15$$

$$\frac{K_0(-i\,au)}{K_1(-i\,au)} = -i\, \frac{J_0J_1 + Y_0Y_1 - i\,J_0Y_1 + i\,J_1Y_0}{J_1^2(au) + Y_1^2(au)} \qquad 3.16$$

$$I_1 = 2\int_{\sqrt{\epsilon}}^{\infty} e^{-\tau u^2} \frac{J_0J_1 + Y_0Y_1 + i\,(J_1Y_0 - J_0Y_1)}{J_1^2 + Y_1^2}\, du \qquad 3.17$$

where the argument for all the Bessel functions is au.

$$I_3 = \int_{\sqrt{\epsilon}}^{\infty} e^{-\tau u^2} \frac{K_0(i\,au)}{K_1(i\,au)} \frac{(-2)}{i}\, du = 2\,i \int_{\sqrt{\epsilon}}^{\infty} e^{-\tau u^2} \frac{K_0(i\,au)}{K_1(i\,au)}\, du \qquad 3.18$$

Thus, $I_3 = -\bar{I}_1$ 3.19

where $\bar{I}_1$ is the complex conjugate of I_1.

Therefore,

$$I_1 + I_3 = 4\,i \int_{\sqrt{\epsilon}}^{\infty} e^{-\tau u^2} \frac{J_1 Y_0 - J_0 Y_1}{J_1^2 + Y_1^2}\, du = \frac{8i}{\pi a} \int_{\sqrt{\epsilon}}^{\infty} e^{-\tau u^2} \frac{du}{u(J_1^2 + Y_1^2)}$$

3.20

Let $\epsilon \to 0$. Then

$$q_2(a) = \frac{2S}{\pi^2} \int_0^{\infty} \frac{\exp\left(-\frac{\tau}{a^2} u^2\right)}{J_1^2(u) + Y_1^2(u)} \frac{du}{u} \qquad 3.21$$

4. Case II. One medium; Laplace transformation

Here, $D_1 = D_2$, $\xi_1 \Sigma_{s1} = \xi_2 \Sigma_{s2}$

from which

$$\alpha = 1 \qquad 4.1$$

$$\beta = 1 \qquad 4.2$$

Eq. 2.25 becomes

$$N = K_1(a\sqrt{p})\, I_0(a\sqrt{p}) + K_0(a\sqrt{p})\, I_1(a\sqrt{p}) = \frac{1}{a\sqrt{p}} \qquad 4.3$$

and, in accordance with 2.27

$$q_2 = \frac{Sa}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{p\tau} I_1(a\sqrt{p})\, K_0(r\sqrt{p}) \frac{dp}{\sqrt{p}} \qquad 4.4$$

The path of integration is treated as before.

$$I_1 = \int_{-\infty}^{-\epsilon} e^{p\tau} I_1(a\sqrt{p})\, K_0(r\sqrt{p})\, \frac{dp}{\sqrt{p}} = \int_{\infty}^{\sqrt{\epsilon}} e^{-\tau u^2}\, I_1(-iau)\, K_0(-iru)\, \frac{2}{i}\, du =$$

$$= 2i \int_{\sqrt{\epsilon}}^{\infty} e^{-\tau u^2}\, \frac{1}{i}\, J_1(au)\, \frac{\pi i}{2} \left[J_0(ru) + i\, Y_0(ru) \right] du \qquad 4.5$$

$$I_3 = \int_{-\epsilon}^{-\infty} e^{p\tau} I_1(a\sqrt{p})\, K_0(r\sqrt{p})\, \frac{dp}{\sqrt{p}} = \int_{\sqrt{\epsilon}}^{\infty} e^{-\tau u^2}\, I_1(i\,au)\, K_0(i\,ru)\, (-\frac{2}{i})\, du =$$

$$= 2i \int_{\sqrt{\epsilon}}^{\infty} e^{-\tau u^2}\, (-\frac{1}{i})\, J_1(au)\, (-\frac{\pi i}{2}) \left[J_0(ru) - i\, Y_0(ru) \right] du \qquad 4.6$$

$$I_1 + I_3 = 2\pi i \int_{\sqrt{\epsilon}}^{\infty} e^{-\tau u^2}\, J_1(au)\, J_0(ru)\, du \qquad 4.7$$

$$I_2 = \int_{-\pi}^{+\pi} e^{\tau\,\epsilon\, e^{i\varphi}}\, I_1(a\sqrt{\epsilon}\, e^{\frac{i\varphi}{2}})\, K_0(r\sqrt{\epsilon}\, e^{\frac{i\varphi}{2}})\, i\sqrt{\epsilon}\, e^{\frac{i\varphi}{2}}\, d\varphi \qquad 4.8$$

For small ϵ, I_2 is of the order of $\epsilon \log \epsilon$. If $\epsilon \to 0$, we get

$$q_2 = Sa \int_0^{\infty} e^{-\tau u^2}\, J_1(au)\, J_0(ru)\, du \qquad 4.9$$

Put $r = a$,

$$q_2(a) = Sa \int_0^{\infty} e^{-\tau u^2}\, J_1(au)\, J_0(au)\, du =$$

$$= S \int_0^{\infty} e^{-\frac{\tau}{a^2} u^2}\, J_1(u)\, J_0(u)\, du \qquad 4.10$$

Here the integral function is known (4)

$$\int_0^\infty e^{-\frac{\tau}{a^2}u^2} J_1(u)\, J_0(u)\, du =$$

$$= \frac{1}{2} \, \overset{\infty}{\underset{0}{I}} \, e^{-\frac{\tau}{a^2}u^2} \left(- J_0^2(u) \right) - \frac{1}{2} \int_0^\infty e^{-\frac{\tau}{a^2}u^2} J_0^2(u) \; 2\frac{\tau}{a^2} u \, du =$$

$$= \frac{1}{2}\left[1 - e^{-\frac{a^2}{2\tau}} \, I_0 \left(\frac{a^2}{2\tau} \right) \right] \qquad 4.11$$

Finally, we get

$$q_2(a) = \frac{S}{2} \left[1 - e^{-\frac{a^2}{2\tau}} \, I_0 \left(\frac{a^2}{2\tau} \right) \right] \qquad 4.12$$

5. Case II. Hankel transformation

The age equation for case II is

$$\frac{\partial q}{\partial \tau} = \nabla^2 q + S\, \delta\, (\tau) \qquad 5.1$$

where S = const. for $0 \leq r \leq a$, and S = 0 for $r > a$.
The Hankel transform of q is

$$\bar{q} = \int_0^\infty q \; r \; J_0\, (\xi\, r)\, dr \qquad 5.2$$

Eq. 5.1 becomes

$$\frac{\partial \bar{q}}{\partial \tau} = \int_0^\infty r\, J_0(\xi\, r) \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial q}{\partial r} \right) dr + \delta\, (\tau) \int_0^\infty S(r)\, r\, J_0(\xi\, r)\, dr \qquad 5.3$$

$$\int_0^\infty J_0(\xi r) \frac{\partial}{\partial r}\left(r \frac{\partial q}{\partial r}\right) dr = \overset{\infty}{\underset{0}{I}} J_0(\xi r) \; r \frac{\partial q}{\partial r} + \xi \int_0^\infty J_1(\xi r)\, r \frac{\partial q}{\partial r} \; dr =$$

$$= \xi \overset{\infty}{\underset{0}{I}} J_1(\xi r)\, r \;\; q - \xi \int_0^\infty (J_1(\xi r) + \xi J_0(\xi r) \;\; r - J_1(\xi r))\, q\, dr =$$

$$= - \xi^2 \bar{q} \qquad 5.4$$

$$\int_0^\infty S(r) \;\; r J_0(\xi r)\, dr = S \int_0^a J_0(\xi r)\, r\, dr = S \frac{a}{\xi} J_1(\xi a) \qquad 5.5$$

The transformed age equation is thus

$$\frac{\partial \bar{q}}{\partial \tau} = - \xi^2 \bar{q} + \delta(\tau) S \frac{a}{\xi} J_1(\xi a) \qquad 5.6$$

from which

$$\bar{q} = S \frac{a}{\xi} J_1(\xi a)\, e^{-\xi^2 \tau} \qquad 5.7$$

and

$$q = \int_0^\infty \xi J_0(r\xi)\; \bar{q}\; d\xi = a \;\; S \int_0^\infty e^{-\xi^2 \tau} J_0(r\xi)\, J_1(a\xi)\, d\xi \qquad 5.8$$

This equation is identical with 4.9

6. Case II. Slowing-down kernel for line source

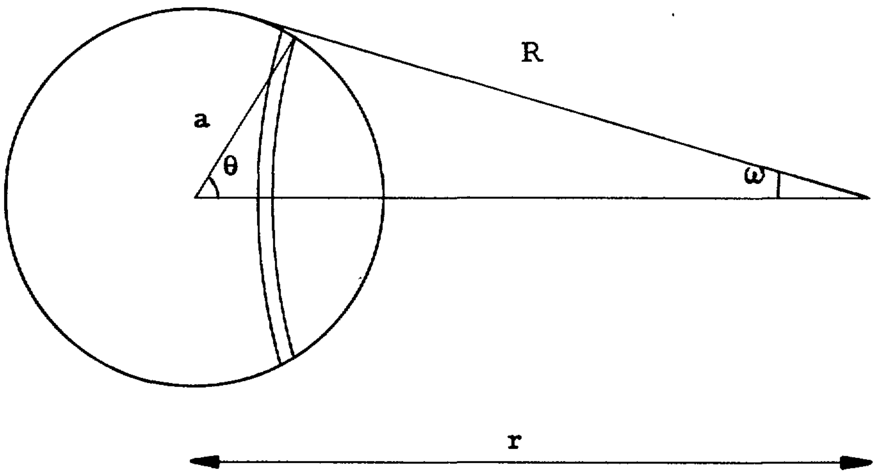


Fig. 3

Suppose $r > a$. The meaning of the symbols is indicated in fig. 3.

The slowing-down kernel is

$$\frac{S_\ell}{4\pi\tau} e^{-\frac{R^2}{4\tau}}$$

where S_ℓ is the source strength per unit length.

$$q(r) = \frac{2S}{4\pi\tau} \int_{r-a}^{r+a} e^{-\frac{R^2}{4\tau}} R\, \omega\, dR \qquad 6.1$$

A new variable of integration θ may be introduced:

$$R^2 = r^2 + a^2 - 2\,ar\cos\theta \qquad 6.2$$

$$RdR = ar\sin\theta\,d\theta \qquad 6.3$$

$$q = \frac{2\,arS}{4\pi\tau}\,e^{-\frac{r^2+a^2}{4\tau}}\int_0^{\pi} e^{\frac{ar}{2\tau}\cos\theta}\,\omega\sin\theta\,d\theta =$$

$$= \frac{2\,arS}{4\pi\tau}\,e^{-\frac{r^2+a^2}{4\tau}}\left\{\Big|_0^{\pi} - \frac{2\tau}{ar}\,e^{\frac{ar}{2\tau}\cos\theta}\cdot\omega + \right.$$

$$\left. + \int_0^{\pi}\frac{2\tau}{ar}\,e^{\frac{ar}{2\tau}\cos\theta}\,\frac{\partial\omega}{\partial\theta}\,d\theta\right\}$$

$$= \frac{S}{\pi}\,e^{-\frac{r^2+a^2}{4\tau}}\int_0^{\pi} e^{\frac{ar}{2\tau}\cos\theta}\,\frac{\partial\omega}{\partial\theta}\,d\theta \qquad 6.4$$

since $\omega = 0$ when $\theta = 0$ and $\theta = \pi$

$$\omega = \arctan\frac{a\sin\theta}{r-a\cos\theta} \qquad 6.5$$

$$\frac{\partial\omega}{\partial\theta} = \frac{a\,(r\cos\theta - a)}{a^2 + r^2 - ar\cos\theta} = \frac{1}{2}\,\frac{a}{r}\,\frac{e^{i\theta} - \frac{a}{r} + e^{-i\theta} - \frac{a}{r}}{(1-\frac{a}{r}e^{i\theta})\,(1-\frac{a}{r}e^{-i\theta})} =$$

$$= \frac{1}{2}\,\frac{a}{r}\left[\frac{e^{i\theta}}{1-\frac{a}{r}e^{i\theta}} + \frac{e^{-i\theta}}{1-\frac{a}{r}e^{-i\theta}}\right] = \sum_{\nu=1}^{\infty}\left(\frac{a}{r}\right)^{\nu}\cos\nu\theta \qquad 6.6$$

Substituting from 6.6 in 6.4,

$$q = \frac{S}{\pi} e^{-\frac{r^2 + a^2}{4\tau}} \int_0^{\pi} e^{\frac{ar}{2\tau}\cos\theta} \cdot \sum_{\nu=1}^{\infty} \left(\frac{a}{r}\right)^{\nu} \cos\nu\theta \, d\theta \doteq$$

$$= S\, e^{-\frac{r^2 + a^2}{4\tau}} \sum_{\nu=1}^{\infty} \left(\frac{a}{r}\right)^{\nu} I_{\nu}\left(\frac{ar}{2\tau}\right) \qquad 6.7$$

When $r = a$ we have, either from fig. 3 or from eq. 6.5,

$$\omega = \arctan \frac{\sin\theta}{1 - \cos\theta} = \frac{\pi}{2} - \frac{\theta}{2} \qquad 6.8$$

$$\frac{\partial\omega}{\partial\theta} = -\frac{1}{2} \qquad 6.9$$

Thus $\omega \neq 0$ when $\theta = 0$.
From eq. 6.4,

$$q(a) = \frac{S}{\pi} e^{-\frac{a^2}{2\tau}} \left\{ \frac{\pi}{2} e^{\frac{a^2}{2\tau}} - \frac{\pi}{2} I_0\left(\frac{a^2}{2\tau}\right) \right\} =$$

$$= \frac{S}{2} \left[1 - e^{-\frac{a^2}{2\tau}} I_0\left(\frac{a^2}{2\tau}\right) \right] \qquad 6.10$$

Alternatively, we may put $r = a$ in 6.7:

$$q(a) = S\, e^{-\frac{a^2}{2\tau}} \sum_{\nu=1}^{\infty} I_{\nu}\left(\frac{a^2}{2\tau}\right) = \frac{S}{2} \left[1 - e^{-\frac{a^2}{2\tau}} I_0\left(\frac{a^2}{2\tau}\right) \right] \qquad 6.11$$

An analogous treatment may be applied when $r < a$, but will be omitted here.

Comparison of 6.7 with the solution obtained with the aid of the Laplace or the Hankel transformation shows that for $r > a$,

$$\int_0^\infty e^{-\tau u^2} J_1(au) J_0(ru)\, du = \frac{1}{a} e^{-\frac{r^2 + a^2}{4\tau}} \sum_{\nu=1}^{\infty} \left(\frac{a}{r}\right)^\nu I_\nu\left(\frac{ar}{2\tau}\right) \qquad 6.12$$

A direct derivation of 6.12 is given in appendix 1. It is also shown there that 6.12 is valid even when $r < a$.

7. Summary

In case 1, where the slowing-down power is very small inside the cylinder, the slowing-down density in the outer medium has at the surface of the cylinder the value

$$q_2(a) = \frac{2S}{\pi^2} \int_0^\infty \frac{\exp\left(-\frac{\tau}{a^2} u^2\right)}{J_1^2(u) + Y_1^2(u)} \frac{du}{u} \qquad 7.1$$

In case II, where the two media are identical, the slowing-down density is

$$q(r) = Sa \int_0^\infty e^{-\tau u^2} J_1(au) J_0(ru)\, du = Se^{-\frac{r^2 + a^2}{4\tau}} \sum_{\nu=1}^{\infty} \left(\frac{a}{r}\right)^\nu I_\nu\left(\frac{ar}{2\tau}\right) \qquad 7.2$$

and at the surface of the cylinder, in particular,

$$q(a) = \frac{S}{2}\left[1 - e^{-\frac{a^2}{2\tau}} I_0\left(\frac{a^2}{2\tau}\right)\right] \qquad 7.3$$

If we now define

$$y = \frac{a^2}{4\tau} \qquad 7.4$$

$$\psi_V(y) = \frac{2}{\pi^2} \int_0^\infty \frac{\exp\left(-\frac{u^2}{4y}\right)}{J_1^2(u) + Y_1^2(u)} \frac{du}{u} \qquad 7.5$$

$$\psi_M(y) = \frac{1}{2}\left[1 - e^{-2y} I_0(2y)\right] \qquad 7.6$$

the slowing-down density at the surface of the cylinder in the two cases may be written

$$\text{Case I:} \quad q_2(a) = S \cdot \psi_V(y) \qquad 7.7$$

$$\text{Case II:} \quad q\,(a) = S \cdot \psi_M(y) \qquad 7.8$$

The functions $\psi_V(y)$ and $\psi_M(y)$ have previously been used in AEF-71 (eqs. 7.8 and 7.9). They are tabulated in table 1. The calculation of $\psi_V(y)$ is described in appendix 2.

Appendix 1. Direct derivation of eq. 6.12

A formula which is as general as 6.12 is

$$\int_0^\infty e^{-u^2} J_1(\alpha u) J_0(\beta u)\, du =$$

$$= \frac{1}{\alpha} e^{-\frac{\alpha^2+\beta^2}{4}} \sum_{\nu=1}^{\infty} \left(\frac{\alpha}{\beta}\right)^\nu I_\nu \left(\frac{\alpha\beta}{2}\right) \qquad (\alpha < \beta) \qquad \text{A.1.1}$$

As shown in (4)

$$\int_0^\infty e^{-u^2} J_n(\alpha u) J_n(\beta u)\, u\, du = \frac{1}{2} e^{-\frac{\alpha^2+\beta^2}{4}} I_n\left(\frac{\alpha\beta}{2}\right) \qquad \text{A.1.2}$$

Integration of the left-hand side by parts gives

$$\int_0^\infty e^{-u^2} J_n(\alpha u) J_n(\beta u)\, u\, du = -\frac{1}{2} \overset{\infty}{\underset{0}{I}} e^{-u^2} J_n(\alpha u) J_n(\beta u) +$$

$$+\frac{1}{2}\int_0^\infty e^{-u^2}\Big[-\alpha J_{n+1}(\alpha u) J_n(\beta u) + \frac{n}{u} J_n(\alpha u) J_n(\beta u) +$$

$$+\beta J_n(\alpha u) J_{n-1}(\beta u) - \frac{n}{u} J_n(\beta u)\Big]\, du$$

$$= \frac{1}{2}\delta_{0,n} - \frac{\alpha}{2}\int_0^\infty e^{-u^2} J_{n+1}(\alpha u) J_n(\beta u)\, du + \frac{\beta}{2}\int_0^\infty e^{-u^2} J_n(\alpha u) \cdot J_{n-1}(\beta u)\, du \qquad \text{A.1.3}$$

where the following equations have been used:

$$\frac{d}{du} J_n(\alpha u) = -\alpha J_{n+1}(\alpha u) + \frac{n}{u} J_n(\alpha u) \qquad \text{A.1.4}$$

$$\frac{d}{du} J_n(\beta u) = \beta J_{n-1}(\beta u) - \frac{n}{u} J_n(\beta u) \qquad \text{A.1.5}$$

A.1.3 and A.1.2 give

$$\int_0^\infty e^{-u^2} J_n(\alpha u)\, J_{n-1}(\beta u)\, du = \frac{\alpha}{\beta} \int_0^\infty e^{-u^2} J_{n+1}(\alpha u)\, J_n(\beta u)\, du$$

$$- \frac{1}{\beta}\, \delta_{0,n} + \frac{1}{\beta}\, e^{-\frac{\alpha^2+\beta^2}{4}}\, I_n\left(\frac{\alpha\beta}{2}\right) \qquad \text{A.1.6}$$

Iteration yields

$$\int_0^\infty e^{-u^2} J_1(\alpha u)\, J_0(\beta u)\, du = \frac{1}{\alpha}\, e^{-\frac{\alpha^2+\beta^2}{4}} \sum_{\nu=1}^{n} \left(\frac{\alpha}{\beta}\right)^\nu I_\nu\left(\frac{\alpha\beta}{2}\right) +$$

$$+ \left(\frac{\alpha}{\beta}\right)^n \int_0^\infty e^{-u^2} J_{n+1}(\alpha u)\, J_n(\beta u)\, du \qquad \text{A.1.7}$$

The last integral is limited as n increases. Thus, when $\alpha < \beta$, the last term approaches zero as n increases, which establishes A.1.1. But the last term also approaches zero when $\alpha > \beta$, as may be shown in the following way.

If α and β are interchanged in A.1.7

$$\int_0^\infty e^{-u^2} J_1(\beta u)\, J_0(\alpha u)\, du = \frac{1}{\beta}\, e^{-\frac{\alpha^2+\beta^2}{4}} \sum_{\nu=1}^{\infty} \left(\frac{\beta}{\alpha}\right)^\nu I_\nu\left(\frac{\alpha\beta}{2}\right) \qquad \text{A.1.8}$$

$$(\alpha > \beta)$$

If A.1.7 is multiplied by α and A.1.8 by β, and the products added, we get

$$\int_0^\infty e^{-u^2} \left[\alpha\, J_1(\alpha u)\, J_0(\beta u) + \beta\, J_1(\beta u)\, J_0(\alpha u) \right] du =$$

$$= e^{-\frac{\alpha^2+\beta^2}{4}} \left\{ \sum_{\nu=-\infty}^{n} \left(\frac{\alpha}{\beta}\right)^\nu I_\nu\left(\frac{\alpha\beta}{2}\right) - I_0\left(\frac{\alpha\beta}{2}\right) \right\}$$

$$+ \alpha \left(\frac{\alpha}{\beta}\right)^n \int_0^\infty e^{-u^2} J_{n+1}(\alpha u)\, J_n(\beta u)\, du \qquad \text{A.1.9}$$

Integration of the left-hand side gives

$$VM = \overset{\infty}{\underset{0}{I}} e^{-u^2} \left[- J_0(\alpha u)\, J_0(\beta u) \right] - 2 \int_0^\infty e^{-u^2} J_0(\alpha u)\, J_0(\beta u)\, u\, du =$$

$$= 1 - e^{-\frac{\alpha^2 + \beta^2}{2}} I_0\left(\frac{\alpha\beta}{2}\right) \qquad \text{A.1.10}$$

Eq. A.1.9 is then transformed to

$$\left(\frac{\alpha}{\beta}\right)^n \int_0^\infty e^{-u^2} J_{n+1}(\alpha u)\, J_n(\beta u)\, du = 1 - e^{-\frac{\alpha^2 + \beta^2}{4}} \cdot \sum_{\nu = -\infty}^{n} \left(\frac{\alpha}{\beta}\right)^\nu I_\nu \left(\frac{\alpha\beta}{2}\right) \qquad \text{A.1.11}$$

Let n now approach ∞. The right-hand side of A.1.11 then approaches

$$1 - e^{-\frac{\alpha^2 + \beta^2}{4}} \sum_{\nu = -\infty}^{+\infty} \left(\frac{\alpha}{\beta}\right)^\nu I_\nu \left(\frac{\alpha\beta}{2}\right) = 1 - e^{-\frac{\alpha^2 + \beta^2}{4}} \cdot e^{\frac{1}{2}\frac{\alpha\beta}{2}\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)} = 0 \qquad \text{A.1.12}$$

Thus, for all values of α/β

$$\lim_{n \to \infty} \left(\frac{\alpha}{\beta}\right)^n \int_0^\infty e^{-u^2} J_{n+1}(\alpha u)\, J_n(\beta u)\, du = 0 \qquad \text{A.1.13}$$

Appendix 2. Calculation of $\psi_V(y)$

According to eq. 7.5

$$\frac{\pi^2}{2}\,\psi_V(y) = \int_0^\infty \frac{e^{-\frac{u^2}{4Y}}}{J_1^2(u) + Y_1^2(u)}\,\frac{du}{u} \qquad \text{A.2.1}$$

The function $1/u\left[J_1^2(u) + Y_1^2(u)\right]$ varies as shown in fig. 4.

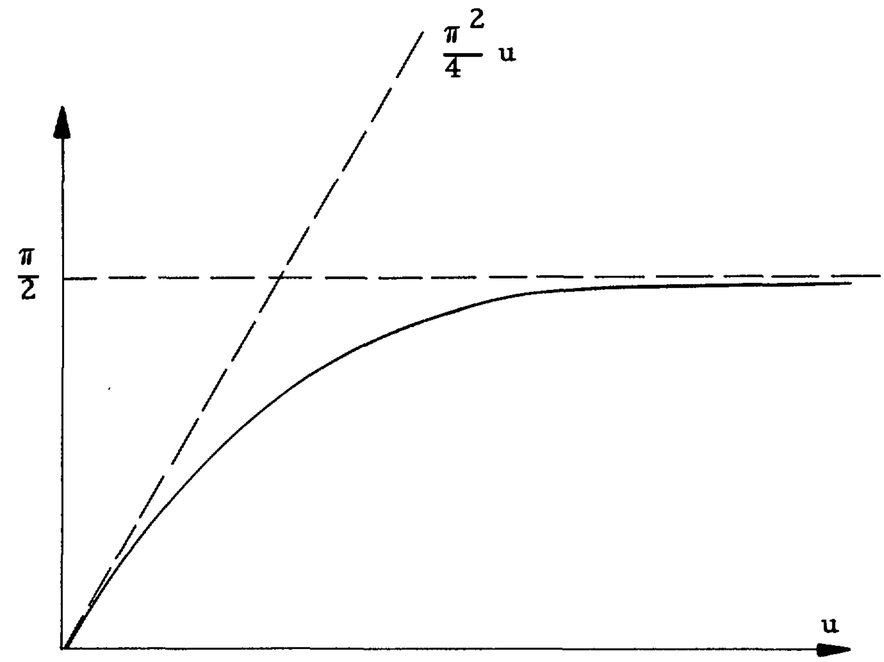


Fig. 4

Thus for small and large values of y respectively, we may write

y << 1

$$\frac{\pi^2}{2}\,\psi_V(y) = \int_0^\infty e^{-\frac{u^2}{4y}}\,\frac{\pi^2}{4}\,u\,du = \frac{\pi^2 y}{2} \qquad \text{A.2.2}$$

$$\psi_V(y) = y \qquad (y << 1) \qquad \text{A.2.3}$$

y >> 1

$$\frac{\pi^2}{2}\,\psi_V(y) = \int_0^\infty e^{-\frac{u^2}{4y}}\,\frac{\pi}{2}\,du - \int_0^\infty \left[\frac{\pi}{2} - \frac{1}{u\,[J_1^2(u) + Y_1^2(u)\,]}\right] du =$$

$$= \frac{\pi\sqrt{\pi}}{2}\sqrt{y} - \lim_{R\to\infty}\left[\frac{\pi}{2}\,R - \frac{\pi}{2}\,(\text{ar ctg}\,\frac{Y_1(R)}{J_1(R)} - \frac{\pi}{2}\,)\right] =$$

$$= \frac{\pi\sqrt{\pi}}{2}\sqrt{y} - \lim_{R\to\infty}\frac{\pi}{2}\left[R - (R - \frac{3}{4}\,\pi + \frac{\pi}{2})\right] = \frac{\pi\sqrt{\pi}}{2}\sqrt{y} - \frac{\pi^2}{8}$$

A. 2. 4

$$\psi_V(y) = \sqrt{\frac{y}{\pi}} - \frac{1}{4} \qquad (y >> 1)$$ A. 2. 5

For intermediate values of y we have to resort to a numerical method. The following simple method has been used.
The function

$$g(u) = \frac{1}{J_1^2(u) + Y_1^2(u)} \cdot \frac{1}{u} = \frac{1}{\left|H_1^{(1)}(u)\right|^2 u}$$ A. 2. 6

has the following properties:

$$u << 1 \quad g(u) \sim \frac{\pi^2}{4}\,u$$

$$u >> 1 \quad g(u) \sim \frac{\pi}{2}\,(1 - \frac{3}{8\,u^2}\,)$$ A. 2. 7

$$\int_0^\infty \left[\frac{\pi}{2} - g(u)\right] du = \frac{\pi^2}{8}$$

g(u) is approximated by a function f(u) -

$$f(u) = \frac{\pi}{2}\left[\frac{\alpha u}{\sqrt{a^2+u^2}} + \frac{\beta u}{\sqrt{b^2+u^2}}\right] \qquad \text{A.2.8}$$

We want f(u) also to have the properties A.2.7. Thus we get

$$\frac{\alpha}{a} + \frac{\beta}{b} = \frac{\pi}{2}$$

$$\alpha + \beta = 1$$

$$\alpha a + \beta b = \frac{\pi}{4} \qquad \text{A.2.9}$$

$$\alpha a^2 + \beta b^2 = \frac{3}{4}$$

Solution of these equations yields values of the constants:

$$\alpha = \frac{1}{2} + \frac{\pi}{4}\,\frac{\pi^2 - 10}{R} = 0.425\,773$$

$$\beta = \frac{1}{2} - \frac{\pi}{4}\,\frac{\pi^2 - 10}{R} = 0.574\,227$$

$$a = \frac{\pi + R}{2\pi^2 - 16} = 1.209\,161 \qquad \text{A.2.10}$$

$$b = \frac{\pi - R}{2\pi^2 - 16} = 0.471\,190$$

$$R = \sqrt{2\pi^4 - 39\pi^2 + 192} = 1.379\,714$$

Let

$$g(u) = f(u) + \delta(u) \qquad \text{A.2.11}$$

$$\int_0^\infty e^{-\frac{u^2}{4y}}\, g(u)\, du = \int_0^\infty e^{-\frac{u^2}{4y}}\, f(u)\, du + \int_0^\infty e^{-\frac{u^2}{4y}}\, \delta(u)\, du \qquad \text{A.2.12}$$

The integral of f(u) may be expressed in terms of tabulated functions. Put

$$\Delta(y) = \frac{2}{\pi^2} \int_0^\infty e^{-\frac{u^2}{4y}} \delta(u)\, du \qquad \text{A. 2. 13}$$

Then

$$\psi_V(y) = \frac{2}{\pi^2} \int_0^\infty e^{\frac{u^2}{4y}} f(u)\, du + \Delta(y)$$

$$= \sqrt{\frac{y}{\pi}} \left\{ \alpha\, e^{\frac{a^2}{4y}} \left[1 - H\left(\frac{a}{2\sqrt{y}}\right) \right] + \beta\, e^{\frac{b^2}{4y}} \left[1 - H\left(\frac{b}{2\sqrt{y}}\right) \right] \right\} + \Delta(y)$$

A. 2. 14

where H(x) is defined by

$$H(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du \qquad \text{A. 2. 15}$$

$\delta(u)$ has been calculated with the help of a table for $\left| H_1^{(1)}(x) \right|$ in Watson (2). Some values of $\Delta(y)$ have been obtained by numerical integration, whereupon other values were obtained by graphical interpolation. Since $\left| \Delta(y) \right|$ is small, 1 % accuracy in $\Delta(y)$ is sufficient when ψ_V is to be determined correct to five decimal figures. $\Delta(y)$ is shown in diagram 1. Note that $\Delta(y) < 0$ for all y except 0.

Finally, $\psi_V(y)$ has been calculated from eq. A. 2. 14, with H(x) taken from (5).

For small y the following formula may be used:

$$\psi_V(y) = y + 2y^2 (C + \ln y) + y^3 \left[6(C + \ln y)^2 + 4(C + \ln y) - 4 - \pi^2 \right] + \ldots$$

where C = 0,577216.

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2. G.N. Watson: Theory of Bessel functions, p. 511

3. G. Petain: La théorie des fonctions de Bessel, p. 85

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Table 1

y	$\psi_V(y)$	$\psi_M(y)$
0	0,00000	0,00000
0,01	00926	00985
0,02	01763	01941
0,03	02545	02869
0,04	03283	03770
0,05	03987	04645
0,06	04661	05494
0,07	05309	06319
0,08	05938	07120
0,09	06546	07898
0,10	07137	08653
0,11	07711	09387
0,12	08272	10100
0,13	08819	10793
0,14	09355	11467
0,15	09879	12121
0,16	10392	12757
0,17	10895	13376
0,18	11390	13977
0,19	11876	14561
0,20	12353	15130
0,21	12823	15683
0,22	13285	16221
0,23	13740	16744
0,24	14189	17253
0,25	14632	17748

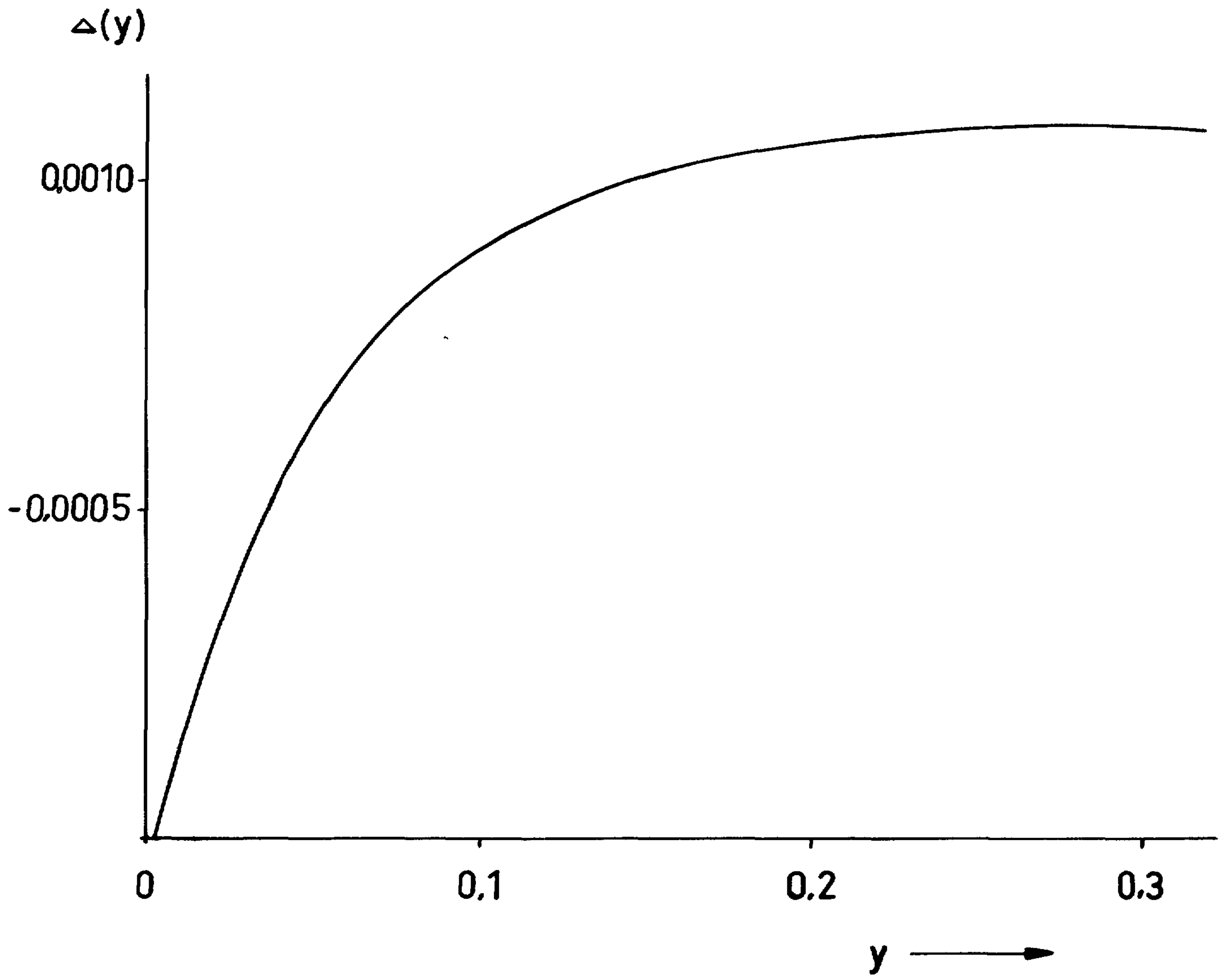

Fig. 1. The function Δ(y). Note that $\Delta(y) < 0$ for all y.

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