Black Hole Dark Energy in 2+1 Gravity

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November 13, 2006

Abstract

In this paper we solve the Einsteins' field equations for a spherically symmetric anisotropic fluid, with kinematic self-similarity of the first kind, in spacetimes (2+1)-dimensional. Considering a null radial pressure we show that the fluid collapses forming a black hole in the end, even if it is constituted by phantom energy.

1 Introduction

Recently, self-similar solutions of the Einstein field equations have attracted a great deal of attention, not only because they can be studied analytically through simplification of the problem, but also because of their relevance in astrophysics and critical phenomena in gravitational collapse.

Lately, we studied self-similar solutions in various spacetimes. In particular, we considered a massless scalar field in $2+1$-dimensional circularly symmetric spacetimes with kinematic self-similarity of the second kind in the context of Einstein's theory, and found all of such solutions. We investigated their local and global properties and found that some of them represent gravitational collapse of a massless scalar field, in which black holes are always formed. In another paper we consider an anisotropic fluid, with the same self-similarity, and we showed that the unique solution, when the radial pressure is vanished, represents a dust fluid which collapses forming both naked singulaties and black holes.

In this paper, we extend the above studies to the case of an anisotropic fluid with zero radial pressure, but now in the self-similarity of the first kind context.

2 Solutions of the field equations

The general metric is given by

$$ds^2 = e^{2\phi}dt^2 - e^{2\psi}dr^2 - r^2S^2d\theta^2, \tag{1}$$

The non-null components of the Einstein's tensor are

$$G_{tt} = \frac{e^{-2\psi}}{r\,S}\left\{e^{2\phi}\psi_{,r}rS_{,r} + e^{2\phi}\psi_{,r}S - e^{2\phi}rS_{,rr} - 2e^{2\phi}\,S_{,r} + e^{2\psi}\psi_{,t}rS_{,t}\right\}, \tag{2}$$

$$G_{tr} = \frac{1}{r\,S}\left\{\phi_{,r}rS_{,t} + \psi_{,t}rS_{,r} + \psi_{,t}S - rS_{,tr} - S_{,t}\right\}, \tag{3}$$

$$G_{rr} = \frac{e^{-2\phi}}{r\,S}\left\{e^{2\phi}\phi_{,r}rS_{,r} + e^{2\phi}\phi_{,r}S + e^{2\psi}\phi_{,t}rS_{,t} - e^{2\psi}rS_{,tt}\right\}, \tag{4}$$

$$\begin{aligned} G_{\theta\theta} \;=\;& r^2S^2\{e^{-2\psi}\phi_{,r}^2 - e^{-2\psi}\phi_{,r}\psi_{,r} + e^{-2\psi}\phi_{,rr} + e^{-2\phi}\phi_{,t}\psi_{,t} \\ & -e^{-2\phi}\psi_{,t}^2 - e^{-2\phi}\psi_{,tt}\}. \end{aligned} \tag{5}$$

In order to study these solutions with self-similarity of the first kind we introduce two new adimensional variables, χ and τ, through of the relations

$$\chi = \ln(z) = \ln\left[\frac{r}{(-t)}\right] \tag{6}$$

$$\tau = -\ln(-t). \tag{7}$$

Note that we are considering that the range of the time coordinate is given by $-\infty < t \leq 0$. Using these transformations, and imposing that the metric must to depend only on the variable χ, we find

$$G_{tt} = -\frac{1}{r^2 S e^{2\psi}} \left\{ e^{2\phi} \left[S_{,\chi\chi} + S_{,\chi} - \psi_{,\chi} \left(S_{,\chi} + S \right) \right] - \frac{r^2}{t^2} \psi_{,\chi} S_{,\chi} e^{2\psi} \right\}, \tag{8}$$

$$G_{tr} = \frac{1}{tr\, S} \left\{ S_{,\chi\chi} - \psi_{,\chi} \left(S_{,\chi} + S \right) - S_{,\chi} \left(\phi_{,\chi} - 1 \right) \right\}, \tag{9}$$

$$\begin{aligned} G_{rr} \;\; = \;\; & \frac{1}{r^2\, S\, e^{2\phi}} \left\{ e^{2\phi} \left[\phi_{,\chi} \left(S_{,\chi} + S \right) \right] \right. \\ & \left. - \frac{r^2\, e^{2\psi}}{t^2} \left(S_{,\chi\chi} - S_{,\chi}\, \phi_{,\chi} \right) - \frac{r^2}{t^2} e^{2\psi} S_{,\chi} \right\}, \end{aligned} \tag{10}$$

$$\begin{aligned} G_{\theta\theta} \;\; = \;\; & S^2 \left\{ e^{-2\psi} \left[\phi_{,\chi\chi} + \phi_{,\chi} \left(\phi_{,\chi} - \psi_{,\chi} - 1 \right) \right] - \frac{r^2 e^{-2\phi}}{t^2} [\psi_{,\chi\chi} + \right. \\ & \left. - \psi_{,\chi} \left(\phi_{,\chi} - \psi_{,\chi} - 1 \right)] \right\}. \end{aligned} \tag{11}$$

The momentum-energy tensor is given by

$$T_{\mu\nu} \; = \; \rho\, u_\mu\, u_\nu \; + \; p_r\, r_\mu\, r_\nu + p_\theta\, \theta_\mu \theta_\nu, \tag{12}$$

where ρ is the energy density, p_r and p_θ are the radial and the tangential pressures, respectively, with u_μ, r_μ and θ_μ being given by

$$u_\mu = e^{\phi(\chi)} \delta^t_\mu \quad , \quad r_\mu \; = \; e^{\psi(\chi)} \delta^r_\mu \quad , \quad \theta_\mu \; = \; r\, S\left(\chi\right)\, \delta^\theta_\mu, \tag{13}$$

where we adopted the comoving frame.

Substituting (8)-(11) and (12) into the field equations given bellow

$$G_{\mu\nu} \; = KT_{\mu\nu}, \tag{14}$$

we have

$$\rho = -\frac{1}{K} \{ \frac{1}{r^2} e^{-2\psi} \left[y_{,\chi} + (y+1)(y - \psi_{,\chi}) \right] - \frac{1}{t^2} e^{-2\phi} \psi_{,\chi}\, y \}, \tag{15}$$

$$y_{,\chi} \; = \; y \phi_{,\chi} + (y+1)(\psi_{,\chi} - y), \tag{16}$$

$$p_r = \frac{1}{l^2 K} \{ \frac{1}{r^2} e^{-2\psi} \phi_{,\chi} (y+1) - \frac{1}{t^2}\, e^{-2\phi} \left[y_{,\chi} \; + \; y(y - \phi_{,\chi} + 1) \right] \}, \tag{17}$$

$$
\begin{aligned}
p_\theta &= \frac{1}{l^2 K}\{\frac{e^{-2\psi}}{r^2}\left[\phi_{,\chi\chi} + \phi_{,\chi}(\phi_{,\chi} - \psi_{,\chi} - 1)\right] - \frac{e^{-2\phi}}{t^2}[\psi_{,\chi\chi} + \\
&\quad -\psi_{,\chi}(\phi_{,\chi} - \psi_{,\chi} - 1)]\}. \qquad (18)
\end{aligned}
$$

Note that in the above equations we make

$$
y\,(\chi) = \frac{S_{,\chi}}{S}. \qquad (19)
$$

Then we have 4 equations and 6 functions to determine, that are ϕ ψ, S, ρ, p_r and p_θ. Thus, we consider two additional equations, which are $p_r = 0$ and $p_\theta = \omega\rho$, in order to solve the system. Doing the coordinates transformation (6), the state equations furnish 2 more equations, that are

$$
\phi_{,\chi}(\,y+1) - e^{2(\chi-\phi+\psi)}\left[\,y_{,\chi} + \,y(\,y - \phi_{,\chi} + 1)\right] = 0, \qquad (20)
$$

$$
\begin{aligned}
&[\phi_{,\chi\chi} + \phi_{,\chi}(\phi_{,\chi} - \psi_{,\chi} - 1)] \\
&+e^{2(\chi-\phi+\psi)}[-\omega\psi_{,\chi}\,y - \psi_{,\chi\chi} + \psi_{,\chi}(\phi_{,\chi} - \psi_{,\chi} - 1)]\} \\
&+\omega\,[y_{,\chi} + (\,y+1)(\,y - \psi_{,\chi})] = 0 \qquad (21)
\end{aligned}
$$

Substituting equation (**??**)into (**??**) equation we find

$$
(y+1)\left[\phi_{,\chi} - e^{(\chi-\phi+\psi)}\psi_{,\chi}\right] = 0. \qquad (22)
$$

Then, we have 2 possible solutions, which are

$$
(i)\ y+1 = 0 \qquad (23)
$$

and

$$
(ii)\ \phi_{,\chi} - e^{(\chi-\phi+\psi)}\psi_{,\chi} = 0. \qquad (24)
$$

We are not be able to solve analyticaly the second case, but starting of (i), and coming back into the other field equations, we have the solutions

$$
S\,(\chi) \;=\; S_0 e^{-\chi}, \qquad (25)
$$

$$
\psi\,(\chi) \;=\; \psi_0 + \ln\left[a + e^{(\omega-1)\chi}\right], \qquad (26)
$$

and

$$
\phi\,(\chi) \;=\; \phi_0, \qquad (27)
$$

where ϕ_0, ψ_0, S_0 and a are integration constants.

Finaly, the metric which represents this solution is given by

$$
ds^2 = dt^2 - \left[a + \left(\frac{r}{-t}\right)^{(\omega-1)}\right]^2 dr^2 - S_0^2 t^2 d\theta^2, \tag{28}
$$

since we done $\phi_0 = \psi_0 = 0$, without lost of gererality.

The Kretschmann's scalar, for these solutions are given by

$$
K(r,t) = \frac{4(\omega-1)^2(\omega^2+1)}{t^4\left[1 + ar^{(1-\omega)}(-t)^{(\omega-1)}\right]}. \tag{29}
$$

So, there is the possibility of formation of until two singularities, that are

$$
t = t_{sing} = 0 \tag{30}
$$

and

$$
r = r_{sing} = \left[\frac{(-t)}{|a|^{1/(1-\omega)}}\right]. \tag{31}
$$

The geometric radius Rg is defined by

$$
Rg = S\, r = -S_0\, t. \tag{32}
$$

Note that the geometric radius decreases with the time coordinate, indicating that the process can represents a collapse.

The expansion of an ingoing and outgoing null geodesics congruence is usefull to understand the global properties of the solutions. They are given by

$$
\theta_l = \theta_n = -S_0. \tag{33}
$$

3 The energy conditions

For $a > 0$, the signs of the energy density and of the tangential pressure

$$
\rho = \frac{(1-\omega)}{l^2Kt^2\left[1 + \left(\frac{r}{-t}\right)^{(1-\omega)}\right]a}, \tag{34}
$$

and

$$
p_\theta = \frac{\omega(1-\omega)}{l^2Kt^2\left[1 + \left(\frac{r}{-t}\right)^{(1-\omega)}\right]a}, \tag{35}
$$

are determined by the terms $1-\omega$ and $\omega(1-\omega)$, respectively. Then it is easy to see that for $-1 \leq \omega \leq 1$ all the energy conditions are satisfied. Moreover, if we admit the dark energy scenario, we can construct a collapse model from of a phantom anisotropic fluid, with $\omega \leq -1$. In this last case, all the energy conditions are violated, although the energy density preserves its positivity.

4 Conclusion

We obtain a solution of the Einsteins equations for an anisotropic and spherically symmetric, self-similar of the first kind fluid in a spacetime (2+1)-dimensional. We need to introduce the state equations, $p_r = 0$ and $p_\theta = \omega\rho$, in order to solve the problem and showed that there is a solution which represents a collapse process, resulting in a normal or phantom black hole in the end. In fact, to assure that we have a black hole, it is still necessary to match our solution with the BTZ vacuum solution. This is done for us now.

5 References

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