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GENERALIZED CHRISTOFFEL-DARBOUX FORMULA FOR SKEW-ORTHOGONAL POLYNOMIALS AND RANDOM MATRIX THEORY

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Abstract

We obtain a generalized Christoffel-Darboux (GCD) formula for skew-orthogonal polynomials. Using this, we present an alternative derivation of the level density and two-point function for Gaussian orthogonal ensembles and Gaussian symplectic ensembles of random matrices.

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Random matrices have found applications in different branches of physics mainly due to the 'universality' in their correlation function under certain scaling limits. In this context, although unitary ensembles of random matrices, corresponding to systems with broken time-reversal symmetry (for example, a mesoscopic conductor in the presence of a magnetic field) have been extensively studied [1, 2], much less is known about Orthogonal Ensembles (OE) and Symplectic Ensembles (SE) of random matrices.

In his phenomenal paper [3] Dyson stressed the importance of a good understanding of the skew-orthogonal polynomials (SOP) to study the OE and SE of random matrices. Some progress has been made in this regard [1, 4, 5] to develop the theory of SOP. Alternative approaches were taken by various authors [6–13] to study these ensembles. For example, Deift and Gioev [14, 15] have recently used the Widom's representation [16] to prove Universality for a wide class of polynomial potentials of the OE and SE of random matrices. In all these work, the authors have used the well known properties of orthogonal polynomials to study these ensembles. Here we use the more elegant SOP, evolving naturally from OE and SE of random matrices.

In this paper we obtain recursion relations for SOP. Using this we derive the GCD formula. We use them to obtain the level density and two-point correlation function for Gaussian orthogonal ensembles and Gaussian symplectic ensembles of random matrices.

We consider ensembles of $2N$ dimensional matrices H with probability distribution

$$P_{\beta,N}(H)dH = \frac{1}{\mathcal{Z}_{\beta N}} \exp[-[\mathrm{Tr}V(H)]]dH, \tag{1}$$

where the parameter $\beta = 1$ and 4 corresponds to H real symmetric or quaternion real self dual.(Note that we have considered the OE of even dimension. The odd dimension can be easily generalized.) dH is the standard Haar measure. $\mathcal{Z}_{\beta N}$ is the so called 'partition function', and is proportional to the product of skew-normalization constants [1]:

$$\mathcal{Z}_{\beta N} = \int \exp[-[\mathrm{Tr}V(H)]]dH = N! \prod_{j=0}^{n_F-1} g_j^{(\beta)}. \tag{2}$$

Here n_F is called the 'Fermi level' by analogy with a system of fermions. In our case, it takes the value $n_F = 2N$ for $\beta = 1$ and 4 respectively. It is assumed that most of the physical quantities (like level density and correlation functions) are related to properties of $g_n^{(\beta)}$ around the 'vicinity' [18] of the Fermi level. In this paper, we give rigorous justification for such a claim.

To study different correlations among the eigenvalues of random matrices, we need to study certain kernel functions [1, 3]. For example, the two-point correlation function for $\beta = 1$ and 4 can be expressed in terms of the 2×2 matrix

$$\sigma_2^{(\beta)}(x,y) \;=\; \begin{pmatrix} S_{2N}^{(\beta)}(x,y) & D_{2N}^{(\beta)}(x,y) \\ I_{2N}^{(\beta)}(x,y) & S_{2N}^{(\beta)}(y,x) \end{pmatrix}, \tag{3}$$

while the level-density

$$\rho^{(\beta)}(x) = S_{2N}^{(\beta)}(x,x). \tag{4}$$

Here, the kernel functions are defined as

$$\begin{aligned} S_{2N}^{(\beta)}(x,y) &= -\widehat{\Phi}^{(\beta)}(x)\prod_{2N}\Psi^{(\beta)}(y), \\ &= \widehat{\Psi}^{(\beta)}(y)\prod_{2N}\Phi^{(\beta)}(x), \end{aligned} \tag{5}$$

$$D_{2N}^{(\beta)}(x,y) = \widehat{\Phi}^{(\beta)}(x)\prod_{N}\Phi^{(\beta)}(y), \tag{6}$$

$$I_{2N}^{(\beta)}(x,y) = -\widehat{\Psi}^{(\beta)}(x)\prod_{2N}\Psi^{(\beta)}(y) + \delta_{1,\beta}\frac{\epsilon(x-y)}{2}, \tag{7}$$

where

$$\Phi^{(\beta)} = (\Phi_0^{(\beta)}\ldots\Phi_n^{(\beta)}\ldots)^t, \;\; \widehat{\Phi}^{(\beta)} = -\Phi^{(\beta)t}Z, \tag{8}$$

(similarly for $\Psi^{(\beta)}$) are semi-infinite vectors. They are formed by quasi-polynomials

$$\Phi_n^{(\beta)}(x) = \begin{pmatrix} \phi_{2n}^{(\beta)}(x) \\ \phi_{2n+1}^{(\beta)}(x) \end{pmatrix}, \tag{9}$$

where

$$\phi_n^{(\beta)}(x) = \frac{1}{\sqrt{g_n^{(\beta)}}}\Pi_n^{(\beta)}(x)\exp[-V(x)], \tag{10}$$

and

$$\Pi_n^{(\beta)}(x) = \sum_{k=0}^{n} c_k^{(n,\beta)}x^k, \tag{11}$$

is the SOP of order n.

$$Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dot{+}\ldots\dot{+} \tag{12}$$

is a semi-infinite anti-symmetric block-diagonal matrix with $Z^2 = -1$ and

$$\epsilon(r) = \frac{|r|}{r}.$$

δ is the kronecker delta. The matrix

$$\prod_{2N} = \mathrm{diag}(\underbrace{1,\ldots,1}_{2N},0,\ldots,0) \tag{13}$$

has $2N$ entries. Finally, we define

$$\Psi_n^{(4)}(x) = \Phi_n'^{(4)}(x), \quad \Psi_n^{(1)}(x) = \int \Phi_n^{(1)}(y)\epsilon(x-y)dy, \tag{14}$$

which satisfy skew-orthonormality relation [1, 4, 5]:

$$(\widehat{\Psi}_n^{(\beta)}, \Phi_m^{(\beta)}) \equiv \int_\Gamma \widehat{\Psi}_n^{(\beta)} \Phi_m^{(\beta)} dx = \delta_{nm}. \tag{15}$$

The contour of integration 'Γ' is the real axis. Here, one must mention that under suitable assumptions on $V(x)$ that ensures convergence of the integral (15), for a given $V(x)$, these SOP are unique up to the addition of a lower even order polynomial to the odd ones.

For the unitary ensemble, study of correlation function involve similar kernel function, which is calculated [4, 5] using the well known Christoffel Darboux formula [17]. To study the kernel functions arising in OE and SE (5,6,7), we derive a GCD formula.

To this effect, we expand $x\Phi^{(\beta)}(x)$, $(\Phi^{(\beta)}(x))'$ and $(x\Phi^{(\beta)}(x))'$ in terms of $\Phi^{(\beta)}(x)$ (and hence introduce the semi-infinite matrices $Q^{(\beta)}$, $P^{(\beta)}$ and $R^{(\beta)}$ respectively):

$$x\Phi^{(\beta)}(x) = Q^{(\beta)}\Phi^{(\beta)}(x), \tag{16}$$

$$\Psi^{(4)}(x) = P^{(4)}\Phi^{(4)}(x), \qquad x\Psi^{(4)}(x) = R^{(4)}\Phi^{(4)}(x), \tag{17}$$

$$\Phi^{(1)}(x) = P^{(1)}\Psi^{(1)}(x), \qquad x\Phi^{(1)}(x) = R^{(1)}\Psi^{(1)}(x), \tag{18}$$

where (18) is obtained by multiplying the above expansion by $\epsilon(y-x)$ and integrating by parts. They satisfy the following commutation relations:

$$[Q^{(\beta)}, P^{(\beta)}] = 1, \qquad [R^{(\beta)}, P^{(\beta)}] = P^{(\beta)}. \tag{19}$$

Using $(\psi_n^{(4)}(x), \psi_m^{(4)}(x))$ and $(x\psi_n^{(4)}(x), \psi_m^{(4)}(x))$ for $\beta = 4$, and replacing $\psi^{(4)}(x)$ by $\phi^{(1)}(x)$ for $\beta = 1$, we get

$$P^{(\beta)} = -{P^{(\beta)}}^D, \qquad R^{(\beta)} = -{R^{(\beta)}}^D, \tag{20}$$

where dual of a matrix A is defined as

$$A^D = -ZA^tZ. \tag{21}$$

However starting with $(x\phi_n^{(\beta)}(x), \psi_m^{(\beta)}(x))$ and using (19), we get

$$Q^{(\beta)} = {Q^{(\beta)}}^D + (P^{(\beta)})^{-1}. \tag{22}$$

We will use the matrices $P^{(\beta)}$ and $R^{(\beta)}$ to obtain the GCD formula. For $\beta = 4$, using (5),

(17) and (20), we get

$$
\begin{aligned}
S_{2N}^{(4)}(x,y)-S_{2N}^{(4)}(y,x) &= \left[\Phi^{(4)^t}(x)\prod_{2N}Z\prod_{2N}\Psi^{(4)}(y)+\Psi^{(4)^t}(x)\prod_{2N}Z\prod_{2N}\Phi^{(4)}(y)\right],\\
&= \left[\Phi^{(4)^t}(x)\prod_{2N}Z\prod_{2N}P\Phi^{(4)}(y)+\Phi^{(4)^t}(x)P^t\prod_{2N}Z\prod_{2N}\Phi^{(4)}(y)\right],\\
&= \Bigg[-\Phi^{(4)^t}(x)ZZ\prod_{2N}Z\prod_{2N}P\Phi^{(4)}(y)\\
&\quad +\Phi^{(4)^t}(x)ZZP^tZZ\prod_{2N}Z\prod_{2N}\Phi^{(4)}(y)\Bigg],\\
&= \widehat{\Phi}^{(4)}(x)\left[P^{(4)},\prod_{2N}\right]\Phi^{(4)}(y), \qquad (23)
\end{aligned}
$$

where we have used $Z\prod_{2N}Z\prod_{2N}=-\prod_{2N}$. Similarly, using (5), (14), (17) and (20) we get

$$
\begin{aligned}
yS_{2N}^{(4)}(x,y)-xS_{2N}^{(4)}(y,x) &= \left(x\frac{d}{dx}+y\frac{d}{dy}\right)\Phi^{(4)^t}(x)\prod_{2N}Z\prod_{2N}\Phi^{(4)}(y),\\
&= \left[\Phi^{(4)^t}(x)R^t\prod_{2N}Z\prod_{2N}\Phi^{(4)}(y)+\Phi^{(4)^t}(x)\prod_{2N}Z\prod_{2N}R\Phi^{(4)}(y)\right],\\
&= \widehat{\Phi}^{(4)}(x)\left[R^{(4)},\prod_{2N}\right]\Phi^{(4)}(y). \qquad (24)
\end{aligned}
$$

Finally using (23) and (24), the GCD for the SE ($\beta=4$) is given by

$$
S_{2N}^{(4)}(x,y)=\frac{x\widehat{\Phi}^{(4)}(x)\left[P^{(4)},\prod_{2N}\right]\Phi^{(4)}(y)-\widehat{\Phi}^{(4)}(x)\left[R^{(4)},\prod_{2N}\right]\Phi^{(4)}(y)}{x-y}. \qquad (25)
$$

Following a similar procedure, the GCD for the OE ($\beta=1$) is given by

$$
S_{2N}^{(1)}(x,y)=\frac{y\widehat{\Psi}^{(1)}(x)\left[P^{(1)},\prod_{2N}\right]\Psi^{(1)}(y)-\widehat{\Psi}^{(1)}(x)\left[R^{(1)},\prod_{2N}\right]\Psi^{(1)}(y)}{y-x}. \qquad (26)
$$

Here, one might recall, that for orthogonal polynomials, the Christoffel-Darboux sum takes the form

$$
S_N^{(2)}(x,y)=\phi^t(x)\prod_N\phi(y)=\frac{\phi^t(x)\left[Q,\prod_N\right]\phi(y)}{x-y}, \qquad (27)
$$

where Q is a tri-diagonal Jacobi matrix, and ϕ are the normalized orthogonal quasi-polynomials. Any matrix of the form $[A,\prod_N]$ has only off-diagonal blocks whose size depends on the number of bands above and below the diagonal of A. Thus the 'size' of $P^{(\beta)}$ and $R^{(\beta)}$ decides the number of terms around the 'Fermi-level' that will ultimately contribute to the correlation. For example, for the orthogonal polynomials, the Christoffel-Darboux has only two terms.

Having derived the GCD for arbitrary weight, we will study ensembles with

$$
V(x)=\sum_{l=1}^{d+1}\frac{u_l}{l}x^l, \qquad (28)
$$

where $V(x)$ is a polynomial of order $d+1$. u_l is called the deformation parameter [2]. We will show that for such ensembles, the matrix $P^{(\beta)}$ and $R^{(\beta)}$ are finite band matrices, i.e. have finite number of bands below and above the principal diagonal.

For $\beta = 4$,

$$\begin{aligned}
\phi'^{(4)}_n(x) &= \sum P^{(4)}_{n,m}\phi^{(4)}_m(x) \\
&= \left[-V'(x)\phi^{(4)}_n(x) + \phi^{(4)}_{n-1}(x) + \ldots\right] \\
&= -\sum (V'(Q))_{n,m}\phi^{(1)}_m(x) + \phi^{(1)}_{n-1}(x) + \ldots,
\end{aligned} \tag{29}$$

while for $\beta = 1$,

$$\begin{aligned}
\phi^{(1)}_n(x) &= \sum P^{(1)}_{nm}\psi^{(1)}_m(x) \\
&= \int \frac{d}{dy}\left[\phi^{(1)}_n(y)\right]\epsilon(x-y)dy \\
&= \int \left[-\sum (V'(Q))_{n,m}\phi^{(1)}_m(y)\right]\epsilon(x-y)dy + \psi^{(1)}_{n-1}(x) + \ldots.
\end{aligned} \tag{30}$$

This gives

$$\left[P^{(\beta)} + V'(Q^{(\beta)})\right] = \text{lower}. \tag{31}$$

Similarly, for $\beta = 4$, we have

$$\begin{aligned}
x\psi^{(4)}_n(x) &= \sum R^{(4)}_{n,m}\phi^{(4)}_m(x) = x\frac{d}{dx}\phi^{(4)}_n(x) \\
&= [-xV'(x)\phi^{(4)}_n(x) + n\phi^{(4)}_n(x) + \ldots \\
&= -(\sum_{m,l} Q^{(4)}_{nm}[(V'(Q^{(4)}))_{ml}\phi^{(4)}_l(x)) + \ldots,
\end{aligned} \tag{32}$$

while for $\beta = 1$, we get

$$\begin{aligned}
x\phi^{(1)}_n(x) &= \sum R^{(1)}_{nm}\psi^{(1)}_m(x) \\
&= \int \frac{d}{dy}\left[y\phi^{(1)}_n(y)\right]\epsilon(x-y)dy \\
&= \int \left[-yV'(y)\phi^{(1)}_n(y)\right]\epsilon(x-y)dy \\
&\quad +(n+1)\psi^{(1)}_n(x) + \ldots \\
&= -(\sum_{m,l} Q^{(1)}_{nm}[(V'(Q^{(1)}))_{ml}\psi^{(1)}_l(x)) + \ldots.
\end{aligned} \tag{33}$$

Thus we get

$$\left[R^{(\beta)} + Q^{(\beta)}V'(Q^{(\beta)})\right] = \text{lower}_+, \tag{34}$$

where 'lower' denotes a strictly lower triangular matrix and 'lower'_+' a lower triangular matrix with the principal diagonal. Since $Q^{(\beta)}$ has only one band above the diagonal, Eqs.(31,34) confirm that $P^{(\beta)}$ and $R^{(\beta)}$ has d and $d+1$ bands above and below (since they are anti-self dual) the principal diagonal. However, unlike $\beta=2$, $Q^{(\beta)}$ for $\beta=1$ and 4 is not a finite band matrix.

$Q^{(\beta)}$ can be calculated in terms of the normalization constant $g_n^{(\beta)}$ and the coefficients of the polynomials $c_j^{(k,\beta)}$. For SOP, with $g_{2N}^{(\beta)}=g_{2N+1}^{(\beta)}$, we have

$$Q_{j,j+1}^{(\beta)} = \frac{c_j^{(j,\beta)}}{c_{j+1}^{(j+1,\beta)}}\sqrt{\frac{g_{j+1}^{(\beta)}}{g_j^{(\beta)}}}, Q_{j,j}^{(\beta)} = \frac{c_{j-1}^{(j,\beta)}}{c_j^{(j,\beta)}} - \frac{c_j^{(j+1,\beta)}}{c_{j+1}^{(j+1,\beta)}}.$$

Finally using Eqs.(25), (26) and the asymptotic results for these SOP [5], we present an alternative derivation of the level-density and the 'two-point' function for the Gaussian orthogonal and symplectic ensembles. For the Gaussian Ensembles, with $d=1$, (25) and (26) give

$$S_{2N}^{(4)}(x,y) = \frac{[xP_{(1,2)}^{(4)} - R_{(1,2)}^{(4)}]\phi_{(0,2)}^{(4)} + R_{(0,2)}^{(4)}\phi_{(1,2)}^{(4)} - R_{(1,3)}^{(4)}\phi_{(0,3)}^{(4)}}{x-y}, \tag{35}$$

and

$$S_{2N}^{(1)}(x,y) = \frac{[yP_{(1,2)}^{(1)} - R_{(1,2)}^{(1)}]\psi_{(0,2)}^{(1)} + R_{(0,2)}^{(1)}\psi_{(1,2)}^{(1)} - R_{(1,3)}^{(1)}\psi_{(0,3)}^{(1)}}{y-x}, \tag{36}$$

respectively, where

$$\phi_{(j,k)}^{(4)} \equiv [\phi_{2N+j}^{(4)}(x)\phi_{2N+k}^{(4)}(y) - \phi_{2N+j}^{(4)}(y)\phi_{2N+k}^{(4)}(x)] \tag{37}$$

(and similarly for $\psi_{(j,k)}^{(1)}$) and

$$A_{(j,k)}^{(\beta)} \equiv A_{2N+j,2N+k}^{(\beta)}. \tag{38}$$

For $\beta=4$, using $g_{2n}^{(4)}=(2n+1)!\pi^{1/2}2^{2n}$ and $c_n^{(n,4)}=(\sqrt{2})^{3n-1}$ [5], we have $Q_{(0,1)}^{(4)}=1/2\sqrt{2}$ and $Q_{(1,2)}^{(4)}=\sqrt{2}N$. From Eq.(31) and (34), with $u_2=2$ and $u_1=0$, we get

$$P_{(1,2)}^{(4)} = -2Q_{(1,2)}^{(4)} = -4N/\sqrt{2}, \tag{39}$$

$$R_{(0,2)}^{(4)} = -2Q_{(0,1)}^{(4)}Q_{(1,2)}^{(4)} = -N, \tag{40}$$

$$R_{(1,3)}^{(4)} = -2Q_{(1,2)}^{(4)}Q_{(2,3)}^{(4)} = -N. \tag{41}$$

For large N, we use the asymptotic results (with Gaussian weight) for the SOP [5]:

$$\phi_{2n+1}^{(4)}(x) = \frac{\sin\left[f^{(4)}(n,\theta)\right]}{n^{1/4}\sqrt{\pi\sin\theta}}, \tag{42}$$

$$\phi_{2n}^{(4)}(x) = \frac{1}{(4n)^{1/4}}\left[\frac{\cos\left[f^{(4)}(n,\theta)\right]}{2\sqrt{2n\pi}\sin^3\theta} + \frac{1}{2}\right], \tag{43}$$

where

$$\begin{aligned} f^{(4)}(n,\theta) &= (n+3/4)(\sin 2\theta - 2\theta) + \frac{3\pi}{4}, \\ &= 2\int_{-\sqrt{2n}}^{x}\rho(n,x)dx + \frac{3\pi}{4}, \end{aligned} \tag{44}$$

and $x = (2n + 3/2)^{1/2}\cos\theta$. For a given x, θ depends on n; for example $\theta_n - \theta_{n\mp1} \simeq \pm(2n\tan\theta_n)^{-1}$. Then with $y = x + \Delta x$ and $\theta \equiv \theta_n$, where $\theta_n \equiv (\theta_{2n}\theta_{2n+1})^t$, and expanding in θ and N, we get from the first term in Eq.(35):

$$xP^{(4)}_{(1,2)}[\phi^{(4)}_{(0,2)}] = -\frac{1}{4\pi}\frac{\cos\theta}{\sin^3}\sin\left(\Delta\theta\frac{\partial f^{(4)}(N,\theta)}{\partial\theta}\right)\sin 2\theta. \tag{45}$$

The second and third terms in Eq.(35) give

$$\begin{aligned} R^{(4)}_{(0,2)}[\phi^{(4)}_{(1,2)}] &= -R^{(4)}_{(1,3)}[\phi^{(4)}_{(0,3)}] \\ &= \frac{\cos 2\theta}{4\pi\sin^2\theta}\sin\left(\Delta\theta\frac{\partial f^{(4)}(N,\theta)}{\partial\theta}\right). \end{aligned} \tag{46}$$

We know that the odd SOP is arbitrary to the addition of a lower even order polynomial. This choice cancels the term $R^{(4)}_{(1,2)}[\phi^{(4)}_{(0,2)}]$. Collecting all the terms, we get

$$S^{(4)}_{2N}(x,y) = \frac{\sin[2\sqrt{(2N-x^2)}\Delta x]}{2\pi\Delta x}, \qquad |x| < \sqrt{2N}. \tag{47}$$

Taking $\Delta x \to 0$, we get the famous 'semi-circle', while

$$\frac{S^{(4)}_{2N}(x,y)}{S^{(4)}_{2N}(x,x)} = \frac{\sin[2\pi\Delta x S^{(4)}_N(x,x)]}{2\pi\Delta x S^{(4)}_{2N}(x,x)} = \frac{\sin 2\pi r}{2\pi r}, \tag{48}$$

where $r = \Delta x S^{(4)}_{2N}(x,x)$, gives the universal sine-kernel in the bulk of the spectrum.

For $\beta = 1$, we have from [5] $g^{(1)}_{2n} = (2n)!\pi^{1/2}2^{2n}$ and $c^{(2n,1)}_{2n} = -c^{(2n+1,1)}_{2n+1} = 2^{2n}$, which gives $Q^{(1)}_{(0,1)} = -1$, $Q^{(1)}_{(1,2)} = -N$. For $u_2 = 1$ and $u_1 = 0$, Eqs.(31) and (34) gives

$$P^{(1)}_{(1,2)} = -Q^{(1)}_{(1,2)} = N, \tag{49}$$

$$R^{(1)}_{(0,2)} = -Q^{(1)}_{(0,1)}Q^{(1)}_{(1,2)} = -N, \tag{50}$$

$$R^{(1)}_{(1,3)} = -Q^{(1)}_{(1,2)}Q^{(1)}_{(2,3)} = -N. \tag{51}$$

For large N, we use the asymptotic results for the SOP [5]:

$$\psi^{(1)}_{2n+1}(x) = \frac{\sin\left[f^{(1)}(n,\theta)\right]}{n^{1/4}\sqrt{\pi\sin\theta}}, \tag{52}$$

$$\psi^{(1)}_{2n}(x) = -\frac{1}{2n^{1/4}}\left[\frac{\cos\left[f^{(1)}(n,\theta)\right]}{\sqrt{n\pi\sin^3\theta}}\right], \tag{53}$$

where

$$\begin{aligned} f^{(1)}(n,\theta) &= (n+1/4)(\sin 2\theta - 2\theta) + \frac{3\pi}{4}, \\ &= \int_{-\sqrt{4n}}^{x}\rho(n,x)dx + \frac{3\pi}{4}, \end{aligned} \tag{54}$$

and $x = (4n+1)^{1/2}\cos\theta$. Writing $y = x + \Delta x$, and expanding in θ and N in (36), we get

$$S^{(1)}_{2N}(x,y) = \frac{\sin[\sqrt{(4N-x^2)}\Delta x]}{\pi\Delta x}, \qquad |x| < \sqrt{4N}. \tag{55}$$

Taking $\Delta x \to 0$, we get the level density, while

$$\frac{S_{2N}^{(1)}(x,y)}{S_{2N}^{(1)}(x,x)} = \frac{\sin[\pi S_{2N}^{(1)}(x,x)\Delta x]}{\pi \Delta x S_{2N}^{(1)}(x,x)} = \frac{\sin \pi r}{\pi r}, \tag{56}$$

where $r = \Delta x S_{2N}^{(1)}(x,x)$, gives the universal sine-kernel in the bulk of the spectrum.

For general d the correlation function corresponding to a weight with single support can be obtained using the asymptotic results for the SOP [18]. However, one needs to understand in greater detail the structure of the finite-band matrices and hence the matrix $Q^{(\beta)}$ to come up with a proof.

In conclusion, the unitary ensembles of random matrices, which involve the orthogonal polynomials have been well studied in recent years. In contrast, barring a few specific weights, nothing much is known about the OE and SE. This is mainly due to the hurdles created by the SOP.

In this paper, we have made some progress in understanding some of the basic properties of these SOP. In this context, we would like to emphasize that the GCD formula, derived in this paper, can at best be considered as the first step for a systematic study of the OE and SE of random matrices. One still needs to develop the theory further to come to an equal footing with the unitary ensemble of random matrices. For example, one would like to understand in greater details the asymptotic behavior of these SOP [18] to study different correlations for a larger family of OE and SE. We would also like to point out the similarity in the GCD formula for $\beta = 1$ and 4 with the interchange of Φ with Ψ. This may be useful in proving the duality between these two ensembles. We believe that these SOP satisfy a $d \times d$ differential system, which can be used to formulate a Riemann-Hilbert problem for these matrix models. We wish to come back to a few of these questions in a later publication.

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