2009 International Nuclear Atlantic Conference - INAC 2009
Rio de Janeiro,RJ, Brazil, September27 to October 2, 2009
ASSOCIAÇÃO BRASILEIRA DE ENERGIA NUCLEAR - ABEN
***ISBN:* 978-85-99141-03-8**

SEMI ANALYTICAL SOLUTION OF POINT KINETIC EQUATION WITH SOURCE FROM COLD START UP TO DELAYED CRITICAL

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ABSTRACT

Point kinetics equations (P. K. E) are system of differential equations, which is solved simultaneously to get the neutron density as a function of time for a given reactivity input. P. K. E are stiff differential equations, computational solution through the conventional explicit method will give a stable consistent result only for smaller time steps. Analytical solutions are available either with step or ramp reactivity insertion without considering the source power contribution. When a reactor operates at low power, the neutron source gives a considerable contribution to the net reactor power. Similarly, when the reactor is brought to delayed critical with the presence of external source, the sub critical reactor kinetics studies with source power are important to understand the power behavior as a function of reactivity insertion rate with respect to the initial reactivity.

In the present work, P.K.E with one group delayed neutron are solved analytically to determine the reactor power as a function of reactivity insertion rate in the presence of neutron source. The analytical solution is a combination of converging two infinite series. Truncated infinite series is the analytical solution of P.K E. A general formulation is made by Combining both the ramp reactivity and step reactivity solution. So that the analytical solution could be useful in analyzing either step and ramp reactivity insertion exclusively or the combination of both. This general formulation could be useful in analyzing many reactor operations, like the air bubble passing through the core, stuck rod conditions, uncontrolled withdrawal of controlled rod, discontinuous lifting of control rod, lowering of rod and etc.

Results of analytical solutions are compared against the results of numerical solution which is developed based on Cohen's method. The comparisons are found to be good for all kind of positive and negative ramp reactivity insertions, with or without the combination of step reactivity. The methodology is found to be a promising tool for analyzing low power reactor with the inclusion of external source.

1. INTRODUCTION

When a nuclear reactor operates at low power, the neutron source gives a considerable contribution to the net reactor power. Similarly, when the reactor is brought to delayed critical with the presence of external source, the sub critical reactor kinetics studies with source power are important to understand the power behavior as a function of reactivity insertion rate with respect to the initial reactivity. Ziya Akcasu et al, [Ziya Akcasu et al, 1971] have solved the point kinetics equations analytically for any ramp reactivity insertion in to a critical reactor without considering the source power contribution. Prompt jump approximation was used in deriving the analytical expression. One group delayed neutron precursor is considered for the analysis. Fan Zhang et al [Fan Zhang et al , 2008] also used the one group model in deriving the analytical solution of point kinetics equation with source power. But the solution is more useful when the reactivity is linearly introduced discontinuously. Basically here, the ramp reactivity insertion is analysed with a solution which is a simple rearrangement of the one group P.K.E with prompt jump approximation. At the end of the ramp the net reactivity is assumed as a step reactivity input. Analytical solutions are derived based on the first order Bernoulli's differential equation for constant reactivity input (step reactivity input).

In the present work P.K.E with one group delayed neutron are solved analytically to determine the neutron density as a function of reactivity insertion rate with respect to the initial reactivity in the presence of neutron source. Prompt jump approximation is used in deriving the analytical solution. The analytical solution is a combination of converging two infinite series. Truncated infinite series is the analytical solution of P.K.E. A general formulation is made by Combining both the ramp reactivity and step reactivity solution. So that the analytical solution could be useful in analyzing either step and ramp reactivity insertion exclusively or the combination of both.

The general formulation could be useful in many reactor operating conditions, like the air bubble passing through the core, stuck rod conditions (step reactivity inputs), uncontrolled withdrawal of controlled rod (linear reactivity input), discontinuous lifting of control rod (combination of both step and positive ramp reactivity input), lowering of rod (combination of both step and negative ramp reactivity input). Since the formulation is made based on prompt jump approximation, the solution is not valid when the net reactivity is greater then one dollar. So, basically the methodology is a promising tool to analyse low power critical reactor with an external source.

Reactor kinetics analyses are carried out with the analytical solution, the results are found to be matching well with the numerical solution [B.Sharada and Om PalSingh, 1990], which is developed based on Cohen's method [Richard Cohen, 1958.]. The comparisons are found to be good for all kind of positive and negative ramp reactivity insertions, with or without the combination of step reactivity.

2. THE POINT KINETIC EQUATIONS AND ITS ANALYTICAL SOLUTION.

General point kinetics equations with the source power for one group delayed neutron precursor concentrations are,

$$\frac{dP}{dt} = \frac{\rho(t) - \beta}{\Lambda} P(t) + \lambda C(t) + S_P \tag{1}$$

$$\frac{dC(t)}{dt} = \frac{\beta}{\Lambda} P(t) - \lambda C(t) \tag{2}$$

Where P(t) is the reactor power, C(t), is the delayed neutron precursor concentration. ρ (t) is the net reactivity as a function of time and Sp is the source term in Watts/s. β are delayed neutron fraction. λ is the delayed neutron decay constant. Λ is prompt neutron generation time. There are two differential equations and two unknowns. The source term S_p(Watt/s) is related to neutron source strength S (neutrons/s) by the following equation,

$$S_p = \frac{S\mu}{\nu\Lambda} \tag{3}$$

Where, μ-energy released per fission

ν -Number of neutrons emitted per fission

Λ-Prompt neutron generation time

Suppose if the equilibrium sub critical power is a known parameter, the source power can derived by rearranging the point kinetics equations for equilibrium condition.

$$S_P = \frac{\rho(0)}{\Lambda} P_0 = \frac{(k-1)P_0}{k\Lambda} \tag{4}$$

Where P_0 is the equilibrium sub critical power corresponding to the sub critical reactivity $\rho(0)$ or the multiplication factor "k.". If a sub critical reactor is brought to delayed critical, initial power P(0) is calculated from the following equation.

$$P(0) = \frac{\beta P_0 + S\Lambda}{\beta - \rho(0)} \tag{5}$$

The point kinetic equation (1) with prompt jump approximations is,

$$\frac{\rho(t) - \beta}{\Lambda} P(t) + \lambda C(t) + S_P = 0 \tag{6}$$

Differentiating the equation (6) with respect to t, substituting equation (2) gives,

$$[\beta - \rho(t)]\frac{dP(t)}{dt} = P(t)\left[\frac{d\rho(t)}{dt} + \lambda\rho\right] + \Lambda\left[\lambda S_P + \frac{dS_P}{dt}\right] = 0 \tag{7}$$

The rate of change of source power is small in many of the practical application. The above equation can be simplified by neglecting the corresponding term,

$$[\beta - \rho(t)]\frac{dP(t)}{dt} = P(t)\left[\frac{d\rho(t)}{dt} + \lambda\rho(t)\right] + \Lambda\lambda S_P = 0 \tag{8}$$

The above equation is a general equation for any kind of reactivity insertion, with a condition that the maximum reactivity is less than one dollar and the minimum reactivity of any sub critical level.

2.1 Step Reactivity Input

For a step reactivity input the equation (8) will be reduced in to the following form,

$$\frac{dP(t)}{dt} - \frac{\lambda\rho_0}{[\beta - \rho_0]} P(t) = \frac{\Lambda\lambda S_P}{[\beta - \rho_0]} \tag{9}$$

Solution of equation (9) is,

$$P(t)e^{-\frac{\lambda\rho_0}{[\beta - \rho_0]}t} = -\frac{\Lambda S_P}{\rho_0} e^{-\frac{\lambda\rho_0}{[\beta - \rho_0]}t} + C_1 \tag{10}$$

Where C_1 is the integrating constant, can be determined from the steady state condition. i.e at time t=0, the power P(t) is assumed to be the initial power P(0). So, from equation (10) the value of C_1 is,

$$C_1 = P(0) + \frac{\Lambda S_P}{\rho_0} \tag{11}$$

Substituting the value of C1 in to equation (10), the analytical solution of point-kinetics equation with the presence of external source for only the step reactivity input is,

$$P(t) = \frac{\Lambda S_P}{\rho_0}\left[e^{\frac{\lambda\rho_0}{[\beta - \rho_0]}t} - 1 \right] + P(0)e^{\frac{\lambda\rho_0}{[\beta-\rho_0]}t} \tag{12}$$

Equation (12) is valid for any sub critical level to a maximum reactivity of less than one dollar. This equation also could be derived from the generalised equation presented by Fan Zhang et al.

2.2 Ramp Reactivity Insertion.

Bringing the reactor from the sub critical level to delayed critical and then to super critical could be done by either continuous or discontinuous linear reactivity insertion of order of few pcm/s based on the sensitivity of the detectors and sensors. The solution of P.K.E with ramp reactivity is required to analyze the given reactor condition. The analytical solution is formulated for the reactivity insertion $\rho = \rho_0 + \gamma' t$, where ρ, ρ_0 and γ't are in dollars. The general equation (8) with the ramp reactivity input is,

$$\frac{dP(t)}{dt} - \frac{\gamma'[1+\lambda t]}{[1-(\rho_0+\gamma' t)]}P(t) = \frac{\Lambda\lambda S_P}{\beta[1-(\rho_0+\gamma' t)]} \tag{13}$$

Solution of equation (13) is,

$$P(t)\left[[1-(\rho_0+\gamma' t)]^{1+\frac{\lambda}{\gamma'}} e^{\lambda t} \right] = \frac{\Lambda\lambda S_P}{\beta}\int [1-(\rho_0+\gamma' t)]^{\frac{\lambda}{\gamma'}} e^{\lambda t} dt \tag{14}$$

Solution of the R.H.S integral of equation (14) is obtained through integration by parts.

$$R.H.S.I = \frac{e^{\lambda t}}{\lambda}[1-(\rho_0+\gamma' t)]^{\frac{\lambda}{\gamma'}} + \frac{e^{\lambda t}}{\lambda}[1-(\rho_0+\gamma' t)]^{\frac{\lambda}{\gamma'}-1} + \frac{\gamma'}{\lambda}\left(\frac{\lambda}{\gamma'}-1\right)\frac{e^{\lambda t}}{\lambda}[1-(\rho_0+\gamma' t)]^{\frac{\lambda}{\gamma'}-2} + --+$$
$$\left(\frac{\gamma'}{\lambda}\right)^{n-2}\left(\frac{\lambda}{\gamma'}-1\right)\left(\frac{\lambda}{\gamma'}-2\right)--\left(\frac{\lambda}{\gamma'}-n+2\right)\int [1-(\rho_0+\gamma' t)]^{\frac{\lambda}{\gamma'}-n+1}\frac{e^{\lambda t}}{\lambda} + C_3 \tag{15}$$

Where C_3 is the integrating constant, determined from the steady state condition. Substituting the equation (14) from equation (15),

$$P(t) = \frac{\Lambda\lambda S_P}{\beta}\left[\frac{1}{\lambda}[1-(\rho_0+\gamma' t)]^{-1} + \frac{1}{\lambda}[1-(\rho_0+\gamma' t)]^{-2} + \frac{1}{\lambda}\frac{\gamma'}{\lambda}\left(\frac{\lambda}{\gamma'}-1\right)[1-(\rho_0+\gamma' t)]^{-3} + \right.$$
$$\left. \ldots\ldots\ldots\ldots\frac{1}{\lambda}\left(\frac{\gamma'}{\lambda}\right)^{n-2}\left(\frac{\lambda}{\gamma'}-1\right)\left(\frac{\lambda}{\gamma'}-2\right)--------\left(\frac{\lambda}{\gamma'}-n+2\right)[1-(\rho_0+\gamma' t)]^{-n} + \ldots \right] \tag{16}$$
$$C_3[1-(\rho_0+\gamma' t)]^{-\left(1+\frac{\lambda}{\gamma'}\right)} e^{-\lambda t}$$

C_3 is the integrating constant which can be determined from the steady state condition, i.e at time t=0 the initial power is P(0).

$$P(t)=\frac{\Lambda\lambda S_P}{\beta}\left[\frac{1}{\lambda}[1-(\rho_0+\gamma' t)]^{-1}+\frac{1}{\lambda}[1-(\rho_0+\gamma' t)]^{-2}+\frac{1}{\lambda}\frac{\gamma'}{\lambda}\left(\frac{\lambda}{\gamma'}-1\right)[1-(\rho_0+\gamma' t)]^{-3}+\right.$$
$$\left.\cdots\frac{1}{\lambda}\left(\frac{\gamma'}{\lambda}\right)^{n-2}\left(\frac{\lambda}{\gamma'}-1\right)\left(\frac{\lambda}{\gamma'}-2\right)\cdots\left(\frac{\lambda}{\gamma'}-n+2\right)[1-(\rho_0+\gamma' t)]^{-n}+\cdots\right]+$$
$$\left[P(0)-\frac{\Lambda\lambda S_P}{\beta}\left[\frac{(1-\rho_0)^{-1}}{\lambda}+\frac{(1-\rho_0)^{-2}}{\lambda}+\frac{\gamma'}{\lambda}\left(\frac{\lambda}{\gamma'}-1\right)\frac{(1-\rho_0)^{-3}}{\lambda}+\cdots+\left(\frac{\gamma'}{\lambda}\right)^{n-2}\left(\frac{\lambda}{\gamma'}-2\right)\right.\right. \quad (17)$$
$$\left.\left.-\left(\frac{\lambda}{\gamma'}-n+2\right)\frac{(1-\rho_0)^{-n}}{\lambda}+\cdots\right]\right](1-\rho_0)^{\left(1+\frac{\lambda}{\gamma'}\right)}[1-(\rho_0+\gamma' t)]^{-\left(1+\frac{\lambda}{\gamma'}\right)}e^{-\lambda t}$$

equation (17) is the analytical solution for point kinetics equation, which can be used for a reactor from any sub critical level to delayed critical condition. Equation (17) is a combination of two infinite series and the validity of the equation is based on convergence. But the solution should work for all the reactivity in the given range $-\infty<\rho_0+\gamma' t\leq 1$.

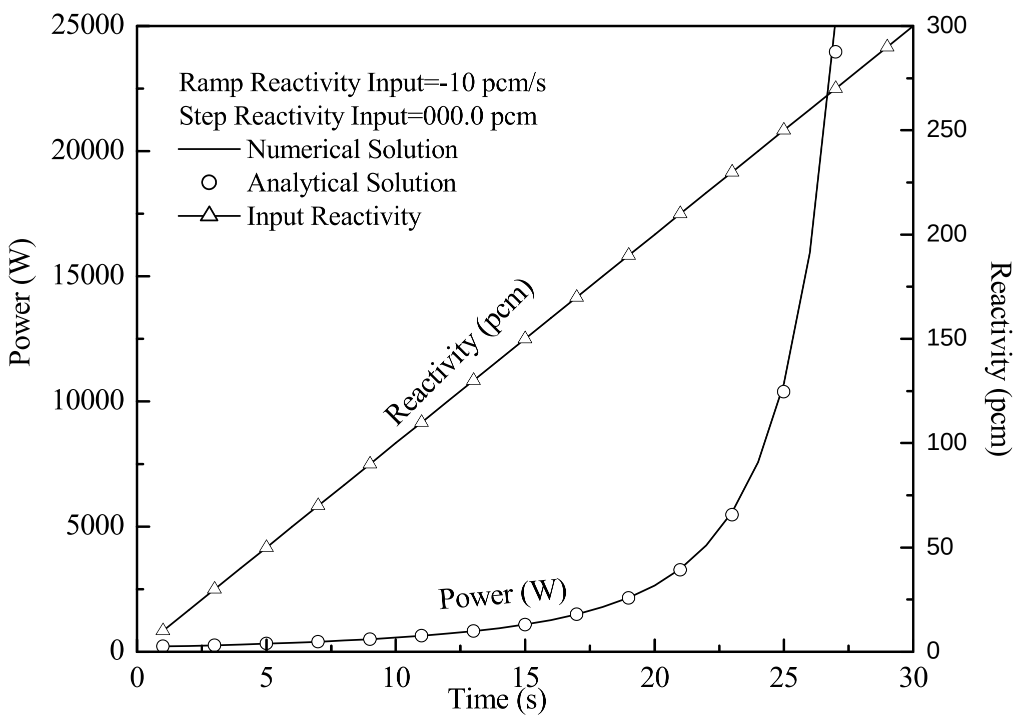

Fig. 1: Comparison of Numerical and analytical solution for 10 pcm/s Ramp Reactivity Input

3. RESULTS.

3.1 Comparisons of Analytical Solutions with Numerical Results

With the validity check results of the analytical solutions are compared against the numerical code POKIN which is developed based on Cohen's method for accuracy. Few cases are taken up for the analysis. Ramp reactivity insertion of 10 pcm/s to a critical reactor is taken for the analysis. The analytical solution is found to be matching well with the numerical results as shown in the Fig.1.

The comparison is found to be good for 10 pcm/s ramp reactivity insertion with -100 pcm step reactivity input also. The results are found to be matching well as shown in the Fig.2. So, from Fig.1 and Fig.2, it is concluded that accuracy of analytical solutions is good for any positive ramp reactivity insertions with or without definite step reactivity input. The analyses are carried out and compared for negative ramp reactivity insertion also. Test case of -1500 pcm step reactivity input and -10 pcm/s ramp reactivity are also analysed. The comparison is found to be good as shown in Fig.3.

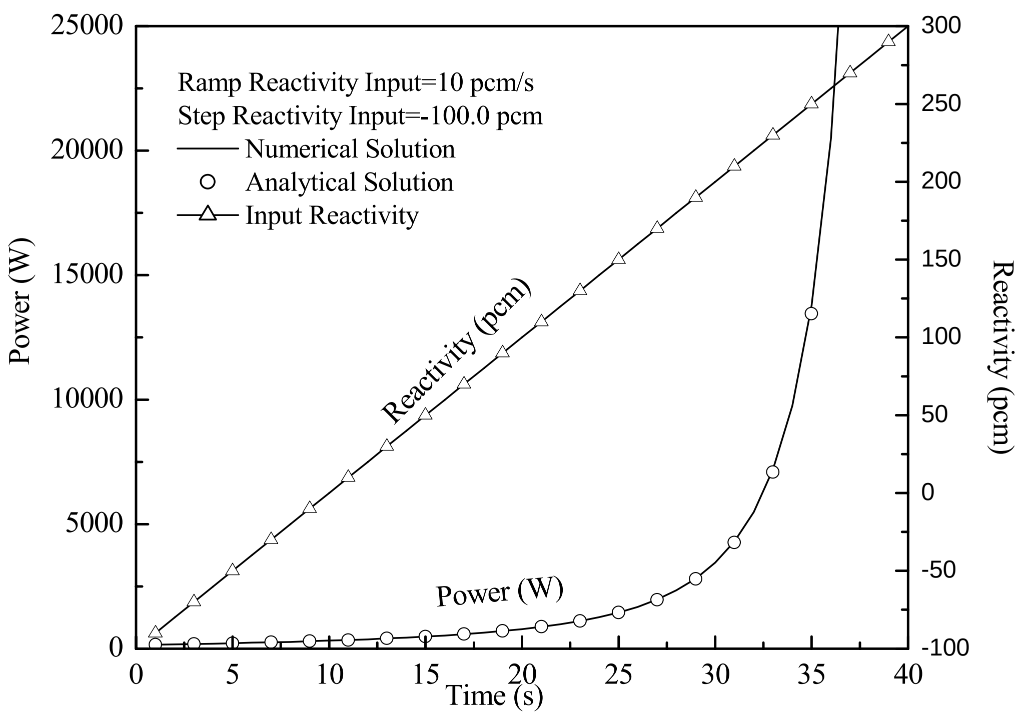

Fig. 2: Comparison of Numerical and analytical solution for 10 pcm/s Ramp Reactivity Input and -100 pcm/s step reactivity input

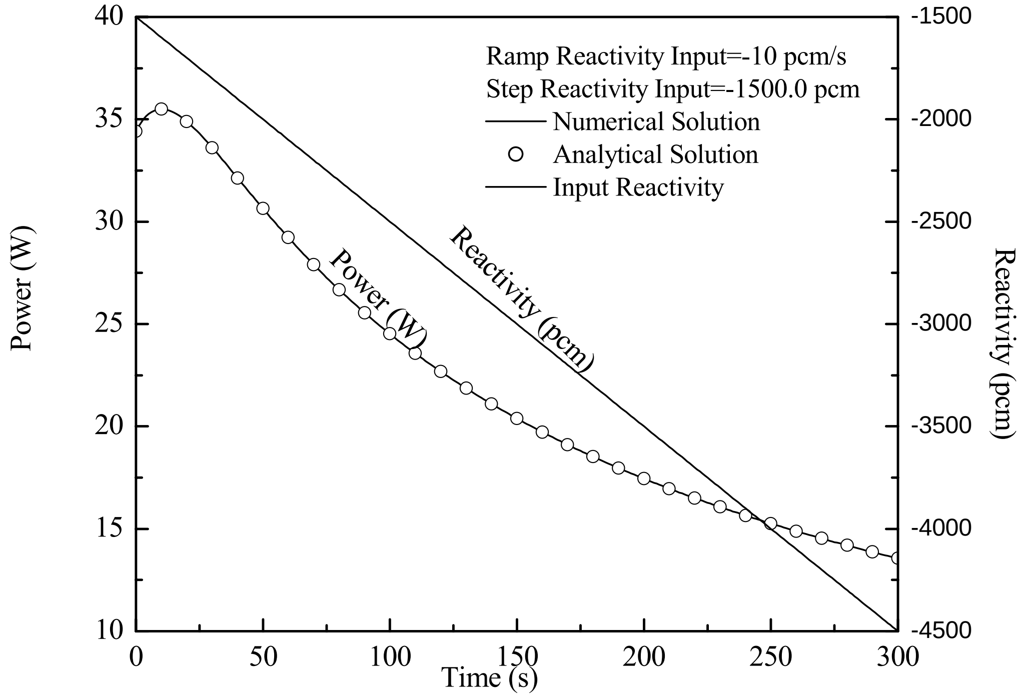

Fig. 3: Comparison of Numerical and analytical solution for -10 pcm/s Ramp Reactivity Input and -1500 pcm/s step reactivity input

3.2 Discussion and Future Development

P.K.E with one group delayed neutron are solved analytically to determine the neutron density as a function of reactivity insertion rate with respect to the initial reactivity in the presence of neutron source. Prompt jump approximation is used in deriving the analytical solution. The analytical solution is a combination of converging two infinite series. Truncated infinite series is the analytical solution of P.K.E. Though equation (17) is general solution for any ramp and step reactivity input, with any number (more number) of terms in the infinite series, the solution should converges to consistent result, with a minimum truncated error. In the present solution, beyond certain number of terms the infinite series doesn't converge to consistent result. If the nth term value R is the product of P and Q, the value of R,P,Q are the following,

$$
\begin{aligned}
.R &= \frac{1}{\lambda}\left(\frac{\gamma'}{\lambda}\right)^{n-2}\left(\frac{\lambda}{\gamma'}-1\right)\left(\frac{\lambda}{\gamma'}-2\right)\text{---}\left(\frac{\lambda}{\gamma'}-n+2\right)\left[[1-(\rho_0+\gamma' t)]^{-n}\right] \\
P &= \frac{1}{\lambda}\left(\frac{\gamma'}{\lambda}\right)^{n-2}\left(\frac{\lambda}{\gamma'}-1\right)\left(\frac{\lambda}{\gamma'}-2\right)\text{---}\left(\frac{\lambda}{\gamma'}-n+2\right): and \\
Q &= [1-(\rho_0+\gamma' t)]^{-n}
\end{aligned} \tag{18}
$$

The condition for convergence is that the value of R should decrease with n and finally go to zero. In case of positive net reactivity $0<(\rho_0+\gamma' t)<1$ at a given time "t", the value of $[1-(\rho_0+\gamma' t)]^{-n}$ is a diverging function with respect to n. In that case if the function P decreases with "n", such that the product of functions P and Q decreases with n, then the series is a converging series. Similarly, for negative ramps, function P is a diverging function, In that case if the function Q decreases with "n", such that the product of the functions P and Q will decrease with n, then the series is a converging series. For example, for a given positive reactivity insertion with a negative step reactivity input, behavior of the functions P, Q and R are shown in Fig.4. If the value of 'R' decreases with 'n', then the solution is expected to converge to the correct solution, but once if n goes beyond certain n, the function R starts diverging and the series may not converged to the expected results. Similarly for negative ramp reactivity insertion with negative step, the behaviour of the fuctions P,Q and R are shown in Fig.5. Convergence of the series and the correct number of terms considered for the analysis are decided based on the numerical value of step and ramp reactivity insertion.

For any positive ramp reactivity insertion the function P decreases up to 'n'number of terms, after that it starts diverging with 'n'. So, with Q, convergence of the series is expected in the region before P reaches n. so the number of terms chosen in the series should satisfy the criteria $n < \frac{2\lambda}{\gamma'}$. For any positive ramp insertion, at n if the value of R is less than 0.1, then the series is a convergent series and the number of terms considered in the series can 'n'. But for, negative ramp reactivity insertions, the function P is a diverging function. So if the series is converging series with Q, then the value of R as a function of n(number of terms) should satisfy the ratio test. If R_1, R_2, R_3…are the terms of the function R, if $RTEST = \frac{R_{n+1}}{R_n} < 1$, then the series is a convergent series. From the equation (18), based on the value of R,

$$
RTEST = \frac{R_{n+1}}{R_n} = \frac{1-\frac{(n-2)\gamma'}{\lambda}}{[1-(\rho_0+\gamma' t)]^{-n}} < 1 \tag{19}
$$

The value of n can be determined from Equation 19, and for that n if the value of R is less than 0.1, than the number of terms considered can be up to n, and the series is expected to converge to the correct solution.

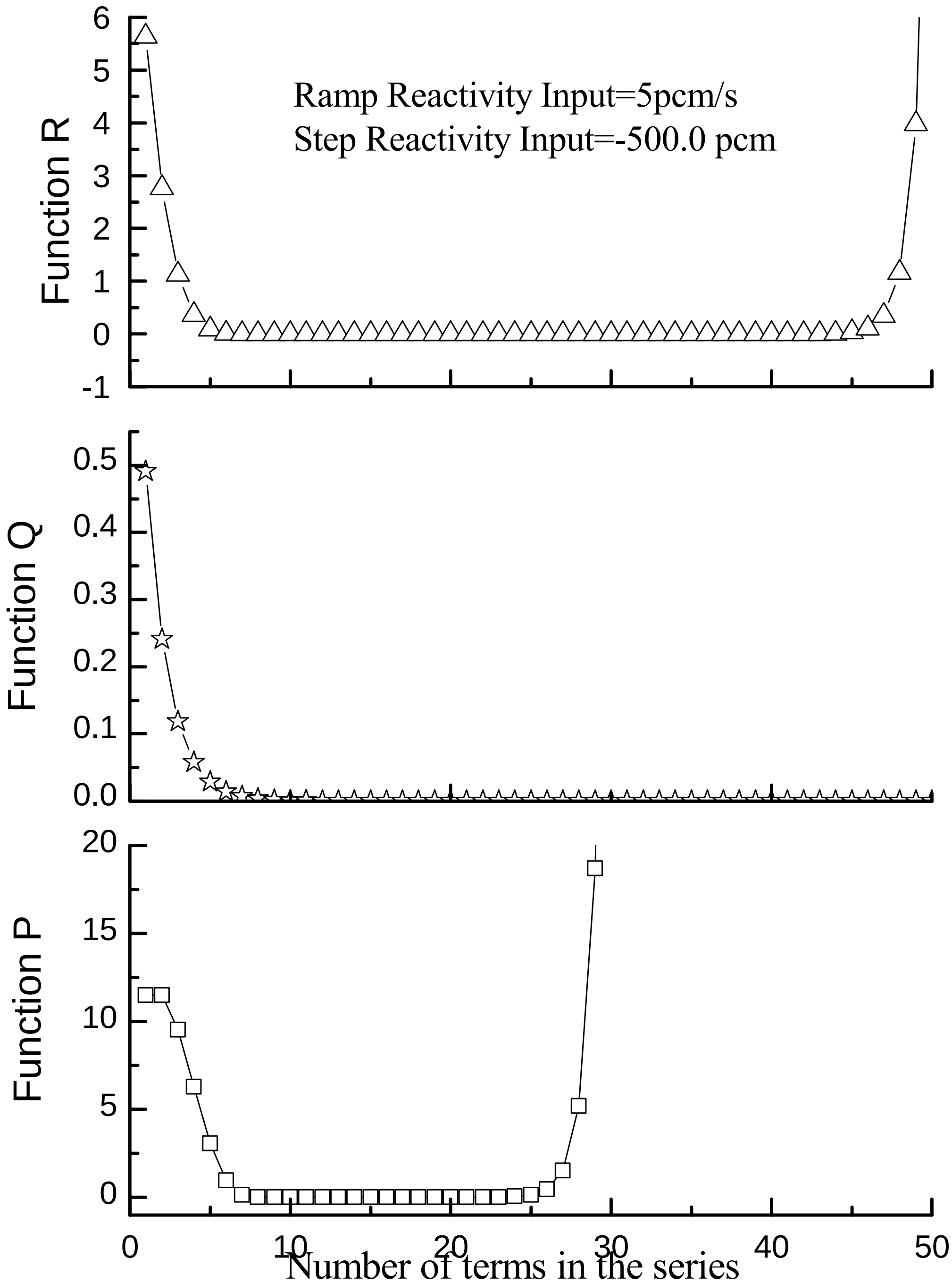

Fig. 4: Function of P, Q and R with the number of terms considered in the series

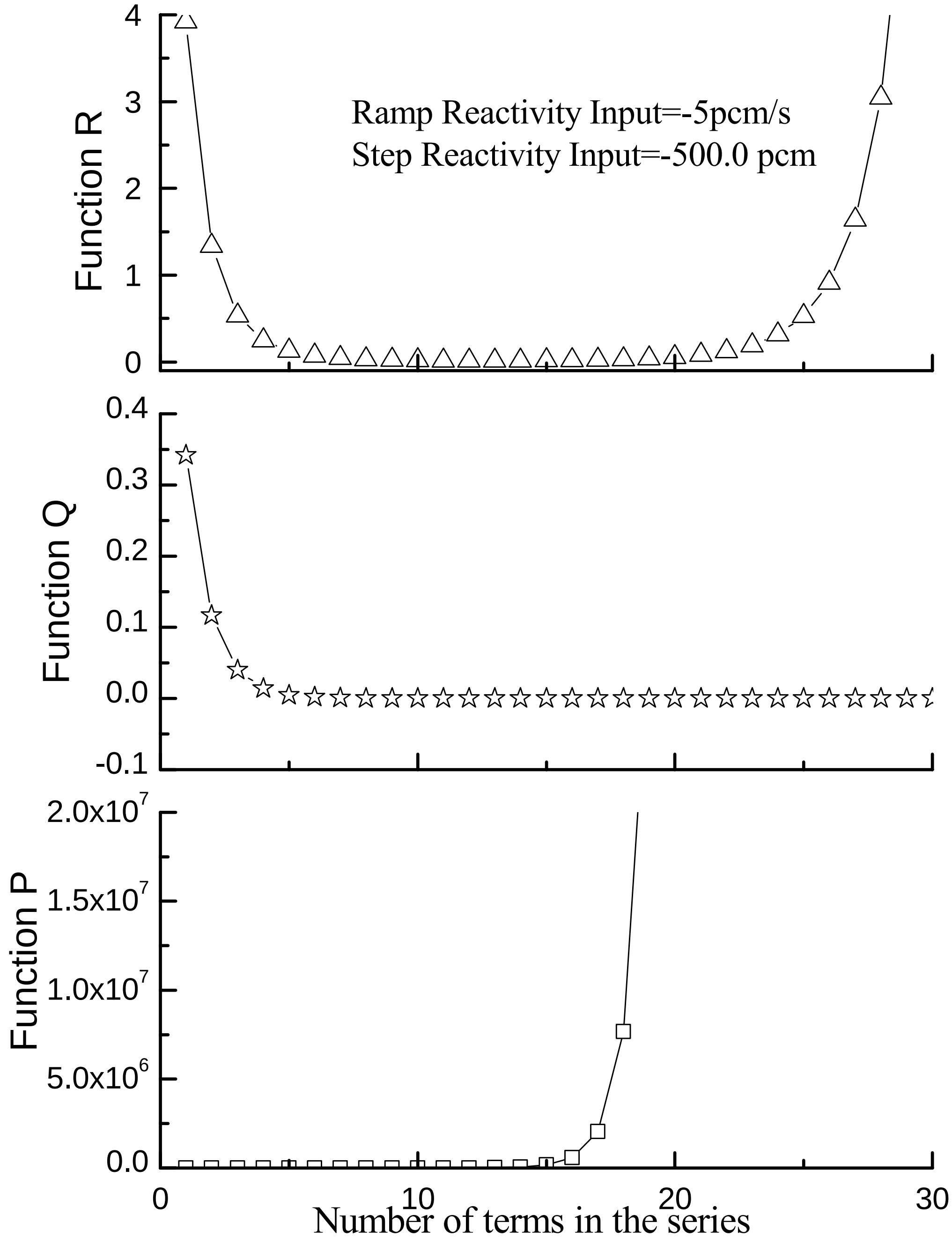

Fig. 5: Function of P, Q and R with the number of terms considered in the series

4 CONCLUSIONS

P.K.E with one group delayed neutron are solved analytically to determine the neutron density as a function of reactivity insertion rate with respect to the initial reactivity in the presence of neutron source. Prompt jump approximation is used in deriving the analytical solution. The analytical solution is a combination of converging two infinite series. Truncated infinite series is the analytical solution of P.K.E. A general formulation is made by Combining both the ramp reactivity and step reactivity solution. So that the analytical solution could be useful in analyzing either step and ramp reactivity insertion exclusively or the combination of both.

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AKNOWLEDGEMENT

The author would like to thank Dr P.Mohanakrishnan Head, RPD for his support and encouragement. The author would like to thank Dr R.Harish RPD/IGCAR for his encouragement useful suggestions.

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