Effective actions and topological strings

Off-shell mirror symmetry and mock modularity of multiple M5-branes

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In memory of my father

Abstract

This thesis addresses two different topics within the field of string theory.

In the first part it is shown how Hodge-theoretic methods in conjunction with open string mirror symmetry can be used to compute non-perturbative effective superpotential couplings for type II/F-theory compactifications with D-branes and fluxes on compact Calabi-Yau manifolds. This is achieved by studying the flat structure of operators which derives from the open/closed B-model geometry. We analyze the variation of mixed Hodge structure of the relative cohomology induced by a family of divisors, which is wrapped by a D7-brane. This leads to a Picard-Fuchs system of differential operators, which can be used to compute the moduli dependence of the superpotential couplings as well as the mirror maps at various points in the open/closed deformation space. These techniques are used to obtain predictions for genuine A-model Ooguri-Vafa invariants of special Lagrangian submanifolds in compact Calabi-Yau geometries and real enumerative invariants of on-shell domain wall tensions.

By an open/closed duality the system of differential equations can also be obtained from a gauged linear σ-model, which describes a non-compact Calabi-Yau four-fold compactification without branes. This is used in the examples of multi-parameter models to study the various phases of the combined open/closed deformation space. It is furthermore shown how the brane geometry can be related to a F-theory compactification on a compact Calabi-Yau four-fold, where the Hodge-theoretic techniques can be used to compute the G-flux induced Gukov-Vafa-Witten potential. The dual F-theory picture also allows to conjecture the form of the Kähler potential on the full open/closed deformation space.

In the second part we analyze the background dependence of theories which derive from multiple wrapped M5-branes. Using the Kontsevich-Soibelman wall-crossing formula and the theory of mock modular forms we derive a holomorphic anomaly equation for the modified elliptic genus of two M5-branes which are wrapped around a rigid divisor inside a compact Calabi-Yau manifold. The non-holomorphicity of the modified elliptic genus in this situation is traced back to the contribution of non-trivial BPS bound states of M5-branes. As a byproduct a result from pure mathematics obtained by Göttsche is re-derived in physical terms, which concerns the structure of the moduli spaces of stable sheaves on certain complex surfaces. The anomaly equation fulfilled by the modified elliptic genus is shown to be consistent with the holomorphic anomaly equations observed in the context of $\mathcal{N} = 4$ topological Vafa-Witten theory on $\mathbb{P}^2$ and theories of E-strings obtained from wrapping M5-branes on del Pezzo surfaces. In selected examples it is argued how the holomorphic anomaly equation supplemented with appropriate boundary conditions can be used to explicitly compute the BPS degeneracies for certain charges.

This work is based on the original publications [1, 2, 3, 4].

Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit zwei verschiedenen Themen aus dem Gebiet der Stringtheorie.

Im ersten Teil werden Hodge-theoretische Methoden und Mirrorsymmetrie offener Strings dazu verwendet um nicht-perturbative effektive Superpotentiale in Typ II/F-Theorie Kompaktifizierungen auf kompakten Calabi-Yau Mannigfaltigkeiten mit D-Branen und Flüssen zu berechnen. Möglich ist dies aufgrund der zugrundeliegenden flachen Ringstruktur der Operatoren im kombiniert offen/geschlossenen B-Modell. Für bestimmte D-Branengeometrien entspricht diese Struktur der mathematischen Theorie von Variationen gemischter Hodge-Strukturen einer relativen Kohomologiegruppe. Im Falle einer D7-Brane, welche um eine Familie von Vierzyklen gewickelt ist, wird hiermit ein System von Picard-Fuchs Gleichungen abgeleitet, welches die relativen Perioden des Deformationsproblems annihiliert. Da die relativen Perioden für diese Geometrien das von den D-Branen induzierte Superpotential, sowie die Mirrorabbildungen beschreiben, können die Picard-Fuchs Gleichungen dazu benutzt werden, um nicht-perturbative Superpotentialkopplungen in der Umgebung zahlreicher Punkte im offen/geschlossenen Deformationsraum explizit zu berechnen.

Mithilfe dieser Techniken trifft die Arbeit Vorhersagen für Ooguri-Vafa Invarianten von speziellen Lagrange-Branen in kompakten Calabi-Yau Geometrien, sowie für reell enumerative Invarianten von Massendichten supersymmetrischer Domänenwände. Eine offen/geschlossene Stringdualität, welche die Branengeometrie auf eine Calabi-Yau Viermannigfaltigkeit ohne Branen abbildet, kann desweiteren dazu benutzt werden, um das Picard-Fuchs System der D-Branengeometrie aus einem 'gauged linear σ-model' herzuleiten. In Modellen mit mehreren Deformationsparametern werden hiermit, unter Zuhilfenahme von Methoden aus der torischen Geometrie, die unterschiedlichen Phasen des kombinierten offen/geschlossenen Deformationsraums untersucht. Zudem erläutert die Arbeit wie die Branengeometrie in eine F-Theorie Kompaktifizierung auf einer kompakten Calabi-Yau Viermannigfaltigkeit ohne D-Branen übersetzt werden kann. Die Hodge-theoretischen Methoden erlauben in diesem Fall die Berechnung von G-Fluss induzierten Gukov-Vafa-Witten Potentialen. Aus dem dualen Bild der F-Theorie wird außerdem die Form des Kähler Potentials auf dem kombiniert offen/geschlossenen Deformationsraum der D7-Brane abgeleitet.

Der zweite Teil der Arbeit ist der Untersuchung der Hintergrundabhängigkeit von Theorien mit mehreren M5-Branen gewidmet. Unter Verwendung der Kontsevich-Soibelman 'wall-crossing' Formel und der Theorie der mock-modularen Formen wird eine holomorphe Anomaliegleichung für den modifizierten elliptischen Genus von mehreren M5-Branen auf einem rigiden Divisor innerhalb einer kompakten Calabi-Yau Mannigfaltigkeit hergeleitet. Die Nichtholomorphizität des modifizierten elliptischen Genus kann dabei auf den Beitrag von nicht-trivialen Bindungszuständen von M5-Branen zurückgeführt werden. Als ein Nebenprodukt wird mithilfe physikalischer Überlegungen ein Resultat aus der reinen Mathematik von Göttsche hergeleitet, welches die Struktur der Moduliräume von stabilen Garben auf bestimmten komplexen Flächen betrifft. Zudem diskutiert die Arbeit die Konsistenz der gefundenen holomorphen Anomaliegleichung mit den holomorphen Anomaliegleichungen, welche im Fall der $\mathcal{N} = 4$ topologischen Vafa-Witten Theorie, sowie im Rahmen von Theorien sogenannter E-Strings, welche von M5-Branen auf del Pezzo Flächen herrühren, abgeleitet wurden. In ausgewählten Beispielen wird gezeigt, wie die holomorphe Anomaliegleichung dazu benutzt werden kann um die BPS-Entartungen für bestimmte Ladungen, unter Zuhilfenahme geeigneter Randbedingungen, explizit zu berechnen.

Diese Doktorarbeit basiert auf den Veröffentlichungen [1, 2, 3, 4].

Contents

Part I

Introduction

1 Chapter 1

Motivation and overview

Contents

1.1. Motivation

This thesis is devoted to topics in string theory, which over the past decades has emerged as the prime candidate for a theory which combines quantum mechanics and general relativity in a sensible way. The original idea of string theory, born in a completely different context in hadronic physics, is that elementary particles might not be described by mathematical points but rather by tiny extended strings, whose excitations are interpreted as different particles. This simple idea has remarkable consequences.

First of all the spectrum of theories with closed strings contains an excitation which corresponds to a massless spin two particle. Such an excitation is the hallmark of gravity and therefore string theories naturally incorporate general relativity and gravity. Amazingly the quantum mechanical scattering amplitudes between strings with such excitations produce UV-finite results, which means that string theories are a sensible UV-completion of quantum gravity. A second consequence of the string idea is that quantum mechanical consistency of string theories often fixes the dimension[1] of the space-time the string propagates in. For the well-studied and phenomenological most interesting theories, the type IIA/B and $E_8 \times E_8$ or $SO(32)$ heterotic strings, the space-time dimension is for example fixed to $D = 10$. At first sight this seems as a huge drawback, as only four space-time dimensions have so far been observed. However with the compactification idea of Kaluza and Klein it can be turned into a partial or at least theoretical very appealing advantage. For this it is usually assumed that the unobserved space-time dimensions are small[2] and their existence therefore does not affect the physics at energies which are accessible by current experiments, such as collider or astro-

[1] Strictly speaking the notion of dimension is tied to a geometric interpretation of the theory. However such a notion might not always immediately exist or even depend on the position in moduli space (see e.g. the Calabi-Yau – Landau-Ginzburg correspondence as in [5]).

[2] In appropriate units.

physical experiments. The shape of the six-dimensional 'inner space' nonetheless governs the effective physics in four dimensions which can be derived from the study of string propagation in such higher dimensional spaces. The geometry and symmetries of the compactification manifold for example determine the particle content and the gauge symmetries which can be observed in the lower dimensions. A further interesting property of string models is that such theories contain only one adjustable parameter, namely the tension of a fundamental string, while the other scales and parameters are set by (possibly complicated) mechanisms and dynamics. This is a very appealing feature, as phenomenological theories of particle physics such as the standard model or models beyond it generically contain a plethora of parameters which can only be fixed by direct comparison with experiment.

All of this hints at the possibility of 'geometrizing' the abstract symmetries which were found in nature and are at the heart of the standard model of elementary particle physics. The hope is thus to derive the properties of the real four dimensional world as a direct consequence of the complicated shape and geometry of a higher-dimensional space-time itself and the objects which can propagate in it.

In summary a theory of strings seems to have a lot of the ingredients one would expect of a fundamental theory of nature, which describes gravity and the other fundamental forces in a unified framework. Unfortunately the current understanding of string theory does not allow to single out a particular preferred background or even a small enough class of backgrounds from which standard-model like physics could be uniquely extracted. One in fact seems to be stuck in a vast landscape of admissible geometries and furthermore it turns out that it is technically challenging to extract the four dimensional physics which is encoded in a particular compactification. Unfortunately it seems therefore hard to connect string theory to the real world.[3]

Nonetheless the ideas originating from string theory have changed the understanding of the inner workings of the theoretical models underlying the most successful theories of nature, such as quantum field theories and general relativity. These insights are not tied to the question whether string theory might describe the real world at very short distances and high energies and remain true even if string theory might be falsified at some point.

Not only in physics but also in the field of pure mathematics ideas from string theory have triggered tremendous progress over the last decades. Among the reasons for this 'unreasonable effectiveness' of string theory in mathematics seems to be the fact that strings and more generally extended objects probe geometries and algebraic structures very different from ordinary point particles. In addition the study of non-perturbative aspects of (supersymmetric) field theories hints at a deep connection between geometry and non-perturbative quantum physics, as was for example shown in the celebrated Seiberg-Witten non-perturbative low-energy solution of $\mathcal{N} = 2$ super-Yang-Mills theory [6, 7]. These two aspects are actually interrelated as can be deduced from remarkable non-perturbative dualities between different

[3]There has however been tremendous progress on these questions in the last decades.

string theories and with the help of the good technical control over quantum corrections due to supersymmetry, which is built into a lot of models coming from string constructions.

D-branes and dualities

String theories generically not only contain strings as fundamental degrees of freedom, but also higher dimensional solitonic objects, called D-branes [8]. These are exact non-perturbative states of string theory[4] and their perturbative excitations are described by open strings which can end on the D-branes. The discovery of D-branes led to the possibility of studying non-perturbative aspects of string theory in great detail. A remarkable consequence of these studies was for example the derivation of the entropy of certain extremal black holes from a microscopic picture of black holes as being built from D-brane micro-states [9]. This can be interpreted as another hint that string theories indeed contain the right degrees of freedom for a sensible theory of quantum gravity.

The study of non-perturbative effects in string theories using D-branes led also to various remarkable dualities which connect seemingly different string theories in ten dimensions with each other. Some of them exchange the perturbative states of one theory with the non-perturbative states of another theory and map weak and strong coupling regimes onto each other. A central theme in this is played by M-theory[5] [12, 13], the conjectured 11-dimensional strong coupling limit of type IIA string theory. Unfortunately up to now the details of M-theory are poorly understood, however some of its properties have been uncovered indirectly, as it can be related to other theories by dualities.[6] The physical degrees of freedom of M-theory are not that of a theory of strings anymore but rather given by a membrane, called M2-brane and its magnetic dual, the M5-brane. The fundamental type IIA string and the various D-brane states arise upon compactification of these M2- and M5-branes on a circle to ten dimensions. By a duality called T-duality, which relates circle compactifications of type IIA and IIB strings, M-theory on an elliptically fibered Calabi-Yau four-fold can be related to yet another theory called F-theory [15], which is a non-perturbative description of type IIB string theory and lives in twelve dimensions[7]. Remarkably the rather abstract F-theory has recently been used to obtain very stringent and (semi-)realistic models of particle physics [16].

Supersymmetry and topological string/field theories

A second property of the above mentioned string models is supersymmetry[8] [17], an extension of Poincaré symmetry which maps bosonic and fermionic degrees of freedom onto each other.

[4] Or better string field theory.

[5] See e.g. [10, 11] for pedagogical introductions.

[6] Recently there has for example been a lot of progress on the question of the world-volume theory which lives on N M2-branes in certain backgrounds and an explanation of the $N^{3/2}$ scaling behavior of its number of degrees of freedom, see e.g. [14] for a detailed account.

[7] Two of these 'dimensions' actually describe the axio-dilaton of the type IIB theory.

[8] Both in the space-time and worldsheet sense.

Supersymmetric theories often contain a sector of states, which is largely protected against quantum corrections and therefore under good technical control. A lot of results about string compactifications and non-perturbative aspects of the theory rely on the rigidity of these so-called BPS-states. For example some of the interesting couplings of the low-energy effective action of a string compactification are protected against quantum corrections by holomorphicity and receive only contributions from BPS-saturated states. In some cases it is even possible to replace the full string theory by another theory, called topological string theory [18, 19, 20, 21], which captures the protected couplings and whose physical degrees of freedom are precisely the BPS-saturated states or a subset of them.[9]

Topological string and field theories are more easier to handle than their physical cousins and have some remarkable properties. These theories are for example generically independent of a large class of deformations of the full underlying string theory. Furthermore correlation functions do not depend on a particular worldsheet metric and the semi-classical approximation is exact. Therefore the correlation functions localize to configurations which are invariant under the topological symmetry of the theory and get only contributions from fixed topological sectors. The contributions in each sector yield topological or enumerative invariants of the target space geometry.

An example of this is the topological A-model on a Calabi-Yau space Z, which can be obtained from the type IIA string by a procedure called topological twisting[10] [20]. The correlation functions of this theory compute the prepotential of the effective $\mathcal{N}=2$ supergravity theory in four dimensions, which a compactification of closed type IIA strings engineers.[11] The topological A-model correlators get only contributions from holomorphic maps and count the number of such maps from a string worldsheet of fixed genus to the Calabi-Yau manifold in a way which can be made mathematically rigorous. The topological A-model is insensitive to complex structure deformations of the target space geometry and in this sense yields topological invariants of the Calabi-Yau manifold Z. The enumerative invariants one obtains are called Gromov-Witten invariants [20, 26, 27] and can be used to distinguish more generally symplectic manifolds. By physical arguments Gromov-Witten invariants can be related to the counting of BPS-states of M2-branes in a M-theory compactification to five dimensions [28, 29]. As the appropriate counting of the BPS-states coming from M2-branes produces integer invariants [28, 29] this led to the remarkable conjecture that there is a hidden integrality structure in Gromov-Witten theory, which is a priori hard to see from the pure mathematical side.

Another example of a topological field theory is the Vafa-Witten gauge theory on a complex surface P [30]. This theory can be obtained by twisting an $\mathcal{N}=4$ supersymmetric Yang-Mills

[9]See [22, 23] for the case of effective couplings in type II compactifications.

[10]This procedure changes the spins of the worldsheet fields. The twisting of the type IIB string leads to the so-called B-model.

[11]See [24, 25] for the closely related engineering of $\mathcal{N}=2$ quantum field theories form type II strings.

theory with $U(r)$ gauge group on the complex surface. The partition function of the Vafa-Witten theory gets only contributions from topological non-trivial instanton configurations and is independent of the complex structure on the surface P. It can be shown that the contributions to the partition function compute the Euler characteristics of moduli spaces of stable sheaves on these surfaces and these are related to the notion of Donaldson invariants in mathematics [31, 32]. Therefore this theory can be used to study on the one hand the abstract theory of stable sheaves on complex surfaces and on the other hand the structure of micro-states of BPS black-holes in supergravity theories, to which the Vafa-Witten theory can be related by dualities.[12]

Albeit the topological theories greatly simplify the technical problems for a particular class of interesting physical quantities and couplings and have deep connections to mathematics it is nonetheless a complicated problem to explicitly compute correlation functions and therefore the quantities of interest. Sometimes these problems can be overcome by an application of the dualities mentioned above. For example for the case of the topological A-model, explicit computations are possible by an application of a duality of Calabi-Yau spaces, called mirror symmetry [35].

Mirror symmetry

In its strongest form the mirror symmetry conjecture states that the full physical type IIA string compactified on a Calabi-Yau manifold Z leads to the same four-dimensional physics as the type IIB string compactified on a mirror Calabi-Yau manifold Z^*. From the worldsheet perspective mirror symmetry can be motivated by a rather trivial symmetry of $U(1)$ currents [21], however from the space-time perspective the geometry of the target space is changed drastically. This means in particular that for a large class of Calabi-Yau manifolds there should exist on physics grounds a mirror Calabi-Yau manifold, which is very surprising from the point of view of pure mathematics. Mirror symmetry can be understood from a target space perspective in terms of the more elementary T-duality symmetry of circle compactifications of type IIA and IIB string theories [36]. This led to the conjecture that any Calabi-Yau manifold with a mirror partner admits a description in terms of a special Lagrangian torus fibration and mirror symmetry is given by T-duality along the torus directions. This is very surprising as both special Lagrangian submanifolds as well as higher dimensional Calabi-Yau manifolds are hard to understand mathematically and it is even very challenging to construct non-trivial examples of both objects.

The equivalence of physical mirror theories has also remarkable consequences for string theories themselves. In particular mirror symmetry implies that not only two particular theories are dual, but one can actually map whole deformation spaces of mirror symmetric theories upon each other. This turns the mirror conjecture into a powerful computational tool in the realm of topological strings. Difficult questions about the type IIA string, such

[12] See e.g. [33]. This is in a smiliar spirit as the famous AdS/CFT correspondence [34].

as the computation of the instanton corrected prepotential, can be turned into a much easier question about the type IIB string on the mirror manifold. This was demonstrated in [37] where the fully worldsheet instanton corrected prepotential of a closed type IIA string theory on the quintic Calabi-Yau was computed by utilizing its mirror description. In particular the prepotential of the topological A-model is given in terms of the B-model mirror theory by geometric integrals on the mirror Calabi-Yau manifold. This means that a question about subtle quantum effects in the A-model can be turned by mirror symmetry into a problem in purely classical geometry on the B-model side.

As the prepotential of the topological A-model encodes the generating function of Gromov-Witten invariants this could be used to solve a longstanding problem about the enumeration of spheres of various degrees in certain Calabi-Yau geometries, which is interesting from a mathematics point of view. An important ingredient in the matching between the families of A- and B-model theories on the mirror symmetric Calabi-Yau manifolds was the identification of the non-trivial mapping between the moduli of both models. These maps are called mirror maps and they in particular reflect the flat structure of the underlying deformation families of topological theories [20].

Mirror symmetry can also be used to gain insights into non-perturbative aspects of other string and field theories. For example mirror symmetry can be used to derive the explicit solution of $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions [38, 39, 25] using a local version of mirror symmetry, which reproduces the Seiberg-Witten prepotential and allows for a generalization to arbitrary gauge groups and the addition of matter fields.[13] Furthermore it is possible to construction non-trivial stable vector bundles on elliptically fibered Calabi-Yau manifolds[14] and to describe their moduli spaces using mirror symmetry constructions [25, 42, 43]. This is a necessary and interesting but in general hard task for semi-realistic compactifications of the heterotic string.

Finally mirror symmetry is also expected to connect the non-perturbative sectors of the involved theories [44]. Therefore there should exist an open string version of mirror symmetry, which exchanges the D-brane states of the type IIA and IIB string theories. This led to an algebraic characterization of D-branes and mirror symmetry as a matching of D-brane categories, known as homological mirror symmetry [45]. This approach captures all the D-brane states of both theories, unfortunately in its current formulation this framework cannot be used as a computational tool for open string mirror symmetry, at least not along the lines of the closed string case, as for example in [37].

This thesis

As sketched above string theory and parts of modern mathematics enjoy a very close and fruitful relationship. This thesis deals with two further examples of this close relationship.

[13]See [40] for the solution via M-theory.

[14]These particular types of bundles on elliptically fibered Calabi-Yau manifolds were constructed in [41].

Part II of this thesis is based on the original publications [1, 2, 3]. In this part we are going to use an off-shell version of mirror symmetry for the computation of certain D-brane induced effective superpotentials. As already mentioned it is natural to contemplate about an open string version of mirror symmetry [44], in order to compute non-perturbative effective couplings by mapping them to easier computable geometric objects on a mirror Calabi-Yau manifold, similar as was done in the case of closed strings in [37].

When adding D-branes and fluxes to a type II compactification on a Calabi-Yau three-fold, the effective theory in four dimensions is generically described by a $\mathcal{N} = 1$ supergravity theory coupled to gauge fields. The vacuum structure of such theories is governed by a superpotential of the chiral superfields, which describe the moduli of the compactification in the supergravity theory. In general it is a hard problem to determine the non-perturbative effective superpotential for a given D-brane configuration.

In order to use mirror for the computation of these superpotential couplings one first has to identify a sensible deformation problem for a given D-brane. A general D-brane has infinitely many non-supersymmetric deformations inside the Calabi-Yau manifold and therefore it is a priori not clear how to single out a well-defined finite dimensional sub-problem. In this work we concentrate on a particular situation which was first described in [46, 47]. On the B-model side we consider a D7-brane which is wrapped around a four-cycle inside the Calabi-Yau manifold. This configuration comes with a natural moduli space of supersymmetric deformations, which correspond to flat directions of the superpotential. Upon the addition of flux on the D7-brane or equivalently the addition of D5-brane charge a superpotential is generated, which obstructs certain directions of the deformation space.

The deformation problem of the D7-brane can be described by the mathematical theory of variation of mixed Hodge structure of a relative cohomology group [46, 47] and the superpotential which is induced by the addition of the D5-brane charge is captured by a geometrical relative period integral, as is discussed in detail in section 3.3. In physics terms this structure is known as $\mathcal{N} = 1$ special geometry [48, 49, 46, 47] and it can be understood as a generalization of the well-known concept of $\mathcal{N} = 2$ special geometry [50], which appears for example in the context of closed string mirror symmetry.

The formulation in terms of the variation of mixed Hodge structure not only allows for the explicit computation of the relative periods, but also for the identification of appropriate mirror maps to the corresponding A-model geometry on the mirror Calabi-Yau manifold, as outlined in section 3.5. The moduli dependence of the relative periods is captured by the solutions to a Picard-Fuchs system of differential equations, similar as in the case of closed string mirror symmetry. Therefore the effective superpotential and the mirror maps can be computed as solutions to a particular set of differential operators, which are derived from the variation of mixed Hodge structure of the relative cohomology group in section 3.4.

On the A-model side the effective superpotential receives non-trivial disc-instanton corrections [51] and encodes enumerative invariants of Lagrangian submanifolds in Calabi-Yau manifolds, which therefore become computable by mirror symmetry techniques [44]. Depending on the point in the deformation space one encounters two different situations. In the first situation the deformation space of the D-brane is classically unobstructed and gets obstructed by non-trivial disc-instanton corrections, only. In this case the superpotential enjoys an expansion in terms of integral Ooguri-Vafa invariants, which enumerate the number of maps from a worldsheet with topology of a disc to the Lagrangian D-brane [52]. In the other case the deformations are classically massive and should therefore be integrated out in the effective action. This leads to a superpotential which depends only on the closed string deformation parameters and describes the tension of domain walls which connect supersymmetric brane vacua. Interestingly such configurations also contain enumerative information and enjoy an Ooguri-Vafa-like expansion [53, 54, 55].

By an open/closed duality [48, 56, 57] the above brane geometry can be mapped to a non-compact Calabi-Yau four-fold without branes and with fluxes only, as will be discussed in sections 3.7 and 3.8. In this situation the relative periods can be used to compute G-flux induced Gukov-Vafa-Witten superpotentials [58]. One has to note that the so obtained superpotentials enjoy a very special structure as compared to generic F-theory superpotentials, as they inherit and reflect the structure of the underlying $\mathcal{N} = 1$ special geometry.

For the special case of divisors which describe toric branes on hypersurfaces in toric ambient spaces[15] the non-compact Calabi-Yau four-fold can be described by a gauged linear σ-model [5], as is discussed in section 3.7. This allows for the study of the combined open/closed deformation space of such configurations by the well-known techniques of toric geometry [27, 59]. We use these in explicit examples in chapter 4 to compute non-perturbative superpotential couplings and verify the integrality properties of the obtained enumerative invariants. This leads to the prediction of genuine Ooguri-Vafa invariants for certain special Lagrangian submanifolds in compact Calabi-Yau geometries, such as the quintic Calabi-Yau three-fold in section 4.1. In models with multiple closed string parameters we furthermore analyze the fate of domain wall tensions through extremal transitions, which change the topology of the target space. These points often correspond to loci of enhanced gauge symmetry for the effective space-time theory and in some examples we encounter tensionless domain walls at such points, which might play a role for the generation of the non-perturbatively enhanced gauge symmetry.

Upon compactification of the non-compact Calabi-Yau four-fold the brane geometry can be related to an honest F-theory compactification on a compact Calabi-Yau four-fold. The details of the compactification are conjectured to capture non-perturbative corrections to the superpotential in the string coupling constant g_s, which are induced by D-instantons [57]. The dual F-theory picture in addition allows to conjecture the form of the Kähler potential on the deformation space of the D7-brane, as will be discussed in section 3.8.

[15] More generally complete intersection Calabi-Yau manifolds in toric ambient spaces.

Part III of this thesis is based on the original publication [4]. Herein we will make use of the duality web of M-theory to connect theories of multiple wrapped M5-branes to the mathematical theories of moduli spaces of stable sheaves and mock modular forms [60, 61, 62]. This part of the work elucidates aspects of the relation between wall-crossing phenomena [63, 31], holomorphic anomaly equations [64, 22, 30] and background-dependence [65] in topological string and field theories.

The duality web of multiple wrapped M5-branes in particular connects the world-volume theory, which lives on multiple M5-branes, to the topological Vafa-Witten gauge theory on a complex surface [30] and links it furthermore to a certain two-dimensional conformal field theory defined on a torus, which is known as MSW-CFT [66].[16]

As a special application of these dualities the appropriate counting of the BPS-states of the dual theories is expected to lead to the same results. In the MSW-CFT an appropriate counting function is given by the modified elliptic genus[17], which gets directly related to the partition function of the dual Vafa-Witten gauge theory. Both objects are expected to enjoy definite transformation properties under the modular group $SL(2,\mathbb{Z})$. From the point of view of the gauge theory this is related to the S-duality invariance of the theory [30], whereas in the CFT picture this property is related to the modular invariance of the modified elliptic genus on the torus [69, 70, 71].

It was observed in [30] that the partition function of Vafa-Witten theory suffers from a holomorphic anomaly, if the theory is put on a rigid complex surface. The occurrence of the holomorphic anomaly equation was attributed to the contribution of reducible gauge connections. Translated into the M5-brane picture this can be interpreted as the contribution of non-trivial BPS bound states of M5-branes.

It is well-known that the spectrum of stable BPS-states depends on the point in moduli space under consideration and that the number of BPS-states can jump when crossing a wall of marginal stability in moduli space [63]. Recently Kontsevich and Soibleman (KS) even gave a rigorous mathematical formula for the change in the BPS-spectrum when crossing such a wall, the celebrated KS wall-crossing formula [72].

In view of the interpretation of the holomorphic anomaly of Vafa-Witten gauge theory as being caused by bound states, it seems therefore reasonable to expect a relation between the wall-crossing phenomenon and the holomorphic anomaly equation. Indeed we will show in chapter 5 that the KS-formula implies the holomorphic anomaly equation for the case of two wrapped M5-branes. In particular the application of the KS-formula in this situation can be viewed as a re-derivation of results obtained in pure mathematics by Göttsche [32] on the structure of sheaves on a complex surface by physical techniques.

The way the holomorphic anomaly equation emerges from the result obtained by the application of the KS-formula is very intriguing and involves the theory of mock modular forms

[16] This is in a similar spirit as the recently studied 4d/2d-correspondences, see e.g. [67].

[17] Technically this is a BPS-index, see e.g. [68].

[60, 61, 62].[18] The jumps in the BPS-numbers are encoded in an indefinite theta-function, first described in the mathematical work of Göttsche [32], which is a highly singular object and furthermore breaks explicitly modular invariance. As demanded by S-duality invariance, a modular invariant object can be constructed by a smoothing procedure, first described by Zwegers in [61], which introduces a mild non-holomorphicity. This turns the modified elliptic genus into a mock modular form and explains the appearance of a non-holomorphic dependence.

Physics-wise this can also be understood as follows. The holomorphic expansion of the modified elliptic genus in the neighborhood of a given point in moduli space, which encodes the degeneracies of the stable BPS-spectrum, does not know about non-trivial bound states, which can potentially contribute when a wall of marginal stability is crossed. However the restoration of the duality invariance, after crossing the wall, forces one to take them into account and leads to a non-holomorphic dependence.

The holomorphic anomaly equation can be used to compute the BPS-degeneracies in favorable cases, when supplemented by appropriate boundary conditions, as is exemplified in section 5.4. Mathematically the BPS-degeneracies encode the Euler characteristics of the moduli spaces of sheaves of fixed topology and the holomorphic anomaly equation can therefore be used to explicitly compute these topological invariants by physical reasoning.

[18]See Appendix H for a short account of mock modular forms.

1.2. Outline

It follows a short outline of the following chapters.

- In **chapter 2**, in order to set the stage for the discussion of open string mirror symmetry, we give a short overview of topological string and field theories and the main techniques used in closed string mirror symmetry. After introducing the general concepts we apply these to the explicit examples of the sextic Calabi-Yau three-fold in $\mathbb{P}^4_{(1,1,1,1,2)}$ and the sextic Calabi-Yau four-fold in $\mathbb{P}^5$ in section 2.6.

In **part II**, based on the original publications [1, 2, 3], we show how the Hodge-theoretic approach to $\mathcal{N} = 1$ mirror symmetry of [46, 47] can be extended to compact Calabi-Yau geometries.

- **Chapter 3** is concerned with off-shell mirror symmetry. After a short review of aspects of topological D-branes in section 3.1, we give a general overview of the concepts of off-shell mirror symmetry in section 3.2. Then we turn to a description of the geometric model for the deformation space of the B-model geometry in section 3.3, which we are going to study further in subsequent sections. Section 3.4 contains a detailed derivation of the Picard-Fuchs operators which annihilate the relative periods. In addition we discuss the integrability conditions which stem from the expected flat structure of the chiral ring of the underlying open/closed topological field theory in section 3.5 and outline connections to CFT correlation functions in section 3.6. In section 3.7 we show how for the case of toric branes the Picard-Fuchs operators can be obtained from a gauged linear σ-model for an open/closed dual Calabi-Yau four-fold. Furthermore in section 3.8 it is pointed out how the brane geometry can be lifted to a F-theory compactification and how this can be used to conjecture the form of the Kähler potential of the effective $\mathcal{N} = 1$ supergravity theory on the full open/closed deformation space.

- In **chapter 4** the techniques presented in the preceding chapter are applied to various brane geometries in hypersurfaces of toric varieties. Particular emphasis lies on the relation between the off-shell techniques presented in chapter 3 and the on-shell computations of [53, 73]. We compute Ooguri-Vafa invariants for certain compact brane geometries and verify integrality of the obtained invariants to high order. For the case of multi-parameter models we study the fate of domain wall tensions through extremal transitions connecting different Calabi-Yau geometries and observe that the web of vacua stays connected for our particular brane geometries.

Part III, based on the original publication [4], is concerned with the relation between wall-crossing and holomorphic anomaly equations in the case of conformal field theories related to multiple M5-brane bound states.

- **Chapter 5** contains a review of the effective descriptions of wrapped M5-brane theories in section 5.2 and discusses the Kontsevich-Soibelman (KS) wall-crossing formula in section 5.3. The KS-formula is then used to reproduce a wall-crossing formula of Göttsche for the Euler numbers of the moduli spaces of sheaves on a divisor. After this we derive the holomorphic anomaly equation for the case of two wrapped M5-branes from the expressions for the modified elliptic genus by an application of the ideas of Zwegers [61] and point out the relation to mock modular forms. The application of the general formulae to particular examples and some speculations about possible extensions in section 5.4 finish this chapter.

The **part IV** is comprised of appendices.

- **Chapter A** contains a short review of techniques from toric geometry, which are relevant for the technical discussions in the main text.
- For better readability we furthermore relegated some of the details from the main text to the **chapters B-G**.
- The last **chapter H** contains a short introduction to the theory of mock modular forms.

1.3. Publications

This thesis is based on the following original publications:

- M. Alim, M. Hecht, P. Mayr and A. Mertens,
  ”Mirror symmetry for toric branes on compact hypersurfaces”,
  JHEP **0909** (2009) 126, [arXiv:0901.2937].

- M. Alim, M. Hecht, H. Jockers, P. Mayr, A. Mertens and M. Soroush,
  “Hints for Off-Shell Mirror Symmetry in type II/F-theory Compactifications”,
  Nucl. Phys. B **841** (2010) 303, [arXiv:0909.1842].

- M. Alim, M. Hecht, H. Jockers, P. Mayr, A. Mertens and M. Soroush,
  “Type II/F-theory Superpotentials with Several Deformations and N=1 Mirror Symmetry”, JHEP **1106** (2011) 103, [arXiv:1010.0977].

- M. Alim, B. Haghighat, M. Hecht, A. Klemm, M. Rauch and T. Wotschke,
  “Wall-crossing holomorphic anomaly and mock modularity of multiple M5-branes”,
  [arXiv:1012.1608].

2 Chapter 2

Closed string mirror symmetry

Contents

In this chapter we review some aspects of closed string mirror symmetry. The extension to open strings, which is the central topic of chapter 3, parallels the closed string case in many respects and in favorable circumstances can even be mapped onto closed string mirror symmetry in one complex dimension higher. Therefore closed string mirror symmetry is a good starting point for the discussions which follow in later chapters.

This chapter follows the expositions given in [22, 35, 27, 74, 75, 76, 77, 78, 79]. For the geometrical tools from toric geometry we refer to the Appendix A and the additional literature given there. We start with a discussion of some aspects of the vacuum geometry of $\mathcal{N} = (2,2)$ SCFTs in two dimensions.

2.1. Vacuum geometry of $\mathcal{N} = (2,2)$ SCFTs

2.1.1. The $\mathcal{N} = 2$ superconformal algebra

The $\mathcal{N} = 2$ superconformal algebra in two dimensions is a supersymmetric extensions of the conformal algebra. In addition to the Virasoro generators one has two fermionic currents $G^{\pm}(z)$ of conformal weight $h = \frac{3}{2}$ together with an $U(1)$ R-current $J(z)$ of weight $h = 1$ under which the fermionic currents carry charges ± 1. These fields together with the energy-momentum tensor $T(z)$ of weight $h = 2$ fit into an $\mathcal{N} = 2$ multiplet. In addition one has to specify boundary conditions for the fermionic fields. There are two possibilities: periodic boundary conditions, which lead to the Neveu-Schwarz (NS) sector and anti-periodic boundary conditions, which lead to the Ramond (R) sector. The OPEs between the above currents are given by[1]

$$
\begin{aligned}
T(z)\cdot T(w) &\sim \frac{c}{2}\frac{1}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial_w T(w)}{(z-w)}\ ,\\
T(z)\cdot G^{\pm}(w) &\sim \frac{3}{2}\frac{G^{\pm}(w)}{(z-w)^2}+\frac{\partial_w G^{\pm}(w)}{(z-w)}\ ,\\
T(z)\cdot J(w) &\sim \frac{J(w)}{(z-w)^2}+\frac{\partial_w J(w)}{(z-w)}\ ,\\
G^{+}(z)\cdot G^{-}(z) &\sim \frac{2c}{3}\frac{1}{(z-w)^3}+\frac{2J(w)}{(z-w)^2}+\frac{2T(w)+\partial_w J(w)}{(z-w)}\ ,\\
J(z)\cdot G^{\pm}(w) &\sim \pm\frac{G^{\pm}(w)}{(z-w)}\ , \quad J(z)\cdot J(w)\sim \frac{c}{3}\frac{1}{(z-w)^2}\ ,
\end{aligned}
\tag{2.1}
$$

where c denotes the central charge.[2] A highest weight representation of the $\mathcal{N}=2$ superconformal algebra can be constructed from a highest weight state $|\phi\rangle$ which fulfills

$$
L_n|\phi\rangle=0, \qquad G^{\pm}_r|\phi\rangle=0, \qquad J_m|\phi\rangle=0, \qquad n,r,m>0\ . \tag{2.2}
$$

Here $r\in\mathbb{Z}+\frac{1}{2}$ for the NS-sector and $r\in\mathbb{Z}$ for the R-sector and the mode expansions are given by

$$
T(z)=\sum_n L_n\, z^{-n-2}, \qquad J(z)=\sum_n J_n\, z^{-n-1}, \qquad G^{\pm}(z)=\sum_n G^{\pm}_{n\pm a}\, z^{-(n\pm a)-\frac{3}{2}}\ , \tag{2.3}
$$

where $0\leq a\leq 1$ is integral in the Ramond sector and half-integral in the NS-sector. The Cartan subalgebra of the $\mathcal{N}=2$ superconformal algebra is spanned by L_0 and J_0 and thus a general state can be labeled by its eigenvalues with respect to these operators

$$
L_0|\phi\rangle=h_\phi|\phi\rangle, \qquad J_0|\phi\rangle=q_\phi|\phi\rangle\ . \tag{2.4}
$$

An important datum of a supersymmetric theory are the supersymmetric ground states. In this case the supersymmetric ground states lie in the R-sector and are given by highest weight states which are furthermore annihilated by the modes $G^{\pm}_0$.

The $\mathcal{N}=2$ algebra has a large group of automorphisms, which map the NS- and the R-sector onto each other. This property is called spectral flow automorphism. For string constructions the spectral flow automorphism leads to space-time supersymmetry of the corresponding string spectrum. For our purpose the spectral flow will turn out to be important in the related context of the $\mathcal{N}=4$ superconformal algebra in chapter 5.

[1] See e.g. [35, 80] for pedagogical introductions.

[2] In a non-linear σ-model realization the central charge is connected to the dimension of the target space geometry.

2.1.2. Chiral rings

There is an interesting ring structure coming with an $\mathcal{N}=2$ superconformal symmetry algebra [21]. For this let $|\phi\rangle$ be a primary state in the NS-sector which furthermore fulfills

$$\begin{aligned} \text{CHIRAL PRIMARY:} \qquad & G^{+}_{-1/2}|\phi\rangle = 0 \ , \\ \text{ANTI-CHIRAL PRIMARY:} \qquad & G^{-}_{-1/2}|\phi\rangle = 0 \ . \end{aligned} \tag{2.5}$$

By the superconformal algebra one can show that the space of chiral fields is finite dimensional as for such a state [21]

$$h_\phi \leq \frac{c}{6}, \qquad \text{and} \qquad h_\phi = \frac{q_\phi}{2} \ . \tag{2.6}$$

Under the OPE the chiral fields form a closed finite ring, called the chiral ring [21]

$$\phi_i \cdot \phi_j = C_{ij}^k \, \phi_k \ , \tag{2.7}$$

where i, j, k label the finite number of different fields. The structure constants C_{ij}^k are related to the three-point functions C_{ijk} of three chiral primaries on a genus zero worldsheet

$$C_{ijk} \equiv \langle \phi_i \phi_j \phi_k \rangle_0 = \langle \phi_i \phi_m \rangle_0 C_{jk}^m \equiv \eta_{im} \, C_{jk}^m \ . \tag{2.8}$$

Here the two-point function $\langle \phi_i \phi_m \rangle_0 = \eta_{im}$ is called the topological metric and becomes important once one considers deformations of a given theory.

As the $U(1)$-charge is additive the chiral ring comes with a charge filtration [21, 35]. This becomes also apparent when looking at the operator-state correspondence. By spectral flow the R- and the NS-sector can be mapped to each other and the chiral primaries are mapped to the supersymmetric ground states in the R-sector [21]. There is a canonical vacuum $|0\rangle$ from which one can construct the other vacua by the action of the chiral primaries ϕ_i [35][3]

$$|i\rangle = \phi_i |0\rangle \ . \tag{2.9}$$

Therefore the states have a definite $U(1)$-charge. The same steps can also be carried out for the anti-chiral fields. If one considers a theory of closed strings one has a left- and a right moving $\mathcal{N}=2$ superconformal algebra. This will lead to combined chiral rings denoted by (c, c) and (a, c), where c denotes the chiral- and a the anti-chiral ring.[4]

2.2. Topological field theories

Up to now implicitly only flat worldsheets were considered. To define the theory on a closed worldsheet one has to make sure that there exist covariantly constant sections of the spin

[3]Strictly speaking this is not a one-to-one map in general. However for the concrete models we will be interested in (the A-model on a Calabi-Yau manifold and B-twisted LG-models) this statement is true [35].

[4]The other two possible rings are isomorphic to the given ones.

bundle. For higher genus surfaces this is however not guaranteed. One way to solve this problem is to construct a new theory out of the original one by the so-called topological twist procedure [18, 19, 20].[5]

The idea is to combine the local Lorentz/euclidean symmetry with the R-symmetries of the worldsheet supersymmetry and to declare a diagonal subgroup to be the new local Lorentz/euclidean symmetry group. This changes the bundles the original fields were sections of and thus their spins. For the supercharges this procedure generically leads to some fermionic scalar charges and one-form fields. Thus one obtains scalar charges Q which square to zero and lead in particular to the simplifications one knows from supersymmetric theories also for higher genus worldsheets.

2.2.1. Properties of topological field theories

A topological field theory (TFT) obtained from twisting has some very remarkable properties:

- The operators and states of the topological and the superconformal theory do not differ. However the notion of a physical state is changed. In the topological theory the Q-cohomology classes are the physical states and they correspond to the ground states of the supersymmetric theory.

- The energy-momentum tensor $T = \{Q, G\}$ is Q-exact. As a consequence the correlation functions of physical operators do not depend on the metric on the worldsheet. Furthermore the correlation functions are independent of the position of the inserted operators.

- The semi-classical approximation is exact. If the action of the theory is furthermore Q-exact the correlation functions get contributions only from the fixed-points of the Q-symmetry. This important property is called localization.

- The correlation functions are independent of a large class of deformations of the underlying QFT. For example they generically do not depend on D-term deformations. Naive arguments leading to these kind of statements can however have important subtleties, as in the case of the holomorphic anomaly equation of topological string theory at higher genus [64, 22].

In addition a topological theory obtained by twisting an $\mathcal{N} = 2$ superconformal theory has vanishing central charge and is thus automatically conformal at the quantum level. In contrast to physical string models the quantum consistency of the model therefore does not single out a particular space-time dimension anymore. However for the special case of topological non-linear σ-models on Calabi-Yau spaces the correlation functions are non-vanishing only if the ghost number anomaly is cancelled [18, 19, 20] and this condition singles out the

[5] For pedagogical introductions to topological field and string theories see e.g. [35, 76, 78, 79, 81].

complex dimensions three and four as particular interesting dimensions for target spaces of the topological string.[6]

2.2.2. Deformations of topological theories

Deformations of topological theories can be constructed via the so-called descent procedure. This procedure is analogous to the construction of marginal operators from chiral fields of charge $(q,\bar{q})=(1,1)$ in the superconformal theory.[7]

By the topological twist some of the supercharges become scalars, while the others become one-form operators. Now given a Q-closed operator $\phi^{(0)}$ one can find one-form and two-form operators $\phi^{(1)}$ and $\phi^{(2)}$ such that [20][8]

$$\begin{aligned} 0 &= [Q,\phi^{(0)}], \qquad & d\phi^{(0)} &= \{Q,\phi^{(1)}\}, \\ d\phi^{(1)} &= [Q,\phi^{(2)}], \qquad & d\phi^{(2)} &= 0\ . \end{aligned} \tag{2.10}$$

These equations are called descent equations and can be formulated in any dimension. Integrating $\phi^{(1)}$ over a curve γ and $\phi^{(2)}$ over the whole worldsheet Σ one can construct new non-local topological observables by virtue of the descent equations (2.10). The second type of operator with the appropriate $U(1)$-charges can be used to deform the action

$$S(t,\bar{t}) = S_0 + t^i \int_\Sigma \phi_i^{(2)} + \bar{t}^{\bar{j}} \int_\Sigma \bar{\phi}_{\bar{j}}^{(2)}\ , \tag{2.11}$$

where we included for convenience the deformations of the anti-topological theory obtained by twisting for example the (a,a)-ring. The indices $i,\bar{j}=1,\ldots,N$ run over all physical operators with $U(1)$-charges $(q,\bar{q})=(1,1)$. The parameters $(t^i,\bar{t}^{\bar{j}})$ are local coordinates on the deformation space $\mathcal{M}$ of the theory, which is called the moduli space.[9]

The deformations can be used to construct perturbed correlation functions. For example the three-point correlator in the deformed theory can be written as

$$C_{ijk}(t) = \langle \phi_i\phi_j\phi_k \,\exp\left(t^n \int_\Sigma \phi_n^{(2)}\right)\rangle_0\ . \tag{2.12}$$

Although this expression looks asymmetric with respect to its insertions it can be shown to fulfill an integrability condition [77]

$$\partial_i C_{jkl} = \partial_j C_{ikl} \tag{2.13}$$

[6] At least if the theory is coupled to worldsheet gravity and one considers also higher genus worldsheets [78]. For the ghost number anomaly without worldsheet gravity see also section 2.3.

[7] See e.g. [74].

[8] See also e.g. [76] for a pedagogical account.

[9] For non-linear σ-models on Calabi-Yau target spaces, in which we will be mainly interested in the following, this deformation space is classically unobstructed and one therefore obtains a true classical moduli space, see also section 2.4.1.

and thus there exists at least locally a function $\mathcal{F}(t)$ such that

$$C_{ijk}(t) = \partial_i \partial_j \partial_k \mathcal{F}(t) \ . \tag{2.14}$$

The function $\mathcal{F}(t)$ is called the prepotential and can be considered as the generating function of tree-level correlators of the topological field theory. The integrability condition furthermore leads to the identity[10]

$$\partial_i \eta_{mn} = \partial_i C_{0mn} = \partial_0 C_{imn} = 0 \ , \tag{2.15}$$

which shows that in the flat TFT coordinates $(t^i, \bar{t}^{\bar{\jmath}})$ in (2.11) the topological metric is locally constant. In addition it follows from crossing-symmetry that the deformed chiral ring is associative

$$C_{ij}^n C_{nkl} = C_{ik}^n C_{njl} \ . \tag{2.16}$$

These equations are called the WDVV-equations [82, 83] and they yield infinitely many conditions on the expansion coefficients of the prepotential $\mathcal{F}(t)$ and therefore between the various n-point correlation functions. For open topological strings there exist analogous conditions, which lead to the very rich mathematical structure of A_∞-categories [84, 85].

The above conditions (2.14)-(2.16) abstractly define what is called a Frobenius algebra [86, 77]. This is a space $\mathcal{M}$ together with a vector bundle $\mathcal{V}$, which is equipped with a locally constant bilinear form η. Furthermore on each fiber there is an associative multiplication with totally symmetric structure constants C_{ijk}, which can be integrated to a generating function $\mathcal{F}(t)$. From this viewpoint a two-dimensional topological field theory can be considered as a particular representation of a Frobenius algebra [86, 77].

2.2.3. The vacuum bundle and tt^*-geometry

The physical operators of the twisted topological theory are in one-to-one correspondence to the RR ground states of the original $\mathcal{N} = (2, 2)$ theory[11] and furthermore by spectral flow to the chiral primaries of the NSNS-sector. When considering deformations of the theory the space of RR ground states, denoted by $H(t)$, gets fibered over the moduli space $\mathcal{M}$ and one obtains a bundle $\mathcal{V}$ of ground states

$$\begin{array}{ccc} \mathcal{H} \supset H(t) & \longrightarrow & \mathcal{V} \\ & & \downarrow \\ & & \mathcal{M} \end{array} \tag{2.17}$$

Here we denoted by $\mathcal{H}$ the full Hilbert-space of the superconformal theory in which, in view of (2.9), the subspace $H(t)$ sits. The bundle $\mathcal{V}$ is called the vacuum bundle of the theory and

[10] Note that ϕ_0 denotes the identity operator.

[11] In this chapter we consider closed string theories with two copies of the $\mathcal{N} = 2$ superconformal algebra.

it can be written as a direct sum of sub-bundles $\mathcal{V}_q$ with definite $U(1)$-charges

$$\mathcal{V} = \oplus_q \mathcal{V}_q \ . \tag{2.18}$$

There is an interesting catch in the above [87]. As one varies the parameters of the theory the number of ground states is known to stay constant, however the chiral ring (2.7) changes rather drastic, e.g. the structure constants $C_{ij}^k(t)$ will depend on t. This means that the variation of the ground states under a change of parameters should encode the structure of the chiral ring. This turns out to be true and the dependence of the ground states on the deformation parameters is captured by the so-called tt^*-equations [87, 35], which we describe in the following.

For this let $|\bar{i}\rangle$ denote the ground states of the anti-topological theory obtained from acting with the anti-chiral fields $\bar{\phi}_{\bar{i}}$ on the canonical ground state $|0\rangle$. Using the topological and anti-topological theory one can define the tt^*-metric [87]

$$g_{\bar{i}j} = \langle \bar{i}|j\rangle \ . \tag{2.19}$$

The tt^*-metric induces a connection on the bundle $\mathcal{V}$ which is compatible with the natural complex structure of $\mathcal{M}$. There exists a basis, called the holomorphic or topological basis, which makes this explicit. In this basis the connection $D_i = 1\!\!1\partial_i - A_i$ is given by

$$(A_i)^a_b = g^{a\bar{c}}\partial_i g_{b\bar{c}} \ , \qquad (A_{\bar{i}})^a_b = 0 \ . \tag{2.20}$$

The structure constants C_i of the chiral ring and the tt^*-connection are related by the tt^*-equations [87]

$$\begin{aligned} &[D_i, D_j] = 0, && [\bar{D}_{\bar{i}}, \bar{D}_{\bar{j}}] = 0 \ , \\ &[D_i, \bar{C}_{\bar{j}}] = 0, && [\bar{D}_{\bar{i}}, C_j] = 0, \\ &[D_i, C_j] = [D_j, C_i], && [\bar{D}_{\bar{i}}, \bar{C}_{\bar{j}}] = [\bar{D}_{\bar{j}}, \bar{C}_{\bar{i}}], \\ &[D_i, \bar{D}_{\bar{j}}] = -[C_i, \bar{C}_{\bar{j}}], && \end{aligned} \tag{2.21}$$

where $[D_i, \bar{D}_{\bar{j}}] = \partial_i \bar{A}_{\bar{j}} - \partial_{\bar{j}} A_i - [A_i, \bar{A}_{\bar{j}}]$. The tt^*-equations imply that the improved connection $\nabla_i = D_i - C_i$ is flat [87]

$$[\nabla_i, \nabla_j] = [\nabla_i, \overline{\nabla}_{\bar{j}}] = [\overline{\nabla}_{\bar{i}}, \overline{\nabla}_{\bar{j}}] = 0 \ . \tag{2.22}$$

In a certain non-linear σ-model realization of the above structures, the so-called B-model which we will consider in a moment, the improved connection can be given a geometrical interpretation and is in this context known as the Gauss-Manin connection.

2.2.4. Geometry of the moduli space

There exists a preferred basis of sections of the bundle $\mathcal{V}$ which can be used to define a metric on $T\mathcal{M}$. For this one has to remember that the elements ϕ_a, for $a = 1, \ldots, N$ with $U(1)$

charge $q_{\phi_a} = 1$, can be used to deform the theory. Thus these elements can be considered as sections of $T\mathcal{M}$. Denoting the operator which generates the canonical ground state of charge zero by ϕ_0 one can define a basis of the chiral (sub-)ring by taking the fields ϕ_0 and ϕ_a and furthermore their duals with respect to either the topological or the tt^*-metric.[12] We therefore obtain two distinct splits of the vacuum bundle. For the case of $c = 9$, corresponding to Calabi-Yau three-fold, the vacuum bundle can be split for example in a non-holomorphic way as

$$\mathcal{V}_{\mathbb{C}} = \mathcal{L} \oplus (\mathcal{L} \otimes T\mathcal{M}) \oplus (\mathcal{L} \otimes T\mathcal{M})^* \oplus \mathcal{L}^* \ , \tag{2.23}$$

where the ground state is a section of the line bundle $\mathcal{L}$ and * denotes the dual space. Using instead the topological metric leads to a holomorphic split of the bundle and the basis elements can be constructed by successive OPE-multiplication of the operators ϕ_0, ϕ_a. Restricting the tt^*-metric to the directions tangent to $\mathcal{M}$ one obtains the Zamolodchikov metric, denoted by $G_{a\bar{b}}$ and given by

$$G_{a\bar{b}} = \frac{g_{a\bar{b}}}{g_{0\bar{0}}} \ . \tag{2.24}$$

Using the tt^*-equations one can furthermore show that [87]

$$G_{a\bar{b}} = -\partial_a \partial_{\bar{b}} \ln g_{0\bar{0}}$$

and thus the Zamolodchikov metric is Kähler with Kähler potential given by $K = -\ln g_{0\bar{0}}$. The tt^*-equations also imply a certain relation for the curvature of the metric $G_{a\bar{b}}$, see e.g. [22, 35]

$$R_{a\bar{b}c}{}^{d} \equiv -\bar{\partial}_{\bar{b}} \Gamma^d_{ac} = G_{c\bar{b}} \delta^d_a + G_{a\bar{b}} \delta^d_c - e^{2K} C_{acn} G^{n\bar{n}} \bar{C}_{\bar{b}\bar{m}\bar{n}} G^{\bar{m}d} \ . \tag{2.25}$$

This condition together with the Kähler property turn the manifold $\mathcal{M}$ into a special Kähler manifold and the geometry of such spaces is called special geometry [50, 22].

2.3. Non-linear σ-model realizations

The structure outlined above can be realized by a two-dimensional $\mathcal{N} = (2, 2)$ non-linear σ-model with target space a Calabi-Yau manifold X of complex dimension d. It turns out that there are two topological twists possible in this case, called the A- and B-twist respectively [20].[13] The qualitative features of the two theories differ rather drastically, however mirror symmetry will connect the two theories on different Calabi-Yau manifolds, as will be discussed in section 2.5.

[12] Here we will be only interested in situations where the chiral ring contains operators up to charge $q = 4$.
[13] Good introductions to the A- and B-model can be found in [22, 35, 88, 76, 78, 79].

We start with a $\mathcal{N} = (2,2)$ non-linear σ-model, defined on a two-dimensional worldsheet Σ, with action

$$S = \int_\Sigma d^2z\, g_{i\bar{j}} \partial_z x^i \partial_{\bar{z}} \bar{x}^{\bar{j}} + \int_\Sigma x^* B + \text{fermions} \ . \tag{2.26}$$

The fields x^i are interpreted as local coordinates on the target space X and B denotes a possible two-form B-field background to which closed strings can naturally be coupled. The non-linear σ-model can be considered as a theory of maps from the worldsheet Σ to the target space X

$$x^i : \Sigma \longrightarrow X \ . \tag{2.27}$$

From the general discussion above this theory comes with two finite rings, the (a,c) and the (c,c) ring. The elements of the chiral rings of the non-linear σ-model have nice geometric representatives in terms of the target space geometry. The supercharges act as differential operators and thus their cohomology is isomorphic to geometric cohomology groups. One can show that [19, 74]

$$(a,c) \quad \sim \quad \bigoplus_{p,q} H^{0,q}_{\bar{\partial}}(X, \Lambda^p T^*_M) \sim \bigoplus_{p,q} H^{p,q}_{\bar{\partial}}(X) \ , \tag{2.28}$$

$$(c,c) \quad \sim \quad \bigoplus_{p,q} H^{0,q}_{\bar{\partial}}(X, \Lambda^p T_M) \sim \bigoplus_{p,q} H^{d-p,q}_{\bar{\partial}}(X) \ , \tag{2.29}$$

where the second isomorphism for the (c,c) ring follows by contraction with the unique $(d,0)$-form of the Calabi-Yau manifold. In the context of mirror symmetry we will mainly be interested in the sub-ring of operators with equal left and right $U(1)$-charges.

The marginal deformations of the non-linear σ-model correspond to the geometrical deformations of the Ricci-flat Kähler metric of its target space. They fall into two classes: Kähler deformations, whose infinitesimal generators are represented by elements of $H^{1,1}(X)$, and complex structure deformations, which are given by elements of $H^1(TX) \simeq H^{d-1,1}(X)$. Locally the moduli space of the non-linear σ-model therefore is a direct product of the Kähler moduli space $\mathcal{M}_K$ and the complex structure moduli space $\mathcal{M}_{\text{CS}}$ of the target space Calabi-Yau manifold

$$\mathcal{M} = \mathcal{M}_K \times \mathcal{M}_{\text{CS}} \ . \tag{2.30}$$

On the level of the $\mathcal{N} = (2,2)$ superconformal algebra the topological twist can be implemented by the following redefinitions of the energy-momentum tensor and the $U(1)$-current

$$\begin{aligned} T(z) &\to T(z) \pm \frac{1}{2} \partial J(z) \ , \\ J(z) &\to \pm J(z) \ . \end{aligned} \tag{2.31}$$

An analogous redefinition is also made in the right-moving sector of the $\mathcal{N} = (2,2)$ non-linear σ-model.[14] With these redefinitions the superconformal algebra becomes the conformal topological algebra [89]

$$\begin{aligned}
&T(z)\cdot T(w) \sim \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)}\ , \quad T(z)\cdot G(w) \sim \frac{2G(w)}{(z-w)^2} + \frac{\partial_w G(w)}{(z-w)}\ ,\\
&T(z)\cdot Q(w) \sim \frac{Q(w)}{(z-w)^2} + \frac{\partial_w Q(w)}{(z-w)}\ , \quad T(z)\cdot J(w) \sim -\frac{\hat{c}}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{(z-w)}\ ,\\
&J(z)\cdot G(w) \sim -\frac{G(w)}{(z-w)}, \quad J(z)\cdot Q(w) \sim \frac{Q(w)}{(z-w)}\ , \quad J(z)\cdot J(w) \sim \frac{\hat{c}}{(z-w)^2}\ ,\\
&Q(z)\cdot G(w) \sim \frac{\hat{c}}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{T(w)}{(z-w)}\ ,
\end{aligned} \tag{2.32}$$

where $\hat{c} = \frac{c}{3}$ and the quantities Q and G have the following mode expansions

$$G(z) = \sum_n G_n\, z^{-n-2}, \qquad Q(z) = \sum_n Q_n\, z^{-n-1}\ . \tag{2.33}$$

There are two different topological twists possible, which differ by the relative sign chosen for the left- and right-movers in the replacement (2.31) and by the states which become the physical ones. This leads the following topological models [20]

	A-model	B-model
Physical states	(a,c)-ring	(c,c)-ring
Twist	$T \to T - \frac{1}{2}J$	$T \to T - \frac{1}{2}J$
	$\bar{T} \to \bar{T} - \frac{1}{2}\bar{J}$	$\bar{T} \to \bar{T} + \frac{1}{2}\bar{J}$

(2.34)

The two different topological theories are called the A- and B-model respectively. Their properties under deformations and the respective Q-invariant configurations, which contribute to correlation functions, are different and we discuss them in turn.

2.3.1. The A-model

The sub-ring of equal $U(1)$-charges of the A-model is given by elements in $H^{p,p}_{\bar{\partial}}(X)$. The marginal operators have representatives in $H^{1,1}(X)$ and can thus be considered as Kähler deformations of the target space geometry. The A-model is however insensitive to deformations of the complex structure[15] and in this sense its correlation functions produce topological invariants of the target space geometry.

The condition for a correlation function $\langle \mathcal{O}_1^{(p_1,q_1)}\mathcal{O}_2^{(p_2,q_2)}\dots\mathcal{O}_n^{(p_n,q_n)}\rangle_g$ of operators with charges (p_i,q_i) to yield a non-vanishing result is given by the ghost number anomaly cance-

[14] By $(0,2)$-deformations it is also possible to construct so-called half-twisted models [20, 5].

[15] The operators corresponding to complex structure deformations are Q-exact with respect to the Q-operator of the A-model [20].

lation condition [20, 35][16]

$$\sum_i p_i = \sum_i q_i = d(1-g) + \int_X x^* c_1(X) \ . \tag{2.35}$$

Here g denotes the genus of the closed worldsheet Σ. For a Calabi-Yau three-fold and $g=0$ the most interesting correlators are thus the three-point functions of three marginal operators with charge $(p,p) = (1,1)$.

The correlation functions localize to the Q-invariant configurations, which for the A-model are given by holomorphic maps [20]

$$\partial_{\bar{z}} x^i = 0 \ . \tag{2.36}$$

The path-integral thus splits up into topological sectors, labeled by the class in $H_2(X)$ to which the worldsheet is mapped. The three-point correlation functions can be written as

$$C_{ijk} = \langle \mathcal{O}_i^{(1,1)} \mathcal{O}_j^{(1,1)} \mathcal{O}_k^{(1,1)} \rangle_0 = \int_X \omega_i \wedge \omega_j \wedge \omega_k + \sum_{\vec{d}>0} N_{0,\vec{d}}\, d_i d_j d_k \frac{\prod_{i=1}^{h^{1,1}} e^{2\pi i\, t^i d_i}}{1 - \prod_{i=1}^{h^{1,1}} e^{2\pi i\, t^i d_i}} \ . \tag{2.37}$$

Here d_i denotes the degree of the holomorphic map with respect to a basis of divisors D_i of the Calabi-Yau target space, where the Poincaré-dual basis of two-forms is denoted by $\{\omega_i\}_{i=1,\ldots,h^{1,1}(X)} \subset H^{1,1}(X)$. The complexified Kähler class can be decomposed as $B + iJ = \sum_{i=1}^{h^{1,1}} t^i \omega_i$, where B denotes the two-form B-field background. The coefficients $N_{0,\vec{d}}$ are called (closed string) Gromov-Witten invariants [20, 26, 27] and count the number of maps of prescribed degree and homology-image from the worldsheet to the Calabi-Yau manifold. These numbers are enumerative invariants of the target space geometry. The Gromov-Witten invariants can physically be related to the counting of BPS-states coming from M2-branes, wrapped around curves of the Calabi-Yau manifold, in a M-theory compactification down to five dimensions and can be re-summed into integral enumerative invariants of the target space geometry, called Gopakumar-Vafa invariants [28, 29].

The expression for the three-point correlator (2.37) has an intriguing geometrical interpretation. The first term in (2.37), which comes from constant maps, computes the classical triple intersection numbers $\#(D_i \cap D_j \cap D_k)$ of the Poincaré-dual divisors D_i. The Gromov-Witten invariants then "count" the number of rational maps of prescribed degree such that three points are mapped to the divisors which are Poincaré-dual to the forms ω_i in (2.37). This leads schematically to

$$C_{ijk} = \langle \mathcal{O}_i^{(1,1)} \mathcal{O}_j^{(1,1)} \mathcal{O}_k^{(1,1)} \rangle_0 = \#(D_i \cap D_j \cap D_k) + \text{instanton corrections} \ . \tag{2.38}$$

One can interpret this as a stringy deformation of the classical geometrical notion of intersections of submanifolds by stringy worldsheet instanton corrections. This is an example of the

[16] This condition changes when coupling the theory to worldsheet gravity and accessing the higher genus sector, i.e. for topological strings.

fact that strings probe geometry different from point-particles. In mathematics terminology the above is also known as quantum cohomology and can be given a precise mathematical meaning [27, 90, 26].

The two-points functions $\eta_{ij} = \langle \mathcal{O}_i^{(p_i,q_i)} \mathcal{O}_j^{(p_j,q_j)} \rangle_0$ in contrast receive no non-trivial instanton corrections and are in flat TFT coordinates given by classical constant intersection numbers [20]. This follows also from the existence of a prepotential as in (2.15).

2.3.2. The B-model

The physical operators of the B-model of equal left and right $U(1)$-charges are given by elements of $H^{d-p,p}(X)$ and thus the marginal deformations are represented by elements of $H^1_{\bar{\partial}}(X,TX) \simeq H^{d-1,1}(X)$. These describe deformations of the complex structure of the target space X and the theory is independent of Kähler deformations of the target space metric.[17] The topological metric in this case is given by wedging and integrating the form-representatives of the operators over the whole Calabi-Yau manifold

$$\eta_{ij} = \langle \mathcal{Q}_i^{(p_i,q_i)} \mathcal{Q}_j^{(p_j,q_j)} \rangle_0 = \int_X \alpha_i^{(p_i,q_i)} \wedge \alpha_j^{(p_j,q_j)} \ , \tag{2.39}$$

where $\alpha_i^{(p_i,q_i)} \in H^{d-p_i,q_i}(X)$ are the geometric representatives of the B-model operators $\mathcal{Q}_i^{(p_i,q_i)}$. The ghost number anomaly cancellation condition is in this case given by [20]

$$\sum_i p_i = \sum_j q_j = d(1-g) \tag{2.40}$$

and thus the interesting correlators are in the Calabi-Yau three-fold case with $g=0$ again the three-point functions of the marginal operators. In contrast to the A-model the Q-invariant configurations of the B-model are given by constant maps [20]

$$\partial_z x^i = 0 = \partial_{\bar{z}} x^i \ . \tag{2.41}$$

Therefore there are no non-trivial instanton contributions to correlation functions. The three-point correlators are given by classical integrals on the target space geometry, e.g. for $d=3$:

$$C_{abc} = \langle \mathcal{Q}_a^{(1,1)} \mathcal{Q}_b^{(1,1)} \mathcal{Q}_c^{(1,1)} \rangle_0 = \int_X \Omega^{(3,0)} \wedge (\omega_a \wedge \omega_b \wedge \omega_c, \Omega^{(3,0)}), \quad a,b,c = 1,\ldots,h^{2,1}(X) \ . \tag{2.42}$$

Here $\omega_a \in H^1_{\bar{\partial}}(X,TX)$ are the geometric representatives of the marginal operators $\mathcal{Q}_a^{(1,1)}$. The mapping $(.,.) : \Omega^{0,p}(\Lambda^q TX) \to \Omega^{d-q,p}(X)$ is given in any dimension by contraction of vector indices with the holomorphic $(d,0)$-form $\Omega^{(d,0)}$ of the Calabi-Yau manifold

$$(\omega, \Omega^{(d,0)}) = \Omega^{(d,0)}_{i_1,\ldots,i_q,i_{q+1},\ldots i_d} \omega^{i_1,\ldots,i_q}_{\bar{j}_1,\ldots\bar{j}_p} \, dx^{i_{q+1}} \wedge \cdots \wedge dx^{i_d} \wedge dx^{\bar{j}_1} \wedge \cdots \wedge dx^{\bar{j}_p}, \ \omega \in \Omega^{0,p}(\Lambda^q TX) \ . \tag{2.43}$$

[17] The operators corresponding to Kähler-deformations are Q-exact with respect to the Q-operator of the B-model [20].

The classical integrals for the B-model three-point functions can be computed explicitly. This is due to their relation to period integrals and the mathematical theory of variation of mixed Hodge structure, which we describe in the following.

2.4. Variation of mixed Hodge structure and period integrals

In this section we are going to discuss how the B-model three-point functions can be expressed in terms of period integrals on the Calabi-Yau manifold X in question. Furthermore we outline how the periods can be computed by an application of the theory of variation of mixed Hodge structure.

2.4.1. Complex structure deformations of Calabi-Yau manifolds

To express the B-model correlators in terms of period integrals we first review some aspects of complex structure deformations of Calabi-Yau d-folds. Complex structure deformations can be understood as deformations of the Dolbeault operator $\bar{\partial}_{\bar{i}}$ on the Calabi-Yau manifold

$$\bar{\partial}'_{\bar{i}} = \bar{\partial}_{\bar{i}} - \omega_{\bar{i}}^{j}(z_a)\,\partial_j, \qquad \omega_{\bar{i}}^{j}(0) = 0, \qquad \text{for} \quad \omega(z_a) \in H^1(X,TX) \simeq H^1_{\bar{\partial}}(TX) \ , \qquad (2.44)$$

where the $z_a \in \mathbb{C}$, $a = 1,\dots,h^{d-1,1}$ are local coordinates on the complex structure moduli space $\mathcal{M}_{\mathrm{CS}}$. The vector-valued forms $\omega \in H^1(X,TX) \simeq T\mathcal{M}_{\mathrm{CS}}$ can be considered as tangent vectors to the moduli space $\mathcal{M}_{\mathrm{CS}}$. The deformed Dolbeault operator should be nilpotent, e.g. $(\bar{\partial}'_{\bar{i}})^2 = 0$, and this leads to the Kodaira-Spencer equation[18]

$$\bar{\partial}_{[\bar{i}}\omega^{k}_{\bar{j}]} = \omega^{l}_{[\bar{i}}\partial_l\omega^{k}_{\bar{j}]}, \qquad \omega_{\bar{i}}^{j}(0) = 0 \ , \qquad (2.45)$$

which has to be satisfied by the deformations $\omega^k_{\bar{j}}(z)$. In order to solve (2.45) one starts with a formal power-series

$$\omega(z) = \omega^{(1)}_a\, z_a + \omega^{(2)}_{ab}\, z_a z_b + \dots \qquad a = 1,\dots,h^{d-1,1} \qquad (2.46)$$

and solves the Kodaira-Spencer equation iteratively. A priori a generic first-order deformation $\omega^{(1)}_a \in H^1(X,TX)$ could be obstructed at higher orders in the deformation parameters z_a. For the complex structure moduli space of Calabi-Yau manifolds this is not the case and every first-order deformation $\omega^{(1)}_a \in H^1(X,TX)$ can be integrated to a finite complex structure deformation, as is the content of the Tian-Todorov lemma [91, 92]. Therefore the complex structure moduli space of a Calabi-Yau manifold is itself a manifold and its complex dimension is given by

$$\dim_{\mathbb{C}}(\mathcal{M}_{\mathrm{CS}}) = h^{d-1,1} \ . \qquad (2.47)$$

[18] See e.g. [22, 79] and references therein.

For open strings this is generically not the case and open string deformations get obstructed at higher orders in the deformation parameters [84], physically encoded in a superpotential coupling. This will be the central topic of chapter 3.

Complex structure deformations can also be understood more explicitly by considering the theory of variation of mixed Hodge structure to which we turn in the following.

Variation of mixed hodge structure

The structure of the vacuum bundle of the B-model ground states can be phrased in the mathematical framework of variation of mixed Hodge structure [93, 94, 95, 96, 97, 98]. For convenience we restrict the discussion to the case of $d = 3$, while the higher dimensional cases can be treated similarly. The physical states of the B-model are in this case represented by $(3-p,p)$-forms. However the notion of (anti-) holomorphicity changes in $H^3(X)$ when the background complex structure is changed. The vacuum (sub-) bundle $\mathcal{V}$ is in this case known as the Hodge bundle $\mathcal{H}^3$ over the complex structure moduli space $\mathcal{M} = \mathcal{M}_{\text{CS}}$ with fiber

$$H^3(X) = \bigoplus_{p=0}^{3} H^{3-p,p}(X) \ . \tag{2.48}$$

One can define a filtration $\mathcal{F}^p$ of holomorphic sub-bundles of $\mathcal{H}^3(X)$ with fibers F^p

$$H^3(X) = F^0 \supset F^1 \supset F^2 \supset F^3 \supset F^4 = 0, \qquad F^p = \bigoplus_{q \geq p} H^{q,3-q}(X) \ . \tag{2.49}$$

Explicitly these spaces are given by

$$\begin{aligned} F^3 &= H^{3,0}(X) \ , \\ F^2 &= H^{3,0}(X) \oplus H^{2,1}(X) \ , \\ F^1 &= H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \ , \\ F^0 &= H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \oplus H^{0,3}(X) \ . \end{aligned} \tag{2.50}$$

As the fiber $H^3(X) \supset H^3(X, \mathbb{Z})$ of the Hodge bundle is a topological group, which in particular admits a basis of integral sections which does not vary over the complex structure moduli space, one obtains an up to monodromies flat connection on the Hodge bundle. This connection is known as the Gauss-Manin connection ∇ and can be identified with the improved tt^*-connection of the vacuum bundle. The flatness of the Gauss-Manin connection translates into the following condition

$$[\nabla_{\partial_{z_a}}, \nabla_{\partial_{z_b}}] = 0 \ , \tag{2.51}$$

where z_a denotes as above local coordinates on the complex structure moduli space $\mathcal{M}_{\text{CS}}$. Furthermore the Gauss-Manin connection fulfills Griffiths transversality

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes T^*\mathcal{M} \ . \tag{2.52}$$

This property can be understood from the topological field theory viewpoint by noticing that the multiplication of a charge one operator can be replaced by the action of the improved connection in the topological path integral [22]

$$\mathcal{Q}_a^{(1,1)} \cdot \mathcal{Q}^{(p,p)} \to \nabla_{\partial_{z_a}} \mathcal{Q}^{(p,p)} \ . \tag{2.53}$$

More explicitly the action of the Gauss-Manin connection on the holomorphic $(3,0)$-form is given by

$$\nabla_{\partial_{z_a}} \Omega^{(3,0)} = (\omega_a, \Omega^{(3,0)}) + m_a \Omega^{(3,0)} \in F^2 = H^{2,1}(X) \oplus H^{3,0}(X) \ , \tag{2.54}$$

where the m_a are possible moduli-dependent functions. Furthermore Griffiths transversality together with a consideration of the involved Hodge types leads to the following identities[19]

$$\int_X \Omega^{(3,0)} \wedge \Omega^{(3,0)} = 0, \quad \int_X \Omega^{(3,0)} \wedge \nabla_{\partial_{z_a}} \Omega^{(3,0)} = 0, \quad \int_X \Omega^{(3,0)} \wedge \nabla_{\partial_{z_a}} \nabla_{\partial_{z_b}} \Omega^{(3,0)} = 0 \ . \tag{2.55}$$

This allows to rewrite the three-point functions of three marginal operators in the B-model (2.42) in terms of derivatives of the holomorphic $(3,0)$-form

$$\begin{aligned} C_{abc} &= \int_X \nabla_{\partial_{z_a}} \Omega^{(3,0)} \wedge \nabla_{\partial_{z_b}} \nabla_{\partial_{z_c}} \Omega^{(3,0)} = -\int_X \Omega^{(3,0)} \wedge \nabla_{\partial_{z_a}} \nabla_{\partial_{z_b}} \nabla_{\partial_{z_c}} \Omega^{(3,0)} \\ &= -\int_X \Omega^{(3,0)} \wedge \partial_{z_a} \partial_{z_b} \partial_{z_c} \Omega^{(3,0)} \ , \end{aligned} \tag{2.56}$$

which follows upon differentiating (2.54). To proceed further one chooses an explicit parametrization of the complex structure moduli space, given by period integrals of the holomorphic $(3,0)$-form.

2.4.2. Period integrals

We first choose a symplectic basis[20] of the homology group $H_3(X)$ given by closed cycles $(\gamma_A^{(3)}, \gamma_B^{(3)})$ for $A, B = 0, 1, \ldots, h^{2,1}$ such that the intersection pairing η of the cycles is given by the symplectic form

$$\eta = \begin{pmatrix} 0 & \Xi_{h_H^3 \times h_H^3} \\ -\Xi_{h_H^3 \times h_H^3} & 0 \end{pmatrix} \ , \quad \text{where} \quad (\Xi_{N\times N})_{ij} := \delta_{i,(N+1-j)}, \quad i,j = 1, \ldots, N \ , \tag{2.57}$$

and $h_H^3 = h^{2,1}(X) + 1$. One can form the so-called period vector with respect to this symplectic basis as

$$\Pi(z) = (X^A, \mathcal{F}_B) \equiv \left(\int_{\gamma_A^{(3)}} \Omega^{(3,0)}, \int_{\gamma_B^{(3)}} \Omega^{(3,0)} \right) \ . \tag{2.58}$$

[19] For complex even dimensional Calabi-Yau manifolds the analogous identities lead to algebraic relations between the periods to be defined below in (2.58). See e.g. [99, 100] for particular clear discussions in the case $d = 4$.

[20] In $d = 3$ the middle dimensional (co)-homology is a symplectic vector space w.r.t. the topological metric.

Since $\Omega^{(3,0)}$ is only defined up to rescalings, the periods X^A can be considered as projective coordinates on $\mathcal{M}_{\text{CS}}$ and parametrize the complex structure moduli space. The other periods $\mathcal{F}_B$ should therefore be expressible in terms of the coordinates X^A. One defines the projective prepotential $\mathcal{F}(X^A)$ as

$$\mathcal{F}(X^A) = \frac{1}{2} X^A \mathcal{F}_A \ , \tag{2.59}$$

such that $\mathcal{F}_A = \partial_{X^A}\mathcal{F}$. Near a point of maximal unipotent monodromy[21] in the complex structure moduli space, there are also inhomogeneous local coordinates called canonical coordinates t^a with $a = 1, \ldots, h^{2,1}$ on the moduli space [22]. They are given by

$$t^a = \frac{X^a}{X^0} \ . \tag{2.60}$$

Transforming to canonical coordinates and using Griffiths transversality one finds the prepotential $F(t^a)$ in flat coordinates

$$\mathcal{F}(X^A) = (X^0)^2 \, F(t^a) \ . \tag{2.61}$$

The period vector is expressed in terms of flat coordinates as

$$\Pi(t) = X^0(1, t^a, \partial_{t^a} F, 2F - t^a F_a) \ , \tag{2.62}$$

where $F_a = \partial_{t^a} F$. The three-point couplings can also be expressed in terms of flat coordinates, leading to

$$C_{abc} = \frac{\partial^3 F}{\partial t^a \partial t^b \partial t^c} \ , \tag{2.63}$$

upon a canonical normalization $\Omega \to \Omega / X^0$. This means that once the periods (2.58) are known in some symplectic basis and in some local coordinates one can calculate the three-points functions in canonical coordinates up to a symplectic redefinition of the periods.

Also the Kähler potential on the moduli space $\mathcal{M}_{\text{CS}}$ can be written in terms of the period integrals

$$K = -\log i \int_X \Omega^{(3,0)} \wedge \bar{\Omega}^{(0,3)} = -\log 2 \, \text{Im} \left(X^A \bar{\mathcal{F}}_{\bar{A}} \right) \ . \tag{2.64}$$

Using canonical coordinates this can be rewritten with the help of the intersection matrix η in the following way

$$e^{-K} = \Pi^\dagger \cdot \eta \cdot \Pi, \qquad \eta = \begin{pmatrix} 0 & \Xi_{h^3_H \times h^3_H} \\ -\Xi_{h^3_H \times h^3_H} & 0 \end{pmatrix} \ , \tag{2.65}$$

or more explicitly

$$e^{-K} = -X_0 \bar{X}_0 \left((t^a - \bar{t}^a)(F_a - \bar{F}_a) - 2(F - \bar{F}) \right) \ . \tag{2.66}$$

[21] See also section 2.4.5.

2.4.3. Picard-Fuchs equations

In some concrete models the period integrals (2.58) can be computed explicitly by direct integration of the involved integrals [37, 101, 102, 103, 104]. However in general this is technically rather challenging. Alternatively one can use the variation of mixed Hodge structure to compute the period integrals as the solutions to a set of differential operators, which are called Picard-Fuchs operators. For this one notes that starting with the holomorphic $(d,0)$-form of a Calabi-Yau d-fold one can create a basis of $H^d(X)$ by applying successive complex structure deformations, i.e. $\nabla^i_{\partial_{z_a}}\Omega^{(d,0)}(z) \in F^{d-i}$, by Griffiths transversality. This leads to the following schematic diagram

$$\Omega^{(d,0)} \xrightarrow{\nabla_a} \Omega^{(d-1,1)} \xrightarrow{\nabla_a} \cdots \xrightarrow{\nabla_a} \Omega^{(0,d)} \ . \tag{2.67}$$

As the space $H^d(X)$ is finite dimensional there should exist relations between derivatives at different order. For example for the case of a single complex structure deformation $h^{d-1,1} = 1$, the $d+1$-th derivative can be re-expressed in terms of the lower derivatives up to exact pieces, leading to a differential equation of the form

$$\nabla^{d+1}\Omega^{(d,0)} + \sum_{i=0}^{d} g_i(z)\nabla^i\Omega^{(d,0)} = d\beta \ , \tag{2.68}$$

for some $\beta \in \Omega^{d-1}(X)$. The equations (2.68) are called Picard-Fuchs equations and they can be used to compute the moduli dependence of the period integrals by integrating them over a topological basis of closed cycles, such that the term on the right-hand side drops out.

The Picard-Fuchs equations are a manifestation of the flat structure of the Hodge bundle over the moduli space. Choosing a basis of sections of the Hodge bundle given by

$$\vec{\omega} = (\Omega^{(d,0)}, \nabla\Omega^{(d,0)}, \ldots, \nabla^d\Omega^{(d,0)}) \tag{2.69}$$

for the case of $h^{d-1,1} = 1$ the flatness of the Gauss-Manin connection translates into a matrix differential equation

$$\nabla \cdot \vec{\omega} = (1\!\!1_{(d+1)\times(d+1)}\partial_z - A(z)) \cdot \vec{\omega} \sim 0 \ , \tag{2.70}$$

which holds up to exact terms. Choosing a topological basis of closed d-cycles $\gamma^{(d)}_\alpha \in H_d(X)$ one can get rid of the exact terms in (2.70) by constructing the period matrix

$$\Pi^\alpha_\beta(z_a) = \int_{\gamma^{(d)}_\alpha} \vec{\omega}_\beta \ , \qquad \alpha, \beta = 0, \ldots, d \ , \tag{2.71}$$

which fulfills the following differential equation

$$\nabla \cdot \Pi^\alpha_\beta(z) = 0 \ . \tag{2.72}$$

These are coupled first-order differential equations which can be traded for several higher-order equations by eliminating the lower rows of the period matrix. These higher order differential operators are the Picard-Fuchs operators (2.68) and they are schematically given by

$$\mathcal{L}_a(z_b, \partial_{z_b}) \cdot \Pi_0^\alpha(z) = \mathcal{L}_a(z_b, \partial_{z_b}) \cdot \int_{\gamma_\alpha^{(d)}} \Omega^{(d,0)} = 0 \,, \tag{2.73}$$

where the $\Pi^\alpha = \Pi_0^\alpha$ are the components of the period vector, e.g. (2.58) for $d = 3$. When studying open/closed mirror symmetry in chapter 3 we will find similar open/closed Picard-Fuchs operators for relative periods. The closed cycles are in this case replaced by open chains and thus the exact terms of (2.70) will give important contributions.

2.4.4. Computation of Picard-Fuchs operators

For the concrete computation of Picard-Fuchs operators one needs an explicit realization of the Calabi-Yau manifold and the unique holomorphic $(d, 0)$-form. In most cases the Calabi-Yau manifold is given as the zero locus of some transversally intersecting polynomials in a toric ambient space. In this situation the Picard-Fuchs equations can either be derived from a residue expression of the holomorphic $(d, 0)$-form [97, 105, 106] or from a set of generalized hypergeometric differential operators, which express the invariance of the forms on the Calabi-Yau space under the torus actions of the toric ambient space [107, 108, 109, 110]. We will describe both techniques shortly in the following.

Residue representation and Griffiths-Dwork reduction

For the ease of notation we consider the case of a Calabi-Yau d-fold X given as the zero locus of a single homogeneous polynomial P_z of degree $w_0 = \sum_i w_i$ in a weighted projective space $\mathbb{P}^{d+1}_{w_1,\ldots,w_{d+2}}$ as in [111]. The canonical $(d+1, 0)$-form Δ on the ambient space is given by

$$\Delta = \sum_{i=1}^{d+2} (-1)^{i+1} w_i x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{d+2} \,, \tag{2.74}$$

where the hat denotes omission. It has been shown in [97] that a closed d-form $\Xi^{(k)} \in F^{d-k}(X)$ in the Hodge filtration on the Calabi-Yau manifold is given by the following residue integral

$$\Xi^{(k)} = \mathrm{Res}_{P_z=0} \left(\frac{q(x)}{P_z^{k+1}} \Delta \right) \in F^{d-k}(X) \,, \tag{2.75}$$

for some homogeneous polynomial $q(x)$ of degree $w_0 k$. The holomorphic $(d, 0)$-form for example can be written as

$$\Omega^{(d,0)}(z) = \mathrm{Res}_{P_z=0} \left(\frac{\rho(z)}{P_z} \Delta \right) \,, \tag{2.76}$$

for some moduli dependent holomorphic function $\rho(z)$. Given homogeneous polynomials A_i of degree $w_0(k-1) + w_i$ one obtains a $(d-1, 0)$-form $\alpha^{(k)}$ by

$$\alpha^{(k)} = \mathrm{Res}_{P_z=0} \left(\sum_{i<j} (-1)^{i+j+1} \frac{(w_i x_i A_j - w_j x_j A_i)}{P_z^k} dx_1 \wedge \ldots \widehat{dx_i} \ldots \widehat{dx_j} \cdots \wedge dx_{d+2} \right) \tag{2.77}$$

and the exterior derivative is given by

$$d\alpha^{(k)} = \mathrm{Res}_{P_z=0}\left(\left(\sum_i k\frac{A_i\partial_i P_z}{P_z^{k+1}} - \frac{\partial_i A_i}{P_z^k}\right)\Delta\right) \ . \tag{2.78}$$

The above residue expressions can be used to compute the Picard-Fuchs operators by the Griffiths-Dwork reduction procedure [112].[22] By taking derivatives of (2.76) with respect to the complex structure modulus z one spans the whole Hodge filtration $\bigoplus_k F^{d-k}(X)$. By (2.78) any numerator which lies in the Jacobian ideal $\mathbb{C}[x]/(\partial_i P_z)$ can furthermore be traded for a form of lower pole order up to an exact form. The form with lower pole is then an element of $F^{d+1-k}(X)$. Iterating this procedure and using for convenience Gröbner basis techniques one can eventually derive the relations between the complex structure variations of the holomorphic $(d,0)$-form and therefore the Picard-Fuchs equations (2.68).

Generalized hypergeometric systems

For complete intersection Calabi-Yau manifolds in toric ambient spaces the period integrals can also be computed from a generalized hypergeometric system of differential operators, called the GKZ-system [107, 108, 109, 110]. The usefulness of the GKZ-system stems from the fact, that this set of differential operators can be extracted directly from the combinatorial data which defines the toric ambient space. A short overview of toric geometry is contained in Appendix A to which we again refer for further details and additional literature.

Let us first fix our notation: Δ is a reflexive polyhedron in $\mathbb{R}^{d+2}$ defined as the convex hull of p integral vertices $\nu_i \subset \mathbb{Z}^{d+2} \subset \mathbb{R}^{d+2}$ lying in a hyperplane of distance one to the origin.[23] $W = P_{\Sigma(\Delta)}$ is the toric variety with fan $\Sigma(\Delta)$ defined by the set of cones over the faces of Δ. $\Delta^\star$ is the dual polyhedron and W^* the toric variety obtained from $\Sigma(\Delta^\star)$. We denote by Z and Z^* the toric hypersurfaces in the ambient spaces W, W^* respectively.

The p (relevant) integral points of Δ determine the hypersurface $Z^* \subset W^*$ as the vanishing locus of the equation

$$P(Z^*) = \sum_{i=0}^{p-1} a_i y_i = \sum_{\nu_i\in\Delta} a_i X^{\nu_i} \ ,$$

where a_i are complex parameters, y_i are certain homogeneous coordinates [114] on W^*, X_k, $k=1,...,(d+1)$ are inhomogeneous coordinates on an open torus $(\mathbb{C}^*)^{d+1} \subset W^*$ and $X^{\nu_i} := \prod_k X_k^{\nu_{i,k}}$ [109]. The integral points ν_i and the homogeneous coordinates y_i fulfill $h^{d-1,1}(Z^*)$ relations

$$\sum_{i=0}^{p-1} l_i^a \nu_i = 0 \ , \qquad \prod_{i=0}^{p-1} y_i^{l_i^a} = 1 \ , \qquad a = 1,...,h^{d-1,1}(Z^*) \ . \tag{2.79}$$

[22] See [111] for an account from the perspective of mirror symmetry.

[23] We use the standard convention, identify the interior point ν_0 of Δ with the origin, and specify the vertices by $(d+1)$ components $\nu_{i,k}$, $k=1,...,(d+1)$, i.e. $\nu_0 = (0,0,\dots,0)$; see refs. [109, 108, 113] for more details.

The p-dimensional integral vectors l^a specify the charges of the matter fields of the gauged linear σ-model (GLSM) associated with the manifold Z [5].[24] The index 0 refers to the special field p of negative charge which enters linearly in the two-dimensional GLSM superpotential. As discussed above we are interested in the period integrals of the holomorphic $(d,0)$-form $\Omega^{(d,0)}$. The fundamental period integral on Z^* can be defined as [107]

$$\Pi^0(a_i) = \frac{1}{(2\pi i)^{d+1}} \int_{|X_j|=1} \frac{1}{P(Z^*)} \prod_{k=1}^{d+1} \frac{dX_k}{X_k} \ . \tag{2.80}$$

As noted in [107, 109], the period integral is annihilated by a system of differential operators of the so-called GKZ hypergeometric type [115]

$$\begin{aligned} \mathcal{L}(l) &= \prod_{l_i>0} \left(\frac{\partial}{\partial a_i}\right)^{l_i} - \prod_{l_i<0} \left(\frac{\partial}{\partial a_i}\right)^{-l_i}, \qquad l \in K, \\ \mathcal{Z}_k &= \sum_{i=0}^{p-1} \nu_{i,k}\vartheta_i \ , k=1,...,4; \qquad \mathcal{Z}_0 = \sum_{i=0}^{p-1} \vartheta_i + 1 \ , \end{aligned} \tag{2.81}$$

where $\vartheta_i = a_i\partial_{a_i}$ and K denotes the set of integral linear combinations of the charge vectors l^a. The differential equations $\mathcal{L}(l)\,\Pi^0(a_i) = 0$ follow straightforwardly from the definition (2.80). The equations $\mathcal{Z}_k\,\Pi^0(a_i) = 0$ express the invariance of the period integral under the torus actions and imply that the periods depend, up to normalization, only on special combinations of the parameters a_i, $\Pi^0(a_i) \sim \Pi^0(z_a)$, where

$$z_a = (-)^{l_0^a} \prod_i a_i^{l_i^a} \tag{2.82}$$

define $h^{d-1,1}(Z^*)$ local coordinates on the complex structure moduli space of Z^* [108]. One has to note that the GKZ-operators might not immediately lead to the Picard-Fuchs operators. However by appropriate factorization the GKZ-system can often be reduced to the Picard-Fuchs system [108, 110].

The solutions to the GKZ-equations (2.81) can be obtained via the Frobenius method.[25] For this one first rewrites (2.81) in terms of the coordinates (2.82)

$$\mathcal{L}(l) = \prod_{k=1}^{l_0}(\theta_0 - k) \prod_{l_i>0} \prod_{k=0}^{l_i-1}(\theta_i - k) - (-1)^{l_0} z_a \prod_{k=1}^{-l_0}(\theta_0 - k) \prod_{l_i<0} \prod_{k=0}^{-l_i-1} (\theta_i - k) \ , \tag{2.83}$$

where $\theta = z_a\partial_{z_a}$. The solutions of (2.83) for an appropriate choice of vectors l can then be obtained from the following generating function

$$B_{\{l^a\}}(z_a;\rho_a) = \sum_{n_1,\ldots,n_N \in \mathbb{Z}_0^+} \frac{\Gamma\left(1 - \sum_a l_0^a(n_a+\rho_a)\right)}{\prod_{i>0} \Gamma\left(1 + \sum_a l_i^a(n_a+\rho_a)\right)} \prod_a z_a^{n_a+\rho_a} \ , \tag{2.84}$$

[24] See Appendix A for a brief account.
[25] See e.g. [108].

by taking multi-derivatives with respect to the formal parameters ρ_a and evaluating at $\rho_a = 0$. Linear combinations of these solutions capture the moduli dependence of the period integrals in (2.58).

2.4.5. Global properties of the complex structure moduli space

As motivated above linear combinations of the solutions of the Picard-Fuchs equations yield the geometric period integrals. However the particular linear combinations are not fixed as the Picard-Fuchs equations are linear differential equations. To pin down the geometric and in particular integral linear combinations of the periods one can make use of the global properties of the complex structure moduli space. This is possible as the period integrals are sections of a bundle over the complex structure moduli space. Analyzing the monodromies around boundary divisors of $\mathcal{M}_{\mathrm{CS}}$ and demanding global consistency of the transformation behavior of the period sections restricts the possible linear combinations. Looping around a boundary point $z_* \in \mathcal{M}_{\mathrm{CS}}$ the period vector transforms as

$$\Pi(z) \longrightarrow M(z_*) \cdot \Pi(z), \qquad M \in \mathrm{Sp}_\eta(h^d_H(X), \mathbb{Z}) \ , \tag{2.85}$$

where $h^d_H(X)$ denotes the dimension of the primary horizontal subspace

$$H^d_H(X) \subseteq \bigoplus_p H^{d-p,p}(X) \ ,$$

consisting of elements which can be reached by complex structure variations of elements of $H^{d,0}(X)$. The group $\mathrm{Sp}_\eta(h^d_H(X), \mathbb{Z})$ consists of all matrices, which leave the intersection form η invariant. For a Calabi-Yau three-fold η is given by the symplectic form in flat coordinates and for a suitable normalization. The integrality condition in (2.85) stems from charge quantization.

The above is in the same spirit as the techniques used in the geometric solution of $\mathcal{N} = 2$ $SU(2)$ SYM-theory by Seiberg and Witten [7, 6]. In this situation the moduli space is given by the u-plane of the scalar vev of the $\mathcal{N} = 2$ vector-multiplet scalars. The periods of an auxiliary torus, which is fibered over the u-plane, capture the quantum corrected scalar vevs, coupling constants and the fully instanton corrected prepotential. Using the behavior of the periods at the monopole, dyon and weak coupling points together with a positivity condition leads to a particular linear combination of torus periods which yields the physical quantities. The Seiberg-Witten setup can be embedded into string theory and actually be mapped to the complex structure moduli space of a type IIB compactification [50, 38, 39, 25]. Therefore this situation is literally captured by the discussion in this section.

We sketch the argument in the following for the case of a Calabi-Yau d-fold X with a single complex structure deformation z and refer to the explicit examples at the end of this chapter for more details. Models with more than one complex structure deformation can be handled in a similar way, but turn out to be technically more challenging [116, 117]. Generically the

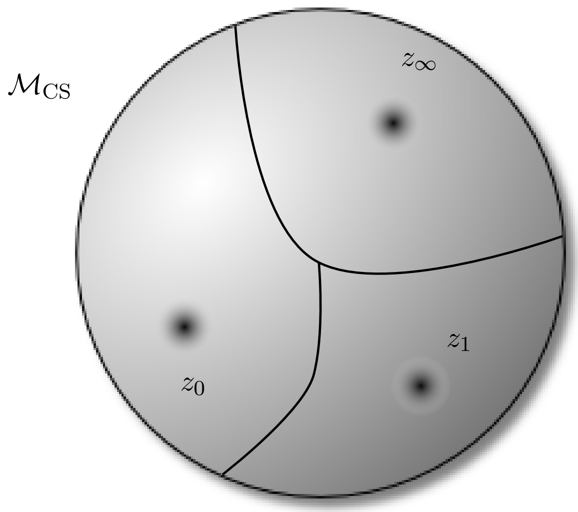

Figure 2.1.: Schematic picture of the generic complex structure moduli space of a Calabi-Yau manifold with a single complex structure deformation.

complex structure moduli space of a Calabi-Yau manifold with a single complex structure modulus is upon compactification topologically equivalent to a $\mathbb{P}^1$ with three special branch points[26]:

- Point of maximal unipotent monodromy z_∞:
  Such a point will be mirror symmetric to a large volume point in the mirror A-model geometry and is therefore a natural base point for mirror symmetry constructions. A point of maximal unipotent monodromy is characterized by the condition [111]

$$(1\!\!1_{h^d_H \times h^d_H} - M_\infty)^{d+1} = 0, \qquad (1\!\!1_{h^d_H \times h^d_H} - M_\infty)^n \neq 0,\ n < d+1\ , \tag{2.86}$$

  on the monodromy M_∞ around z_∞.

- Principal discriminant[27] point z_1:
  At a principle discriminant point the manifold develops a special type of 'singularity'. The nature of such a point depends on the particular form and dimension of the Calabi-Yau manifold under consideration. In the explicit example of the mirror sextic three-fold in section 2.6.1 and generically for Calabi-Yau three-folds the manifold develops at this locus a true geometrical conifold singularity and this can be traced to the vanishing of a 3-cycle of topology S^3 [118, 37]. The monodromy M_1 around such a conifold point is for three-folds generically of infinite order. However for higher-dimensional Calabi-Yau manifolds with $d > 3$ the principal discriminant needs not to be related to a true

[26]Details on the determination of the branch points for models which have an description in terms of a gauged linear σ-model can be found in Appendix A.

[27]For higher dimensional moduli spaces, e.g. $h^{d-1,1}(X) > 1$, the principal discriminant is generically of complex co-dimension one.

conifold singularity. In particular in the example of the sextic Calabi-Yau four-fold in section 2.6.2, we observe a principle discriminant monodromy of order two, which does not seem to be related to a vanishing cycle.

- Landau-Ginzburg point z_0:
  At a Landau-Ginzburg point the compactification can be described by a Landau-Ginzburg orbifold model, which generically has a discrete quantum symmetry $\mathbb{Z}_k$. The periods have to respect this symmetry and therefore the monodromy M_0 around z_0 is of finite order k

$$M_0^k = \mathbb{1}_{h_H^d \times h_H^d} \ . \tag{2.87}$$

The global consistency of the moduli space and the absence of further branch points leads to the following relation between the monodromy matrices [37][28]

$$M_\infty^{-1} \cdot M_1 \cdot M_0 = \mathbb{1}_{h_H^d \times h_H^d} \ . \tag{2.88}$$

Typically one has given the solutions to the Picard-Fuchs equations as a Taylor-series with finite radius of convergence around one of the boundary points and obtains therefore a vector of solutions $\bar{\omega}$. This vector should be related to the geometric integral period vector Π by a (non-integral) change of basis B

$$\Pi = B \cdot \bar{\omega} \ . \tag{2.89}$$

To relate the Taylor-series solutions around one point to the solutions around the other boundary points one has to resort to analytic continuation techniques. It is a non-trivial statement that the analytic continuation of one series can be expressed as a linear-combination of solutions around another point and is a consequence of the well-defined global structure of the deformation problem under consideration. For the case of hypersurfaces in toric varieties the Taylor-series are of hypergeometric type and can be analytically continued with the help of Barnes type integrals, as is shown in the explicit examples of section 2.6.

By demanding integrality of the obtained analytic continuation matrices one can then eventually fix the change of basis B and obtain the geometric integral period integrals.

2.5. Mirror symmetry

The properties of the topological A- and B-models are very different, as summarized in Table 2.1. While for example the correlation functions of one theory receive an infinite tower of worldsheet instanton corrections the other theory receives no such corrections. It is remarkable

[28] This relation depends on a choice of base-point and on the direction in which the boundary points are encircled. It can also be obtained from the Zariski–van Kampen theorem, see e.g. [119, 117].

	A-model	B-model
Topological states ($p=q$)	$H^{p,p}_{\bar\partial}(X)$	$H^{d-p,p}_{\bar\partial}(X)$
Marginal deformations	$H^{1,1}_{\bar\partial}(X)$	$H^{d-1,1}_{\bar\partial}(X)$
Geometrical deformations	Kähler deformations	complex structure deformations
Moduli Space	$\mathcal{M}_{\text{Kähler}}$	$\mathcal{M}_{\text{CS}}$
Q-invariant configurations	$\partial_{\bar z} x^i = 0$ (hol. curves)	$\partial_z x^i = \partial_{\bar z} x^i = 0$ (points)
Topological D-branes	(special) Lagrangian submanifolds (real d-dim.) + flat bundles	holomorphic submanifolds (even dim.) + holomorphic bundles

Table 2.1.: Comparison of the topological A- and B-models on a Calabi-Yau d-fold X. The topological D-branes are discussed in section 3.1.

that nonetheless both theories become equivalent upon compactification on mirror symmetric Calabi-Yau manifolds.

That such a relation exists can be seen from the perturbative string worldsheet point of view by a $\mathbb{Z}_2$ redefinition of the $U(1)$-currents $\bar J \to -\bar J$ of the topological conformal algebra (2.34), which exchanges the A- and B-twists [21]. This simple symmetry has drastic consequences on the target spaces the theories live on, as we will discuss momentarily.

Mirror symmetry can be extended to the full physical type II theories compactified on mirror symmetric Calabi-Yau three-folds. In its most general form the mirror conjecture then states that given a compactification of type IIA string theory on a Calabi-Yau three-fold Z there exists a mirror dual compactification of IIB string theory on a Calabi-Yau three-fold Z^* such that the effective physics in the lower dimensions stays the same.[29] The mirror duality includes not only the operator content and the perturbative correlators, but also the moduli spaces of both theories and the non-perturbative D-branes states [44].

The equivalence between the topological models on the mirror symmetric Calabi-Yau manifolds leads therefore to the following prediction on their respective topology

$$H^{p,p}_V(Z) \simeq H^{d-p,p}_H(Z^*) \ , \tag{2.90}$$

where $H_V(Z)$ denotes the primary vertical subspace of $\bigoplus_p H^{p,p}(Z)$ obtained by wedging the forms in $H^{1,1}(Z)$ and $H_H(Z^*)$ is the primary horizontal subspace of $\bigoplus_p H^{d-p,p}(Z^*)$ spanned by the complex structure deformations of the elements in $H^{d,0}(Z^*)$. This already shows that mirror symmetry acts highly non-trivial on the target space geometry.

Mirror symmetry respects string perturbation theory[30] and therefore it predicts a relation

[29] This should at least be true, if the moduli space of the compactification contains a Gepner point.

[30] See e.g. [114] for a worldsheet derivation of closed string mirror symmetry for models which can be described by a gauged linear σ-model.

between the generating functions of correlation functions on a closed worldsheet with genus g, which is schematically given by

$$F_A^{(g)}(t) \quad \longleftrightarrow \quad F_B^{(g)}(z) \ . \tag{2.91}$$

For the case $g = 0$ one in particular recovers the prepotential

$$F_A(t) \simeq F_B(z(t)) \ . \tag{2.92}$$

Here we collectively denoted by t the Kähler-moduli of the A-model and by z the complex structure moduli of the B-model. As discussed the left-hand side of (2.92) receives a whole tower of worldsheet instanton corrections, whereas the right-hand side can be computed by classical integrals on the target space geometry. To make use of the relation (2.92) one has to find the explicit mappings $z(t)$ between the moduli of both models. These mappings are called mirror maps and we will outline how they can be determined in the following.

Flat coordinates and mirror maps

In order to find an explicit isomorphism between $H_V^{p,p}(Z)$ and $H_H^{d-p,p}(Z^*)$ one can make use of the flat structure of the chiral ring of the underlying topological field theories [20, 75, 22, 107].[31] This isomorphism gives in particular a mapping between the tangent spaces of the respective moduli spaces. The classical moduli space of the A-model is a linear space, given by the Kähler cone

$$\mathcal{M}_K = \left\{ \sum_i t^a \omega_a : \mathrm{Im}\, t^a \geq 0,\ a = 1, \ldots, h^{1,1}(Z) \right\} \ , \tag{2.93}$$

where $\{\omega_a\}_{a=1,\ldots,h^{1,1}(Z)}$ is a basis of $H^{1,1}(Z)$, such that the Kähler volumes of all submanifolds are positive. There is a natural basis for the A-model operators[32] $\mathcal{O}_{k_p}^{(p)}$ near a large volume point $m_A \in \mathcal{M}_K$, generated by wedging the basis elements ω_a. For this we denote by D_a the Poincaré-dual basis of $H_{2d-2}(Z)$. The basis is then given by

$$\mathcal{O}^{(0)} = 1, \quad \mathcal{O}_a^{(1)} = \omega_a, \quad \mathcal{O}_{b_2}^{(2)} = (E_{b_2}^{(2)})^{ab}\, \omega_a \omega_b \quad \ldots \quad \mathcal{O}^{(d)} = (E^{(d)})^{b_1 \ldots b_d} \omega_{b_1} \ldots \omega_{b_d} \ , \tag{2.94}$$

where the form and the number d_p of the coefficient matrices $E_{b_p}^{(p)}$, $p = 0, \ldots, d$, $b_p = 1, \ldots, d_p$ is fixed by the classical intersection properties of the homology basis spanned by the D_a. This basis has the following properties[33]

$$\mathcal{O}_a^{(1)} \cdot \mathcal{O}_b^{(p)} = (C^{(p)})_{ab}^c\, \mathcal{O}_c^{(p+1)}, \quad \eta(\mathcal{O}_a^{(p)}, \mathcal{O}_b^{(q)}) = \delta_{p+q,d}\, \eta_{ab}^{(p)}, \quad \mathcal{O}_a^{(1)} \cdot \mathcal{O}^{(d)} = 0 \ , \tag{2.95}$$

where $\eta_{ab}^{(p)}$ denotes the classical intersections of the respective Poincaré-dual cycles.

[31] See also [99] for a particular clear discussion in the case $d = 4$.

[32] We abbreviate here $\mathcal{O}_{k_p}^{(p,p)} \equiv \mathcal{O}_{k_p}^{(p)}$.

[33] In these equations the indices a, b, c are to be understood as running over all the d_p Operators of charge p respectively.

A mirror symmetric basis of the B-model operators $\mathcal{Q}^{(p)}_{b_p}$ should reproduce the properties of the above A-model basis. The moduli space of the B-model is a curved manifold and therefore the mirror isomorphism will require a choice of base-point around which suitable flat coordinates can be constructed to match the properties of the linear A-model moduli space. The crucial step of [75] in finding a suitable base-point and constructing a basis is to replace the OPE-action of a marginal operator by the action of the unprojected Gauss-Manin connection on the B-model operators

$$\mathcal{O}^{(1)}_a \cdot \mathcal{O}^{(p)}_{b_p} \longrightarrow \nabla_a \mathcal{Q}^{(p)}_{b_p} \ . \tag{2.96}$$

Here the directional derivative is specified in terms of marginal operators $\mathcal{Q}^{(1)}_a$ corresponding to deformations parametrized by the coordinates t^a of the A-model. The properties of the chiral ring of the topological field theory enforces the following non-trivial relation [22]

$$\mathcal{Q}^{(1)}_a \cdot \mathcal{Q}^{(p)}_{b_p} = \partial_{t^a} \mathcal{Q}^{(p)}_{b_p} \stackrel{!}{=} (C^{(p)}_a)_{b_p}^{c_{p+1}}(t)\, \mathcal{Q}^{(p+1)}_{c_{p+1}} \ , \tag{2.97}$$

where $C^{(p)}_a$ are the chiral ring coefficients and the operators $\mathcal{Q}^{(p)}_{b_p}$ have definite $U(1)$-charge, e.g. Hodge-degree. This means that in flat coordinates one can replace an operator product with a marginal operator by a derivative with respect to the flat coordinate in the topological path integral [22]. The Gauss-Manin connection in general coordinates z_a centered near a point $m_B \in \mathcal{M}_{\text{CS}}$ has the following schematic form

$$\nabla_a = 1\!\!1_{h^d_H \times h^d_H} \partial_{z_a} - \Gamma_a(z) - C_a(z), \quad C_a(z) = \begin{pmatrix} 0 & 1\!\!1 & 0 & 0 & 0 & \dots \\ 0 & 0 & C^{(1)}_a(z) & 0 & 0 & \dots \\ 0 & 0 & 0 & C^{(2)}_a(z) & 0 & \dots \\ 0 & 0 & 0 & 0 & C^{(3)}_a(z) & \dots \\ & & \dots & \dots & & \end{pmatrix} . \tag{2.98}$$

In order to implement the flat structure of the chiral ring one therefore has to find a suitable change of coordinates $t(z)$ and possibly moduli-dependent normalizations of the operators such that the transformed Gauss-Manin connection becomes upper triangular, e.g. $\Gamma'_a(t) = 0$.

The suitable basis and coordinate changes can be found by row-manipulations on the period matrix (2.71), which include addition of rows and multiplication with moduli dependent holomorphic functions [75, 107]. As the A-model is invariant under an integral shift of the B-field, e.g. $t \to t+1$ [111, 75], a suitable basepoint m_B for the identification with the A-model is given by a point of maximal unipotent monodromy [37, 111, 75, 107]. This is a boundary point of the complex structure moduli space around which the topological cycles have non-trivial monodromies. Specifically near a point of maximal unipotent monodromy one finds the following schematic leading behavior of the periods (for $h^{d-1,1}(Z^*) = 1$) [37, 111]

$$\int_{\gamma_k} \Omega(z) = S_0(z) \frac{(\log z)^k}{(2\pi i)^k} + \dots, \quad k = 0, \dots, d \tag{2.99}$$

and $S_0(z) = \int_{\gamma_0} \Omega(z) = \text{single} - \text{valued}$.

The flat TFT structure leads then near a point of unipotent maximal monodromy to the condition that the flat coordinates t^a are given by ratios of periods [75]

$$t^a(z) = \frac{\int_{\gamma_a} \Omega^{(d,0)}(z)}{\int_{\gamma_0} \Omega^{(d,0)}(z)} \tag{2.100}$$

and this brings the period matrix into the form

$$(\Pi^\alpha_\beta)(t^a) = \begin{pmatrix} 1 & * & * & * & \dots \\ 0 & 1\!\!1_{d_1\times d_1} & * & * & \dots \\ 0 & 0 & 1\!\!1_{d_2\times d_2} & * & \dots \\ 0 & 0 & 0 & 1\!\!1_{d_3\times d_3} & \dots \\ & & \dots & & \end{pmatrix}, \qquad \alpha, \beta = 1, \dots, h^d_H \,. \tag{2.101}$$

The leading behavior of the periods near a point of maximal unipotent monodromy given in (2.99) then singles out the ratio of the period with linear logarithmic behavior as the candidate for the mirror map near this point. One has to note that the above construction can also be carried out around other points and therefore yields candidates for mirror maps around any point in the closed string moduli space [107].

In the following examples we are going to show how these structures are implemented in explicit examples.

2.6. Examples of closed string mirror symmetry

The aim of this section is to give explicit examples the above general considerations. Both models which we are going to study have already been discussed to great extent in the literature. The sextic Calabi-Yau three-fold was analyzed in [120, 121] along the lines of [37], and in [107], whereas results for the sextic Calabi-Yau four-fold were obtained in [75, 122, 123]. For details on the techniques from toric geometry, which are used in the following, especially the construction of the mirror manifolds, we refer again to Appendix A.

2.6.1. The Calabi-Yau three-fold $\mathbf{X}_6^{(1,1,1,1,2)}$

As a first example we consider the Calabi-Yau three-fold $\mathbf{X}_6^{(1,1,1,1,2)}$, which is given by the generic degree six hypersurface in weighted projective space $\mathbb{P}^4_{(1,1,1,1,2)}$. The ambient space for the A-model geometry is specified by the vertices in Table 2.2.[34]

One has to note that the weighted projective space $\mathbb{P}^4_{(1,1,1,1,2)}$ has singular points, however as a generic hypersurface misses single points, one does not have to smooth out the ambient

[34]Here and in the following we adopt the convention that all vertices are to be understood as lying in an affine hyperplane in one dimension higher, e.g. the true vertices are $\hat{\nu}_i = (1, \tilde{\nu}_i)$, see e.g. [108].

Δ									Δ^*							
	$\tilde{\nu}_0$	= (	0	0	0	0)	,			ν_0	= (	0	0	0	0)
	$\tilde{\nu}_1$	= (	1	0	0	0)				ν_1	= (	5	−1	−1	−1)
	$\tilde{\nu}_2$	= (	0	1	0	0)				ν_2	= (	−1	5	−1	−1)
	$\tilde{\nu}_3$	= (	0	0	1	0)				ν_3	= (	−1	−1	5	−1)
	$\tilde{\nu}_4$	= (	−1	−1	−1	−2)				ν_4	= (	−1	−1	−1	−1)
	$\tilde{\nu}_5$	= (	0	0	0	1)				ν_5	= (	−1	−1	−1	2)

Table 2.2.: Vertices Δ of the toric ambient geometry of the hypersurface $\mathbf{X}_6^{(1,1,1,1,2)}$.

geometry in this case. Triangulating[35] Δ leads to two phases:

1. Orbifold phase: In this phase the cones of the fan are given by

$$\{\langle\tilde{\nu}_1,\tilde{\nu}_2,\tilde{\nu}_3,\tilde{\nu}_4,\tilde{\nu}_5\rangle\}\ . \tag{2.102}$$

   One obtains a Landau-Ginzburg orbifold model with orbifold group $\mathbb{Z}_6$ and superpotential given by the generic hypersurface constraint.

2. Large volume phase: The cones of the fan in this phase are given by

$$\{\langle\tilde{\nu}_0,\tilde{\nu}_1,\tilde{\nu}_2,\tilde{\nu}_3,\tilde{\nu}_4\rangle,\langle\tilde{\nu}_0,\tilde{\nu}_1,\tilde{\nu}_2,\tilde{\nu}_3,\tilde{\nu}_5\rangle,\langle\tilde{\nu}_0,\tilde{\nu}_1,\tilde{\nu}_2,\tilde{\nu}_4,\tilde{\nu}_5\rangle,\langle\tilde{\nu}_0,\tilde{\nu}_1,\tilde{\nu}_3,\tilde{\nu}_4,\tilde{\nu}_5\rangle,\langle\tilde{\nu}_0,\tilde{\nu}_2,\tilde{\nu}_3,\tilde{\nu}_4,\tilde{\nu}_5\rangle\}\ . \tag{2.103}$$

   Computing the Mori-cone leads to the following charge vector

	0	1	2	3	4	5
l	−6	1	1	1	1	2

(2.104)

   and the topological data is given by

$$c(Z)=1+14K^2-68K^3,\quad K^3=3,\quad \chi(X)=-204,\quad \frac{c_2\cdot K}{24}=\frac{7}{4}\ . \tag{2.105}$$

   The Kähler form is expressed as $J=tK$, where K denotes the hyperplane class. The non-vanishing Hodge numbers can be computed to

$$h^{0,0}=1=h^{3,0},\quad h^{1,1}=1,\quad h^{2,1}=103\ . \tag{2.106}$$

We next describe the geometry, which is mirror to the above specified A-model geometry.

The mirror model

The mirror model can be constructed by the Batyrev-Borisov construction [109, 126], see also Appendix A. The hypersurface constraint for the mirror model is given in terms of homogeneous coordinates on $\mathbb{P}^4_{(1,1,1,1,2)}/(\mathbb{Z}_6^2\times\mathbb{Z}_3)$ by

$$P(Z^*)=x_1^6+x_2^6+x_3^6+x_4^6+x_5^3-6\psi\, x_1x_2x_3x_4x_5\ , \tag{2.107}$$

[35] These and the following calculations of topological data were performed with the help of existing computer code [124, 125].

together with an appropriate Greene-Plesser orbifold $\mathbb{Z}_6^2\times\mathbb{Z}_3$ action, which acts by $x_i \to \lambda_k^{g_{k,i}} x_i$ for $\lambda_{1,2}^6 = \lambda_3^3 = 1$ and weights

$$\mathbb{Z}_6 : \; g_1 = (1,-1,0,0,0), \quad \mathbb{Z}_6 : \; g_2 = (1,0,-1,0,0), \quad \mathbb{Z}_3 : \; g_3 = (1,0,0,0,-1) \; . \tag{2.108}$$

The Picard-Fuchs operators can be computed from the general formula (2.83) and are given by

$$\mathcal{L}(l) = \theta^4 - 9z(6\theta+1)(6\theta+2)(6\theta+4)(6\theta+5) \; , \tag{2.109}$$

where $z = (6\psi)^{-6}$ and we factorized the degree six GKZ-operator to a degree four operator.[36] Using the explicit form of the Picard-Fuchs operator one can recover the Gauss-Manin connection expressed in the coordinate z. Choosing a basis $\mathbf{\Omega}_z = (\Omega(z), \partial_z\Omega(z), \partial_z^2\Omega(z), \partial_z^3\Omega(z))^T$ the Gauss-Manin connection is given by

$$\nabla_z \mathbf{\Omega} = (1\!\!1_{4\times 4}\partial_z - G(z))\, \mathbf{\Omega} \; , \tag{2.110}$$

where

$$G(z) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{360}{z^3(11664z-1)} & \frac{1-55080z}{z^3(11664z-1)} & \frac{7z-167508z^2}{z^3(11664z-1)} & \frac{6z^2-93312z^3}{z^3(11664z-1)} \end{pmatrix} . \tag{2.111}$$

The solutions to the Picard-Fuchs equations (2.109) can be found by using the generating function (2.84), which specializes in this case to

$$B_{\{l\}}(z;\rho) = \sum_{n\geq 0} \frac{\Gamma(1+6n+6\rho)\, z^{n+\rho}}{\Gamma(1+n+\rho)^4\, \Gamma(1+2n+2\rho)} \tag{2.112}$$

and leads to a basis of solutions $\bar{\omega}$ near the point $z=0$ given by $\bar{\omega}_k = \frac{1}{(2\pi i)^k}\partial_\rho^k B_{\{l\}}(z;\rho)|_{\rho=0}$ for $k = 0,\ldots,3$

$$\bar{\omega}(z) = \begin{pmatrix} 1 \\ \frac{\log(z)}{2\pi i} \\ \frac{\log(z)^2}{(2\pi i)^2} - \frac{7}{6} \\ \frac{\log(z)^3}{(2\pi i)^3} - \frac{7i\,\log z}{4\pi} - \frac{51 i\zeta(3)}{\pi^3} \end{pmatrix} + \mathcal{O}(z) \; . \tag{2.113}$$

This basis has the following monodromy around the point $z=0$ of maximal unipotent monodromy

$$\bar{\omega}(e^{-2\pi i}z) = \tilde{M}_\infty \cdot \bar{\omega}(z), \qquad \tilde{M}_\infty = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix} . \tag{2.114}$$

[36] This form is expected from the Stanley-Reisner ideal of the mirror manifold Z and the form of the classical mirror map [110].

To fix the particular linear combinations, which correspond to the integral periods we compare the large volume basis with a basis near the orbifold point $\psi = 0$. In order to carry this out we represent the solutions to (2.109) near the orbifold point as

$$\omega_j^{\text{orb}}(\psi) = \psi^j \; {}_4F_3\left(\frac{j}{6},\frac{j}{6},\frac{j}{6},\frac{j}{6};\overbrace{\frac{j+2}{6},\frac{j+3}{6},\frac{j+5}{6},\frac{j+6}{6}};4\psi^6\right), \quad j = 1,2,4,5\ , \tag{2.115}$$

where the overbrace means, that the coefficient which is equal to one is left out. The functions ω_j^{orb} can be represented by a Barnes type integral[37] as

$$\omega_j^{\text{orb}}(\psi) = -a_j\psi^j \oint_{C_+} \frac{\Gamma(s+\frac{j}{6})^4\Gamma(-s)(-1)^s\Gamma(s+1)\ (4\psi^6)^s}{\Gamma(s+\frac{j+2}{6})\Gamma(s+\frac{j+3}{6})\Gamma(s+\frac{j+5}{6})\Gamma(s+\frac{j+6}{6})}\, ds\ , \tag{2.116}$$

where C_+ is a closed contour which picks the poles at $s = n$ for $n \geq 0$ and makes the integrand to vanish exponentially fast at infinity. The coefficients a_j are given by

$$a_j = \frac{\Gamma(\frac{j+2}{6})\Gamma(\frac{j+3}{6})\Gamma(\frac{j+5}{6})\Gamma(\frac{j+6}{6})}{\Gamma(\frac{j}{6})^4}\ . \tag{2.117}$$

In order to analytically continue we close the contour to the left along a path C_- to obtain

$$-\,a_j\psi^j \oint_{C_-} \frac{\Gamma(s+\frac{j}{6})^4\Gamma(-s)(-1)^s\Gamma(s+1)\ (4\psi^6)^s}{\Gamma(s+\frac{j+2}{6})\Gamma(s+\frac{j+3}{6})\Gamma(s+\frac{j+5}{6})\Gamma(s+\frac{j+6}{6})}\, ds\ , \tag{2.118}$$

and pick the poles at $s = -n - \frac{j}{6}$ for $n \geq 0$ and $j = 1,2,4,5$. The result can be matched to the large volume basis $\bar{\omega}$. This leads to an analytic continuation matrix m of the form

$$\bar{\omega}(z) = m\cdot\omega^{\text{orb}}(\psi(z)),\quad m = \begin{pmatrix} -\frac{1296(\sqrt{3}+i)\pi}{\Gamma(-\frac{1}{6})^4\Gamma(\frac{2}{3})} & \frac{9\sqrt[3]{-2}\sqrt{3}\Gamma(\frac{7}{6})}{\pi^{3/2}\Gamma(\frac{2}{3})} & \frac{6-6i\sqrt{3}}{\Gamma(\frac{1}{3})^3} & -\frac{36(\sqrt{3}-i)\pi}{\Gamma(\frac{1}{6})^4\Gamma(\frac{1}{3})} \\ -\frac{2i\pi}{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})^4} & \frac{9i\sqrt[3]{2}\Gamma(\frac{7}{6})}{\pi^{3/2}\Gamma(\frac{2}{3})} & -\frac{4i\sqrt{3}}{\Gamma(\frac{1}{3})^3} & \frac{72i\pi}{\Gamma(\frac{1}{6})^4\Gamma(\frac{1}{3})} \\ -\frac{7776\sqrt{3}\pi}{\Gamma(-\frac{1}{3})\Gamma(-\frac{1}{6})^4} & -\frac{2}{\Gamma(\frac{2}{3})^3} & -\frac{4}{\Gamma(\frac{1}{3})^3} & \frac{72\sqrt{3}\pi}{\Gamma(\frac{1}{6})^4\Gamma(\frac{1}{3})} \\ \frac{12960i\pi}{\Gamma(-\frac{1}{6})^4\Gamma(\frac{2}{3})} & -\frac{2i\sqrt{3}}{\Gamma(\frac{2}{3})^3} & \frac{4i\sqrt{3}}{\Gamma(\frac{1}{3})^3} & \frac{540i\pi}{\Gamma(-\frac{2}{3})\Gamma(\frac{1}{6})^4} \end{pmatrix}. \tag{2.119}$$

The action of the $\mathbb{Z}_6$-orbifold action with $\alpha^6 = 1$ on the solutions ω^{orb} is given by

$$\omega^{\text{orb}}(\alpha\psi) = \tilde{M}_0^{\text{orb}}\cdot\omega^{\text{orb}}(\psi),\qquad \tilde{M}_0^{\text{orb}} = \text{diag}(\alpha,\alpha^2,\alpha^4,\alpha^5)\ . \tag{2.120}$$

Therefore the orbifold monodromy acts on the large volume solutions as

$$\bar{\omega}(\alpha\psi) = \tilde{M}_0\cdot\bar{\omega}(\psi),\qquad \tilde{M}_0 = m\cdot\tilde{M}_0^{\text{orb}}\cdot m^{-1}\ . \tag{2.121}$$

The basis of solutions $\bar{\omega}$ should now be related to the integral periods by a linear transformation

$$\Pi(z) = B\cdot\bar{\omega}(z)\ . \tag{2.122}$$

[37] See e.g. [37].

This change of basis can be fixed up to a symplectic transformation by demanding integrality of the symplectic[38] monodromy matrices M_0, M_∞ and $M_1 = M_\infty \cdot M_0^{-1}$ with respect to the new basis. Under the coordinate change the monodromy matrices transform as $\tilde{M}_a \to M_a = B \cdot \tilde{M}_a \cdot B^{-1}$, where $a \in \{0, 1, \infty\}$. This leads to the following monodromy matrices in the integral basis

$$M_\infty = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ 4 & 0 & 1 & 1 \end{pmatrix}, \; M_0 = \begin{pmatrix} -3 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ 4 & 0 & 1 & 1 \end{pmatrix}, \; M_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.123}$$

and the basis change is fixed to

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & -\frac{7}{2} & 0 & -\frac{1}{2} \end{pmatrix} . \tag{2.124}$$

Using the mirror map $t(z)$ given by

$$2\pi i\, t(z) = \frac{\bar{\omega}_1(z)}{\bar{\omega}_0(z)} = \log(z) + \mathcal{O}(z) , \tag{2.125}$$

this translates into the following integral period vector in flat coordinates

$$\frac{\Pi(t)}{\bar{\omega}_0(t)} = \begin{pmatrix} 1 \\ t \\ F_t \\ F_0 \end{pmatrix} = \begin{pmatrix} 1 \\ t \\ \frac{3t^2}{2} - \frac{3t}{2} - \frac{7}{4} \\ -\frac{t^3}{2} - \frac{7t}{4} + \frac{51 i\zeta(3)}{2\pi^3} \end{pmatrix} + \mathcal{O}(te^{2\pi i t}, e^{2\pi i t}) . \tag{2.126}$$

on which the monodromies act as (2.123). This period vector fulfills the integrability condition $F_0 = 2F - tF_t$. Using the mirror map one can find the following limit for $\psi \to 0$

$$B + iJ = t(z(\psi)) \overset{\psi\to 0}{\longrightarrow} -\frac{1}{2} - i\frac{\sqrt{3}}{6} , \tag{2.127}$$

which gives the expected behaviour of a half-integral B-field at the orbifold point noted in [37, 127]. Now one can compute the instanton corrected three-point correlators, as in [121, 120]

$$C_{ttt} = \partial_t^2 F_t = 3 + \sum_{n\geq 1} N(n)\, n^3 \frac{q^n}{1-q^n}, \quad q = e^{2\pi i t} , \tag{2.128}$$

[38] The symplectic form is given by $\eta = \begin{pmatrix} 0 & \Xi_{2\times 2} \\ -\Xi_{2\times 2} & 0 \end{pmatrix}$, where $\Xi_{2\times 2}$ was defined in (2.57), and a symplectic matrix A fulfills $A^T \cdot \eta \cdot A = \eta$. The matrices $\tilde{M}_a$, $a \in \{0, 1, \infty\}$ are not symplectic.

where the first few $N(n)$ are given by

$$\begin{array}{c|ccccc} n & 1 & 2 & 3 & 4 & 5 \\ \hline N(n) & 7884 & 6028452 & 11900417220 & 34600752005688 & 124595034333130080 \end{array} \tag{2.129}$$

The above result can also be obtained by bringing the connection matrix (2.110) into upper triangular form. For this one chooses a new basis

$$\mathbf{\Omega}'_t = \left(\frac{\Omega(z(t)}{S_0(z(t)}, \partial_t \frac{\Omega(z(t)}{S_0(z(t)}, C(z(t))^{-1}\partial_t^2 \frac{\Omega(z(t)}{S_0(z(t)}, \partial_t \left(C(z(t))^{-1}\partial_t^2 \frac{\Omega(z(t)}{S_0(z(t)}\right)\right)^T \tag{2.130}$$

under which the connection matrix transforms as

$$G'(t) = \partial_t A(t) \cdot A(t) + A(t) \cdot G(z(t)) \cdot A(t)^{-1} , \tag{2.131}$$

where

$$\mathbf{\Omega}'_t(t) = A(t) \cdot \mathbf{\Omega}_z(z(t)) . \tag{2.132}$$

The condition that $G'(t)$ should be upper triangular leads to differential equations which can be solved for $z(t)$, $C(t)$ and $S_0(t)$. This leads to

$$G'(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & C(q) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{2.133}$$

where

$$S_0(q) = 1 + 360q + 1247400q^2 + 6861254400q^3 + \ldots , \tag{2.134}$$

$$z(q) = q - 2772q^2 + 1980126q^3 - 4010268048q^4 + \ldots , \tag{2.135}$$

$$C(q) = c\left(3 + 7884q + 48235500q^2 + 321311272824q^3 \ldots\right) . \tag{2.136}$$

By comparison to (2.128) the coefficient $c \in \mathbb{R}$ is fixed to $c = 1$.

Near the conifold

This model exhibits a conifold singularity at $z = 6^{-4}3^{-2}$. The form of the conifold monodromy M_1 in (2.123) hints at the fact that there is a vanishing cycle associated to this point. To make this more explicit we study the Picard-Fuchs system near the conifold point. In the coordinate $\Delta = 1 - 6^4 3^2 z$ a complete system of solutions near $\Delta = 0$ can be constructed as

$$\begin{aligned} \omega_0^{\text{con}} &= 1 + \frac{5\Delta^3}{1944} + \frac{3485\Delta^4}{839808} + \frac{614113\Delta^5}{120932352} + \mathcal{O}\left(\Delta^6\right), \\ \omega_1^{\text{con}} &= \Delta - \frac{67\Delta^3}{216} - \frac{13513\Delta^4}{31104} - \frac{10953457\Delta^5}{22394880} + \mathcal{O}\left(\Delta^6\right), \\ \omega_2^{\text{con}} &= \Delta^2 + \frac{265\Delta^3}{216} + \frac{118753\Delta^4}{93312} + \frac{84727369\Delta^5}{67184640} + \mathcal{O}\left(\Delta^6\right), \\ \omega_3^{\text{con}} &= \frac{(\omega_1^{\text{con}} + \frac{49}{72}\omega_2^{\text{con}})}{2\pi i}\log\Delta - \frac{6677\Delta^3}{93312} - \frac{9551897\Delta^4}{80621568} - \frac{2673939853\Delta^5}{18139852800} + \mathcal{O}\left(\Delta^6\right) . \end{aligned} \tag{2.137}$$

This set of solutions exhibits around the point $\Delta = 0$ a monodromy of the form

$$\omega_i^{\rm con}(e^{2\pi i}\Delta) = (\hat{M}_1)_{ij}\, \omega_j^{\rm con}(\Delta), \qquad \hat{M}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & \frac{49}{72} & 1 \end{pmatrix} . \tag{2.138}$$

By a Lefschetz argument [128] the period $\omega_3^{\rm con}$ gives the integral of the holomorphic $(3,0)$-form over a vanishing S^3 at $\Delta = 0$. The basis of solutions near the conifold (2.137) should be related to the large volume basis $\Pi(t)$ (2.126) by analytic continuation. The form of the conifold monodromies (2.123),(2.138) restricts the form of the associated analytic continuation matrix to

$$\Pi_i = \hat{m}_{ij} \cdot \omega_j^{\rm con}, \qquad \hat{m} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \frac{72}{49} b_{43} \\ b_{21} & b_{22} & b_{23} & 0 \\ b_{31} & b_{32} & b_{33} & 0 \\ 0 & \frac{72}{49} b_{43} & b_{43} & 0 \end{pmatrix} . \tag{2.139}$$

Furthermore the coefficients $b_{ij} \neq 0$ can be fixed numerically and are given by

$$\hat{m} \approx \begin{pmatrix} 1.06 & -0.0212 & -0.00918 & 0.276i \\ 1.50i & 0.152i & 0.0781i & 0 \\ -5.09 - 2.25i & -0.710 - 0.228i & -0.393 - 0.117i & 0 \\ 0 & 0.276i & 0.188i & 0 \end{pmatrix} . \tag{2.140}$$

This shows that the period F_0 vanishes as $\Delta \to 0$. This is in accord with the expectations from the SYZ-picture of mirror symmetry [36].

2.6.2. The Calabi-Yau four-fold $\mathbf{X}_6^{(1,1,1,1,1,1)}$

The second example is the Calabi-Yau four-fold $\mathbf{X}_6^{(1,1,1,1,1,1)}$, which is given by a generic degree six hypersurface in $\mathbb{P}^5$. The toric ambient space for the A-model geometry is specified by the vertices in Table 2.3. Triangulating Δ leads to two phases:

1. Orbifold phase: In this phase the cones of the fan are given by

$$\{\langle \tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_3, \tilde{\nu}_4, \tilde{\nu}_5, \tilde{\nu}_6 \rangle\} . \tag{2.141}$$

   One obtains a Landau-Ginzburg orbifold model with orbifold group $\mathbb{Z}_6$ and superpotential given by the generic hypersurface constraint.

2. Large volume phase: The cones of the fan in this phase are given by

$$\begin{aligned} &\{\langle \tilde{\nu}_0, \tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_3, \tilde{\nu}_4, \tilde{\nu}_5 \rangle, \langle \tilde{\nu}_0, \tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_3, \tilde{\nu}_4, \tilde{\nu}_6 \rangle, \langle \tilde{\nu}_0, \tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_3, \tilde{\nu}_5, \tilde{\nu}_6 \rangle, \\ &\langle \tilde{\nu}_0, \tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_4, \tilde{\nu}_5, \tilde{\nu}_6 \rangle, \langle \tilde{\nu}_0, \tilde{\nu}_1, \tilde{\nu}_3, \tilde{\nu}_4, \tilde{\nu}_5, \tilde{\nu}_6 \rangle, \langle \tilde{\nu}_0, \tilde{\nu}_2, \tilde{\nu}_3, \tilde{\nu}_4, \tilde{\nu}_5, \tilde{\nu}_6 \rangle\} . \end{aligned} \tag{2.142}$$

Δ												Δ^*									
	$\tilde{\nu}_0$	=	(	0	0	0	0	0	)				ν_0	=	(	0	0	0	0	0	)
	$\tilde{\nu}_1$	=	(	1	0	0	0	0	)				ν_1	=	(	5	−1	−1	−1	−1	)
	$\tilde{\nu}_2$	=	(	0	1	0	0	0	)				ν_2	=	(	−1	5	−1	−1	−1	)
	$\tilde{\nu}_3$	=	(	0	0	1	0	0	)	,			ν_3	=	(	−1	−1	5	−1	−1	)
	$\tilde{\nu}_4$	=	(	0	0	0	1	0	)				ν_4	=	(	−1	−1	−1	5	−1	)
	$\tilde{\nu}_5$	=	(	−1	−1	−1	−1	−1	)				ν_5	=	(	−1	−1	−1	−1	5	)
	$\tilde{\nu}_6$	=	(	0	0	0	0	1	)				ν_6	=	(	−1	−1	−1	−1	−1	)

Table 2.3.: Vertices Δ of the toric ambient geometry of the hypersurface $\mathbf{X}_6^{(1,1,1,1,1,1)}$.

Computing the Mori-cone leads to the following charge vector

$$\begin{array}{c c|c c c c c c} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline l & -6 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \tag{2.143}$$

and the topological data is given by

$$c(Z) = 1 + 15K^2 - 70K^3 + 435K^4, \quad K^4 = 6, \quad \chi(X) = 2610 \,, \tag{2.144}$$

where the Kähler form is $J = tK$, with K the hyperplane class. The non-vanishing Hodge numbers are computed to

$$h^{0,0} = 1 = h^{4,0}, \quad h^{1,1} = 1, \quad h^{3,1} = 426, \quad h^{2,2} = 1752 \,. \tag{2.145}$$

We proceed by discussing the mirror B-model geometry.

The mirror model

The mirror model can again be constructed by the Batyrev-Borisov method [109, 126]. The hypersurface constraint for the mirror model is given in terms of homogeneous coordinates on $\mathbb{P}^5/\mathbb{Z}_6^4$ by

$$P(Z^*) = x_1^6 + x_2^6 + x_3^6 + x_4^6 + x_5^6 + x_6^6 - 6\psi\, x_1x_2x_3x_4x_5x_6 \tag{2.146}$$

and an appropriate Greene-Plesser orbifold $\mathbb{Z}_6^4$ action, which acts by $x_i \to \lambda_k^{g_{k,i}} x_i$ for $\lambda_k^6 = 1$ and weights

$$\begin{aligned} &\mathbb{Z}_6 : \; g_1 = (1,-1,0,0,0,0), \quad \mathbb{Z}_6 : \; g_2 = (1,0,-1,0,0,0), \\ &\mathbb{Z}_6 : \; g_3 = (1,0,0,-1,0,0), \quad \mathbb{Z}_6 : \; g_4 = (1,0,0,0,-1,0) \,. \end{aligned} \tag{2.147}$$

The Picard-Fuchs operators can be computed from the general formula (2.83) and are given by

$$\mathcal{L}(l) = \theta^5 - 6z(6\theta+1)(6\theta+2)(6\theta+3)(6\theta+4)(6\theta+5) \,, \tag{2.148}$$

where $z = (6\psi)^{-6}$ and we factorized the degree six GKZ-operator to a degree five operator.[39] Using the explicit form of the Picard-Fuchs operator one can recover the Gauss-Manin connection expressed in the coordinate z. Choosing a basis

$$\mathbf{\Omega}_z = (\Omega(z), \partial_z\Omega(z), \partial_z^2\Omega(z), \partial_z^3\Omega(z), \partial_z^4\Omega(z))^T \tag{2.149}$$

the Gauss-Manin connection is given by

$$\nabla_z \mathbf{\Omega} = (1\!\!1_{5\times 5}\partial_z - G(z))\,\mathbf{\Omega}\ , \tag{2.150}$$

where

$$G(z) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\frac{720}{z^4(46656z-1)} & \frac{1-331920z}{z^4(46656z-1)} & \frac{15z-1895400z^2}{z^4(46656z-1)} & \frac{25z^2-1976400z^3}{z^4(46656z-1)} & \frac{10z^3-583200z^4}{z^4(46656z-1)} \end{pmatrix} \tag{2.151}$$

The solutions to the Picard-Fuchs equations (2.148) can be found by using the generating function (2.84), which specializes in this case to

$$B_{\{l\}}(z;\rho) = \sum_{n\geq 0} \frac{\Gamma(1+6n+6\rho)\, z^{n+\rho}}{\Gamma(1+n+\rho)^6}\ , \tag{2.152}$$

and leads to a basis of solutions $\bar{\omega}$ near the point $z = 0$ given by $\bar{\omega}_k = \frac{1}{(2\pi i)^k}\partial_\rho^k B_{\{l\}}(z;\rho)|_{\rho=0}$ for $k = 0, \dots, 4$

$$\bar{\omega} = \begin{pmatrix} 1 \\ \frac{\log(z)}{2\pi i} \\ \frac{\log^2(z)}{(2\pi i)^2} - \frac{5}{4} \\ \frac{\log^3(z)}{(2\pi i)^3} + \frac{15i\log(z)}{8\pi} - \frac{105i\zeta(3)}{2\pi^3} \\ \frac{\log^4(z)}{(2\pi i)^4} + \frac{15\log^2(z)}{8\pi^2} - \frac{105\zeta(3)\log(z)}{\pi^4} + \frac{161}{16} \end{pmatrix} + \mathcal{O}(z)\ . \tag{2.153}$$

This basis has the following monodromy around the point $z = 0$ of maximal unipotent monodromy

$$\bar{\omega}(e^{-2\pi i}z) = \tilde{M}_\infty \cdot \bar{\omega}(z), \qquad \tilde{M}_\infty = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}\ . \tag{2.154}$$

[39] This form is expected from the Stanley-Reisner ideal of the mirror manifold Z and the form of the classical mirror map [110].

To fix the particular linear combinations which correspond to the integral periods we compare the large volume basis with a basis near the orbifold point $\psi = 0$. We again represent the solutions of (2.148) near the orbifold point in terms of hypergeometric functions as

$$\omega_j^{\text{orb}}(\psi) = \psi^j \, {}_5F_4\left(\frac{j}{6},\frac{j}{6},\frac{j}{6},\frac{j}{6},\frac{j}{6};\overbrace{\frac{j+1}{6},\frac{j+2}{6},\frac{j+3}{6},\frac{j+4}{6},\frac{j+5}{6}};\psi^6\right) , \quad j = 1,\ldots,5 , \tag{2.155}$$

where the overbrace means, that the coefficient which is equal to one is left out. The functions ω_j^{orb} can be represented by a Barnes type integral as

$$\omega_j^{\text{orb}}(\psi) = -a_j\psi^j \oint_{C_+} \frac{\Gamma(s+\frac{j}{6})^5\Gamma(-s)(-1)^s\Gamma(s+1)\,(4\psi^6)^s}{\Gamma(s+\frac{j+1}{6})\Gamma(s+\frac{j+2}{6})\Gamma(s+\frac{j+3}{6})\Gamma(s+\frac{j+4}{6})\Gamma(s+\frac{j+5}{6})}\,ds , \tag{2.156}$$

where C_+ is a closed contour which picks the poles at $s = n$ for $n \geq 0$ and makes the integrand to vanish exponentially fast at infinity. The coefficients a_j are given by

$$a_j = \frac{\Gamma(\frac{j+1}{6})\Gamma(\frac{j+2}{6})\Gamma(\frac{j+3}{6})\Gamma(\frac{j+4}{6})\Gamma(\frac{j+5}{6})}{\Gamma(\frac{j}{6})^5} . \tag{2.157}$$

In order to analytically continue we close the contour to the left along a path C_- to obtain

$$-a_j\psi^j \oint_{C_-} \frac{\Gamma(s+\frac{j}{6})^5\Gamma(-s)(-1)^s\Gamma(s+1)\,(4\psi^6)^s}{\Gamma(s+\frac{j+1}{6})\Gamma(s+\frac{j+2}{6})\Gamma(s+\frac{j+3}{6})\Gamma(s+\frac{j+4}{6})\Gamma(s+\frac{j+5}{6})}\,ds , \tag{2.158}$$

and pick the poles at $s = -n - \frac{j}{6}$ for $n \geq 0$ and $j = 1,\ldots,5$. The result can be matched to the large volume basis $\bar{\omega}$. This leads to an analytic continuation matrix m of the form[40]

$$\bar{\omega}(z) = m \cdot \omega^{\text{orb}}(\psi(z)) . \tag{2.159}$$

The action of the $\mathbb{Z}_6$-orbifold action with $\alpha^6 = 1$ on the solutions ω^{orb} is given by

$$\omega^{\text{orb}}(\alpha\psi) = \tilde{M}_0^{\text{orb}} \cdot \omega^{\text{orb}}(\psi), \qquad \tilde{M}_0^{\text{orb}} = \text{diag}(\alpha,\alpha^2,\alpha^3,\alpha^4,\alpha^5) . \tag{2.160}$$

Therefore the orbifold monodromy acts on the large volume solutions as

$$\bar{\omega}(\alpha\psi) = \tilde{M}_0 \cdot \bar{\omega}(\psi), \qquad \tilde{M}_0 = m \cdot \tilde{M}_0^{\text{orb}} \cdot m^{-1} . \tag{2.161}$$

The basis of solutions $\bar{\omega}$ should be related to the integral periods by a linear transformation

$$\Pi(z) = B \cdot \bar{\omega}(z) . \tag{2.162}$$

This change of basis can be fixed, up to a transformation which leaves the intersection form invariant, by demanding integrality of the monodromy matrices M_0, M_∞ and $M_1 = M_\infty \cdot M_0^{-1}$

[40] The matrix m is a very complicated constant matrix, therefore we refrain from displaying its explicit form.

with respect to the new basis.[41] Under the coordinate change the monodromy matrices transform as $\tilde{M}_a \to M_a = B \cdot \tilde{M}_a \cdot B^{-1}$, where $a \in \{0, 1, \infty\}$. This leads to the following monodromy matrices in the integral basis (2.166)[42]

$$M_\infty = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 6 & 6 & 1 & 0 & 0 \\ 7 & 3 & 1 & 1 & 0 \\ -4 & -4 & 0 & -1 & 1 \end{pmatrix}, \; M_0 = \begin{pmatrix} -4 & -4 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 6 & 6 & 1 & 0 & 0 \\ 7 & 3 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \; M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.163}$$

and the basis change is fixed to

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 & 0 \\ 0 & -\frac{9}{2} & \frac{3}{2} & -1 & 0 \\ -1 & 0 & -\frac{15}{4} & 0 & -\frac{1}{4} \end{pmatrix} . \tag{2.164}$$

Using the mirror map $t(z)$ given by

$$-2\pi i \, t(z) = \frac{\bar{\omega}_1(z)}{\bar{\omega}_0(z)} = \log(z) + \mathcal{O}(z) , \tag{2.165}$$

this translates into the following integral period vector in flat coordinates

$$\frac{\Pi(t)}{\bar{\omega}_0(t)} = \begin{pmatrix} 1 \\ t \\ F^{(2)} \\ F_t \\ F_0 \end{pmatrix} = \begin{pmatrix} 1 \\ t \\ 3t^2 + 3t - \frac{15}{4} \\ t^3 + \frac{3t^2}{2} + \frac{3t}{4} + \frac{105 i\zeta(3)}{2\pi^3} - \frac{15}{8} \\ -\frac{t^4}{4} - \frac{15t^2}{8} - \frac{105 it\zeta(3)}{2\pi^3} + \frac{3375\zeta(4)}{32\pi^4} \end{pmatrix} + \mathcal{O}\left(te^{2\pi i t}, e^{2\pi i t}\right) , \tag{2.166}$$

on which the monodromies act as (2.163). The linear combinations of integral periods can also be fixed universally from an open string picture [129]. For the case at hand this leads for $F_0(t) = \sum_{n=0}^{4} a_n t^n$ to (see also [123] and Appendix B)[43]

$$a_4 = -\frac{1}{4!}K^3, \quad a_2 = -\frac{c_2 \cdot K^2}{48}, \quad a_1 = \frac{c_3 \cdot K}{(2\pi i)^3}\zeta(3), \quad a_0 = \frac{5c_2^2}{2(2\pi i)^4}\zeta(4) . \tag{2.167}$$

[41] In addition the monodromy matrices should leave invariant the form $\eta = \begin{pmatrix} 0 & 0 & \Xi_{2\times 2} \\ 0 & -\frac{1}{6} & 0 \\ \Xi_{2\times 2} & 0 & 0 \end{pmatrix}$, where $\Xi_{2\times 2}$ was defined in (2.57), e.g. fulfill $M^T \cdot \eta \cdot M = \eta$. The matrices $\tilde{M}_a$, $a \in \{0, 1, \infty\}$ do not fulfill this condition.

[42] The orders of the monodromies are consistent with a direct analysis of the residues of the Gauss-Manin connection as in [111, 116].

[43] Note that the term proportional to $\zeta(3)$ is not fixed by this argument. Also in this case the coefficient a_3 is identically zero.

The above results can alternatively also be obtained from the techniques presented in [129], as is shown in Appendix C.

Using its explicit form one can in addition find the following limit for the mirror map near the orbifold point $\psi \to 0$

$$B + iJ = t(z(\psi)) \overset{\psi\to 0}{\longrightarrow} -\frac{1}{2} - i\frac{\sqrt{3}}{3} \,, \tag{2.168}$$

which shows the same behaviour as in the three-fold case [37]. Furthermore one can compute the instanton corrected three-point correlator as [75][44]

$$C_{ttt} = \partial_t^2 F^{(2)} = 6 + \sum_{n\geq 1} N(n)\, n^2 \frac{q^n}{1-q^n}, \quad q = e^{2\pi i t} \,, \tag{2.169}$$

where the first few $N(n)$ are given by

n	1	2	3	4	5
$N(n)$	60480	440884080	6255156277440	117715791990353760	2591176156368821985600

(2.170)

This reproduces the results of [75, 122]. The above can also be obtained by bringing the connection matrix (2.150) into upper triangular form. For this one chooses a new basis

$$\boldsymbol{\Omega}'_t = \begin{pmatrix} \frac{\Omega(t)}{S_0(t)} \\ \partial_t \frac{\Omega(t)}{S_0(t)} \\ (C^{(1)}(t))^{-1}\partial_t^2 \frac{\Omega(t)}{S_0(t)} \\ (C^{(2)}(t))^{-1}\partial_t \left((C^{(1)}(t))^{-1}\partial_t^2 \frac{\Omega(t)}{S_0(t)} \right) \\ \partial_t \left((C^{(2)}(t))^{-1}\partial_t \left((C^{(1)}(t))^{-1}\partial_t^2 \frac{\Omega(t)}{S_0(t)} \right) \right) \end{pmatrix} \tag{2.171}$$

under which the connection matrix transforms as

$$G'(t) = \partial_t A(t) \cdot A(t) + A(t) \cdot G(z(t)) \cdot A(t)^{-1} \,, \tag{2.172}$$

where

$$\boldsymbol{\Omega}'_t(t) = A(t) \cdot \boldsymbol{\Omega}_z(z(t)) \,. \tag{2.173}$$

The condition that $G'(t)$ should be upper triangular leads to differential equations which can be solved for the unknown functions $z(t)$, $C^{(1)}(t)$, $C^{(2)}(t)$ and $S_0(t)$. This leads to

$$G'(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & C^{(1)}(q) & 0 & 0 \\ 0 & 0 & 0 & C^{(2)}(q) & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \,, \tag{2.174}$$

[44] This correlator is given by $C_{ttt} = \langle \mathcal{O}^{(1)} \cdot \mathcal{O}^{(1)} \cdot \mathcal{O}^{(2)} \rangle$. In this case there is a single operator of charge two and therefore this is the only three-point coupling. See [75, 99] for a discussion of the general case. Note also the modified multi-cover prescription as opposed to the three-fold situation [75, 122].

where

$$
\begin{aligned}
S_0(q) &= 1+720q+7484400q^2+137225088000q^3+\dots\,, && (2.175)\\
z(q) &= q-6264q^2-8627796q^3-237290958144q^4+\dots\,, && (2.176)\\
C^{(1)}(q) &= c_1\left(6+60480q+1763596800q^2+\dots\right)\,, && (2.177)\\
C^{(2)}(q) &= c_2\left(6+60480q+1763596800q^2+\dots\right)\,. && (2.178)
\end{aligned}
$$

By comparison to (2.169) the coefficients $c_1, c_2 \in \mathbb{R}$ are fixed to $c_1 = -\frac{1}{6}$ and $c_2 = 1$.

Near the point $z = 6^{-6}$

The moduli space of this model has a principal discriminant boundary point at $z = 6^{-6}$. The form of the monodromy matrix M_1 in (2.163) shows that the monodromy has order two and exchanges the periods 1 and F_0. To gain more insight we study the Picard-Fuchs system near this point. In the coordinate $\Delta = 1 - 6^6 z$ a complete system of solutions near $\Delta = 0$ can be constructed as

$$
\begin{aligned}
\omega_0^{\text{con}} &= 1-\frac{\Delta^4}{3888}-\frac{269\Delta^5}{489888}-\frac{194101\Delta^6}{238085568}+\mathcal{O}\left(\Delta^7\right),\\
\omega_1^{\text{con}} &= \Delta+\frac{1981\Delta^4}{19440}+\frac{70187\Delta^5}{349920}+\frac{47942173\Delta^6}{170061120}+\mathcal{O}\left(\Delta^7\right),\\
\omega_2^{\text{con}} &= \Delta^2-\frac{41\Delta^4}{48}-\frac{57503\Delta^5}{38880}-\frac{36510907\Delta^6}{18895680}+\mathcal{O}\left(\Delta^7\right),\\
\omega_3^{\text{con}} &= \Delta^3+\frac{125\Delta^4}{72}+\frac{1451\Delta^5}{648}+\frac{16287101\Delta^6}{6298560}+\mathcal{O}\left(\Delta^7\right),\\
\omega_4^{\text{con}} &= \Delta^{3/2}+\frac{17\Delta^{5/2}}{18}+\frac{551\Delta^{7/2}}{648}+\frac{1210999\Delta^{9/2}}{1574640}+\mathcal{O}\left(\Delta^{11/2}\right)\,.
\end{aligned}
\qquad (2.179)
$$

This set of solutions exhibits a monodromy around the point $\Delta = 0$ of the form

$$
\omega_i^{\text{con}}(e^{2\pi i}\Delta) = (\hat{M}_1)_{ij}\,\omega_j^{\text{con}}(\Delta), \qquad \hat{M}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}\,. \qquad (2.180)
$$

By a Lefschetz argument [128] this form of the monodromy matrix would imply that there is either a linear combination of cycles vanishing or there is no vanishing cycle at all at $\Delta = 0$ and therefore no geometric singularity. The basis of solutions near the conifold (2.179) should again be related to the large volume basis $\Pi(t)$ (2.166) by analytic continuation. The form of the conifold monodromies (2.180),(2.163) restricts the form of the associated analytic continuation matrix to

$$\Pi_i = \hat{m}_{ij} \cdot \omega_j^{\text{con}}, \qquad \hat{m} = \begin{pmatrix} b_{51} & b_{52} & b_{53} & b_{54} & \frac{1}{\sqrt{3}\pi^2} \\ b_{21} & b_{22} & b_{23} & b_{24} & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} & 0 \\ b_{51} & b_{52} & b_{53} & b_{54} & -\frac{1}{\sqrt{3}\pi^2} \end{pmatrix}. \tag{2.181}$$

The coefficients $b_{ij} \neq 0$ can be fixed numerically and are given by

$$\hat{m} \approx \begin{pmatrix} 1.02 & -0.0516 & -0.0577 & -0.0524 & 0.0584 \\ -1.72i & -0.155i & -0.0787i & -0.0527i & 0.+i0. \\ -12.54-5.15i & -1.63-0.47i & -0.891-0.236i & -0.619-0.158i & 0.+i0. \\ -6.27+5.74i & -0.82+1.30i & -0.446+0.772i & -0.310+0.561i & 0.+i0. \\ 1.02 & -0.0516 & -0.0576 & -0.0523 & -0.0586 \end{pmatrix}. \tag{2.182}$$

This indicates that there are no vanishing cycles associated to the locus $\Delta = 0$ in this case. This is in contrast to the generic principal discriminant in three-fold case. It would be interesting to understand this type of boundary point better.

Part II

Off-shell mirror symmetry and type II/F-theory superpotentials

3

Chapter 3

$\mathcal{N} = 1$ mirror symmetry and type II/F-theory superpotentials

Contents

In this chapter, which is based on the original publications [1, 2, 3], we turn to the study of open string mirror symmetry on compact Calabi-Yau manifolds. By Hodge-theoretic methods, first described in [46, 47] for the case of non-compact Calabi-Yau manifolds, we are able to compute the fully disc instanton corrected superpotential induced by certain parameter dependent families of D-branes in type II and F-theory compactifications on compact Calabi-Yau manifolds. For this we start with a geometric Gauss-Manin connection for B-type branes and study their integrability and flatness conditions in sections 3.3 and 3.5. Relations to CFT correlators are pointed out in section 3.6. The B-model geometry defines an interesting ring structure of operators. For the mirror A-model this indicates the existence of an open-string extension of the so-called A-model connection, whereas the discovered ring structure should be part of the open-string A-model quantum cohomology.

In section 3.4 we give a derivation of the Picard-Fuchs operators, which annihilate the relative periods associated to a particular off-shell deformation problem. For the case of toric branes we describe how to do explicit and efficient computations by constructing a gauged linear σ-model for an open/closed dual Calabi-Yau four-fold in section 3.7. These techniques allow one to deduce a canonical hypergeometric system, whose solutions determine the open/closed mirror maps and the partition functions for spheres and discs. In section 3.8 we furthermore discuss the lift of the brane compactifications to F-theory on compact Calabi-

Yau four-folds and the effective couplings in the effective supergravity action as determined by the $\mathcal{N} = 1$ special geometry of the open/closed deformation space. We start by reviewing certain aspects of topological D-branes.

3.1. Topological D-branes

Up to now the discussion was mainly concerned with worldsheets without boundaries. There are various reasons why the inclusion of boundaries is very important. From a space-time point of view for example one would like to break part of the supersymmetries the background preserves. This can be achieved by adding supersymmetric D-branes to the compactification, which can break the $\mathcal{N} = 2$ space-time supersymmtry to $\mathcal{N} = 1$ in $d = 4$. From the space-time perspective a D-brane can be wrapped around a non-trivial cycle in space-time and has a physical theory living on top of it, describing its excitations. For the case of supersymmetric D-branes these theories are given by supersymmetric Yang-Mills theories. As is well-known the low-energy excitations of a D-brane are described by the excitations of open strings ending on the D-brane [8]. Thus in order to describe these kind of compactifications from a worldsheet perspective one has to include boundaries.

By analyzing the conditions for unbroken supersymmetry/topological symmetry either from the space-time [130] or the string worldsheet perspective [131] it turns out that there are two different types of topological D-branes.[1]

3.1.1. A-branes

Physical A-branes wrap special Lagrangian (sL) cycles of the compactification manifold [130, 131]. For a Calabi-Yau d-fold Lagrangian submanifolds are real d-dimensional cycles L such that the pull-back of the Kähler class J vanishes on L. This is the main condition[2] on a topological A-brane, which corresponds to a boundary condition on the string worldsheet which is compatible with the topological twist of the A-model. Therefore Lagrangian submanifolds are the topological branes of the A-model [51]. To qualify furthermore as a special Lagrangian and thus a physical A-brane the restriction of the holomorphic $(d, 0)$-form Ω should in addition be proportional to the induced volume form on L:

$$J|_L = 0, \qquad \Omega|_L = e^{i\theta}\,\mathrm{vol}, \qquad F(A) = 0\ , \tag{3.1}$$

where on top of the Lagrangian sits a flat vector bundle E with gauge potential A and $F(A)$ denotes the curvature form of the connection A. The physical open string states of the theory can be represented by elements of the following sheaf-cohomology groups

$$\{\text{open string states}\} \simeq \bigoplus_{n=0}^{3} H^n(L, \mathrm{End}(E))\ , \tag{3.2}$$

[1] A good introduction to topological A- and B-type branes is given in [88].
[2] In addition a topological A-brane should have a trivial Maslov class, see e.g. [88] for an account.

where n denotes the ghost number.[3] Strictly speaking the above geometrical notions are only valid in large volume regimes of a compactification. More general A-branes are more appropriately modeled by objects in a category, called the Fukaya category, where the morphisms are given by the open string states, which stretch between the branes.[4]

The deformations of a sL-cycle are given by Hamiltonian deformations and Wilson line moduli of the flat bundle on L [132, 133]. These marginal deformations are represented by elements in $H^1(L,\mathbb{Z})$ and thus the moduli space of a sL-cycle has dimension $\dim_{\mathbb{C}} \mathcal{M}_L = b_1(L)$. The size of a deformation can naturally be measured by integrating the Kähler form over a disc D with $\partial D = S^1 \subset L$ and pairing it up with the Wilson line of the gauge field A around the S^1 into a complex parameter, similar as in the closed string case

$$\hat{t} = \int_{\partial D} A + i \int_D J \,. \tag{3.3}$$

These open string parameters get in general corrections from closed string worldsheet instantons. The instanton corrected open string parameters together with their closed string counterparts are the natural flat coordinates on the open/closed deformation space of the A-model. We are going to show in certain examples in chapter 4, how the closed string instanton corrections to the open string parameters can in some cases be computed explicitly by an application of off-shell mirror symmetry.

From a space-time perspective there exists a functional $f_{\mathbb{R}}$ on the space of submanifolds, whose critical points are given by the Lagrangian submanifolds. This functional is a variant of the real Chern-Simons functional and for $d = 3$ given by [35]

$$f_{\mathbb{R}}(L) = \int_{L_0}^{L} (F_{\mathbb{A}} + J)^2 \tag{3.4}$$

where the integration is over a four-chain connecting L to a homologous three-cycle L_0. The connection $\mathbb{A}$ on the four-chain is such that $\mathbb{A}|_{L_0} = A_0$ and $\mathbb{A}|_L = A$ for some fixed A_0 on L_0.

Ooguri-Vafa invariants and quantum geometry

As in the closed string case the open string A-model disc correlators receive worldsheet instanton corrections [51, 134, 135], which in this case come from worldsheets whose topology is that of a disc and which end on the Lagrangian A-brane L inside the Calabi-Yau Z under consideration. The topological path integral for the generating function of disc correlators splits into instanton contributions from different topological sectors

$$F_A^{(0,1)}(t,\hat{t}) = p_{\rm cl}^{(2)}(t,\hat{t}) + \sum_{n_i,\hat{n}_j \geq 0} N_{\{n_i,\hat{n}_j\}} \,\mathrm{Li}_2\Big(\prod_{i=1}^{h^{1,1}(Z)} e^{2\pi i\, n_i t^i} \prod_{j=1}^{r} e^{2\pi i\, \hat{n}_j \hat{t}^j} \Big) \,, \tag{3.5}$$

[3] The ghost number is subtle to define in the open string A-model as it is subjected to an anomaly. The cancellation condition is that the Maslov class of the Lagrangian should vanish, see e.g. [88] for a discussion.

[4] See [88] for a more detailed account.

where $p_{\rm cl}^{(2)}(t,\hat{t})$ denotes a possible classical polynomial contribution quadratic in the moduli and r is the total number of open string deformation parameters. The integers $n_i, \hat{n}_j$ denote the degrees of images of the disc worldsheet in $H_2(Z,L)$ with respect to an arbitrary basis $\{C_i, D_j\}$ such that $\partial D_j \subset L$. The parameters $t^i, \hat{t}^j$ measure the (complexified) classical volume of the fundamental spheres and discs respectively as in (3.3)

$$t^i = \int_{C_i} B + iJ, \qquad \hat{t}^j = \int_{\partial D_j} A + i\int_{D_j} J\ , \tag{3.6}$$

where B denotes the NSNS two-form B-field, which naturally couples to the closed string worldsheet. Remarkably the coefficients $N_{\{n_i,\hat{n}_j\}}$ are integers and define enumerative invariants of the open/closed target space geometry. They are called Ooguri-Vafa invariants and can be related to the counting of BPS-states in five dimensions coming from M2-branes which end on M5-branes [52, 136]. They are the open string analogs of the Gopakumar-Vafa invariants [28, 29], which count M2-brane states.

The classical polynomial contribution in (3.5) derives from point-like worldsheet instantons and can in principle be given a meaning in terms of classical intersections of appropriate cycles. In the closed string case the analog of (3.5) can be interpreted as a stringy deformation of the classical wedge product of forms. This structure is called quantum cohomology and it shows that strings probe geometry very different from point particles. The geometry derived from the study of string propagation in manifolds is called quantum geometry and leads in the A-model to a foamy picture of space-time [137]. Up to now no proper generalization of the notion of quantum geometry to the open string case has been given, however this notion is expected to incorporate a stringy deformation of a certain classical intersection theory, yet to be defined.

3.1.2. B-branes

B-type branes wrap holomorphic submanifolds Y of the compactification space X, together with a holomorphic bundle on top of them [130, 131]. Therefore they can correspond to physical $D(2p)$-branes $p = 0, 1, 2, 3$. The proper mathematical framework to describe the lower dimensional B-branes and bound-states of branes of various dimensions is given by the bounded derived category of coherent sheaves [45]. By the Calabi-Yau – Landau-Ginzburg correspondence the category of B-branes can alternatively also be characterized by the category of matrix factorizations [138, 139].[5] The physical open string states in the B-model are furthermore described by Ext-groups [141], as is briefly discussed in section 3.6.

On general grounds, as in the closed string case, the B-model amplitudes are expected to be computable by classical integrals on the target space geometry Z^*, which we take as a Calabi-Yau three-fold. It is known [51] that the generating function of disc correlation

[5]See [140] for an overview.

functions for a B-brane which wraps the whole Calabi-Yau manifold Z^* is computed by the holomorphic Chern-Simons functional

$$F_B^{(0,1)} = S_{\rm hCS} = \int_{Z^*} \Omega \wedge {\rm Tr} \left(A \wedge \bar{\partial} A + \frac{2}{3} A \wedge A \wedge A \right) \ , \tag{3.7}$$

where Ω denotes the nowhere-vanishing holomorphic $(3,0)$-form on Z^* and A is the anti-holomorphic part of the gauge connection. For lower dimensional branes one can find similar expressions by dimensional reduction [51, 142, 143, 144]. For a B-brane wrapping a divisor D inside the Calabi-Yau this leads for example to

$$S_{\rm hCS}|_D = \int_D {\rm Tr} \left(i_\phi \Omega \wedge F \right) \ , \tag{3.8}$$

where $F = dA$ is the field-strength on the brane and ϕ is a section of the normal bundle $N_{D|Z^*}$ of $i : D \hookrightarrow Z^*$. The holomorphic Chern-Simons functional can locally also be rewritten as a geometrical chain integral [145]

$$S_{\rm hCS}|_D(z, \hat{z}) = \int_{\gamma(\hat{z})} \Omega(z) \ , \tag{3.9}$$

where γ is a three-chain in Z^* such that $\partial\gamma = C \subset D$ and the curve C is the Poincaré-dual of F in the divisor D. The critical points of (3.8) are given by flux which satisfies $F^{(0,2)} = 0$ and therefore correspond to holomorphic curves. With trivial flux $F = 0$ the infinitesimal geometrical deformations of the brane wrapped on D, which preserve supersymmetry, can be represented by holomorphic deformations of the divisor D and are given by elements of $H^0(D, N_{D|Z^*}) \simeq H^{2,0}(D)$, where the isomorphism follows upon contraction with the unique $(3,0)$-form of the Calabi-Yau manifold.

The representation (3.9) of the holomorphic Chern-Simons action as a geometrical chain integral will be of prime interest for us in the following.

3.1.3. Effective couplings for $\mathcal{N} = 1$ compactifications

The addition of branes and fluxes into a Calabi-Yau compactification changes the low-energy effective action, which is obtained in the remaining four-dimensional non-compact space-time. Such a compactification leads to an effective $\mathcal{N} = 1$ supergravity theory in four dimensions coupled to various gauge fields, whose two-derivative part is specified by the gauge kinetic function $f(\phi)$, the Kähler potential $K(\phi, \bar{\phi})$ and the holomorphic superpotential $\mathcal{W}(\phi)$. Here ϕ denotes collectively the chiral superfields, which describe some of the moduli of the compactification in the $\mathcal{N} = 1$ supergravity theory. The superpotential $\mathcal{W}(\phi)$, in which we will be mainly interested in the following, encodes the obstructions of the deformations described by the fields ϕ and has a 'closed string' part coming from fluxes and an 'open string' part induced by D-branes

$$\mathcal{W}_{\mathcal{N}=1}(z, \hat{z}) = \mathcal{W}_{\rm flux}(z) + \mathcal{W}_{\rm brane}(z, \hat{z}) \ , \tag{3.10}$$

where we denoted by $z/\hat{z}$ collectively the closed/open moduli respectively. Explicit computations of the superpotential couplings turn out to be possible as these quantities are known to be related to topological string amplitudes. More generally the topological string computes the following F-terms in the effective supergravity theory [23, 22, 52]

$$S_{\text{sugra}} \supset \int d^4x \int d^2\theta\, F^{(g,h)}(\phi)(\mathbb{W}^2)^g(\mathbb{W}\cdot v)^{h-1} , \tag{3.11}$$

where g denotes the genus of the worldsheet, h the number of boundaries and $\mathbb{W}$ the graviphoton multiplet. The superpotential is therefore captured by the topological disc amplitude [23, 22, 52, 51]. As discussed above, depending on whether one considers the type IIA or IIB compactification one has to consider either the topological A-model or B-model and the quantum corrections and properties of the topological disc amplitudes are very different in nature. Nonetheless mirror symmetry is expected to identify the D-brane states and open string correlation functions of the two topological models upon compactification on mirror symmetric manifolds [44].

3.2. Off-shell mirror symmetry

In view of the successes for closed strings it is natural to contemplate about an open string version of mirror symmetry [44], which in particular relates the generating functions of the A- and B-model worldsheet correlators on a general worldsheet of genus g with h holes schematically as

$$F_A^{(g,h)}(t,\hat{t}) \quad \longleftrightarrow \quad F_B^{(g,h)}(z,\hat{z}) . \tag{3.12}$$

This seems also natural from the mathematical viewpoint of the homological mirror symmetry conjecture [45]. The Fukaya category of A-branes is notoriously hard to describe and it would be desirable to obtain an explicit description in terms of the more manageable bounded derived category of coherent sheaves or equivalently the category of matrix factorizations on a mirror Calabi-Yau manifold.

The action of mirror symmetry is expected to map a given configuration of B-branes to a configuration of mirror-symmetric A-branes. Generalizing the approach of closed mirror symmetry, in order to compute instanton corrected couplings in the open/closed A-model by mapping it to classically computable couplings in the open/closed B-model, one would therefore be tempted to map deformation families of branes upon each other.

This however leads to an immediate puzzle. The deformation space $\mathcal{M}$ of a generic B-brane configuration is obstructed in the physical theory by a superpotential $\mathcal{W}(\phi)$ in the resulting $\mathcal{N}=1$ space-time theory which is computed by the holomorphic Chern-Simons action (3.7). On the A-model side the expected mirror-symmetric superpotential is in certain circumstances non-perturbatively generated and receives a whole tower of non-trivial disc-instanton corrections.

A topological B-brane with a true flat deformation space however has a vanishing superpotential along the flat directions and therefore these deformations cannot be used to study non-trivial superpotential couplings. In contrast the generic off-shell deformations of the brane, which are obstructed by a superpotential and therefore represent heavy fields in spacetime that should be integrated out in the effective action, span an infinite dimensional space. To write an off-shell superpotential $\mathcal{W}(\phi)$ on a true deformation space $\mathcal{M}$, one therefore needs to specify extra data, in particular a concrete parametrization for the off-shell configurations. In order to define a suitable finite dimensional space $\mathcal{M}$ with obstruction potential $\mathcal{W}(\phi)$ one needs to choose an appropriate set of 'light' fields and integrate out infinitely many others. The result at the critical locus is independent of the parametrization of the off-shell directions, but the off-shell values depend, in a well-defined way, on the parametrization. A priori it is thus not clear how one can consistently single out a sensible deformation problem out of the infinite dimensional space of non-supersymmetric deformations. In this sense one is in need of a non-trivial off-shell version of open string mirror symmetry.

In this work we will be interested in the situation of a B-brane which is wrapped around a two-cycle C inside a Calabi-Yau manifold Z^*. Generically also in this situation, there are many consistent choices for the set of light fields, corresponding to local coordinate patches of the off-shell deformation space of different dimension and range of validity.[6] The way this problem is overcome in the present work was pioneered in [46, 47]. We are going to study a preferred class of parametrizations favored equally well by mathematics and physics, which arises from the following construction motivated by duality to M/F-theory. For this we embed C into a four-cycle D and define $\mathcal{M}$ as the *unobstructed* deformation space of a holomorphic family $\mathcal{D}$ of such 4-cycles. Adding a D-brane charge on $C \subset H_2(D)$ induces a superpotential $\mathcal{W}(C)$ on $\mathcal{M}$ [145, 142]. Physics-wise this can be viewed as perturbing the true moduli space $\mathcal{M}$ of an F-theory compactification with an unobstructed family of D-branes wrapped on the four-cycles D by adding a D-brane charge on a two-cycle C in D [2, 56].

Therefore we actually do not study the generic deformation of a D5-brane wrapped on a two-cycle C, but only those deformations which can be described by the holomorphic deformations of a divisor D, which contains the curve C. Physically this can be understood as a D7-brane, whose unobstructed moduli space gets obstructed upon the addition of a D5-brane charge on the curve $C \in H_2(D)$, where the obstruction is captured by (3.8) [56].

It was observed in [46, 47], that this class of parametrizations is the one preferred by the topological open/closed string theory, as it leads to flat coordinates on the open/closed deformation space $\mathcal{M}$, which are in agreement with the expectations from the chiral ring in the topological string theory. Moreover the Hodge-theoretic definition of the open-string mirror map obtained from the reasoning of [46, 47] yields consistent results for the A-model disc invariants, as is shown in examples in chapter 4, which are in agreement with localization

[6]Each choice of parametrization corresponds to a slightly different formulation as a relative cohomology problem as is discussed below in section 3.3.

computations in the A-model, if available. Mathematically, this class of parametrizations furthermore derives directly from the on-shell meaning of the superpotential as an Abel-Jacobi invariant measuring rational equivalence of the cycles C and C_* in (3.13), as explained below in section 3.3.3.

In the approach of [46, 47] the superpotential $\mathcal{W}(C)$ is derived in view of (3.9) from the period integrals $\underline{\Pi}(z,\hat{z})$ on certain relative cohomology groups defined by the branes, as is discussed in more detail in section 3.3. Schematically this means

$$\underline{\Pi}(z,\hat{z}) = \int_\gamma \Omega = \mathcal{W}(C) - \mathcal{W}(C_*) = \mathcal{W}_{\text{brane}}(z,\hat{z})\,, \quad \partial\gamma = C - C_*\,, \tag{3.13}$$

where $(z,\hat{z})$ are local coordinates on the open/closed deformation space $\mathcal{M}$, further specified in section 3.3. The 3-chain γ has at its two boundaries a 2-cycle $C \subset Z^*$ wrapped by the D-brane and a 2-cylce C_*, which is a reference cycle in the same homology class, $[C_*]=[C]$. The above expression is equal to the tension of a domain wall interpolating between the configurations obtained by wrapping the D-brane either on C or on C_*. The relative periods also capture the 3-form flux superpotential $W_{\text{flux}} = \int_{Z^*} G_3 \wedge \Omega$ of [146, 58], leading to a unified expression of the four-dimensional $\mathcal{N}=1$ superpotential in terms of a general linear combination of all relative[7] period integrals [46, 47]

$$\mathcal{W}_{\mathcal{N}=1}(z,\hat{z}) = \sum \underline{N}_\Sigma\, \underline{\Pi}^\Sigma(z,\hat{z}) = \mathcal{W}_{\text{flux}}(z) + \mathcal{W}_{\text{brane}}(z,\hat{z})\,. \tag{3.14}$$

The coefficients $\underline{N}_\Sigma$ are determined by the topological charges of the brane and flux background. Solving the vacuum condition $\frac{d}{d\hat{z}}\mathcal{W}_{\mathcal{N}=1} = 0$ in the open-string direction gives the *on-shell* (in the open-string direction) superpotential $W_{\text{crit}}(z)$ as a function of the closed string moduli and the topological data $\underline{N}_\Sigma$.

The perturbation idea can also be understood in the framework of a dual M/F-theory compactification on an open/closed dual Calabi-Yau four-fold X_4, which geometrizes the branes to flux [48, 56], as discussed in section 3.8. In this context, $\mathcal{M}$ maps to the *unobstructed* complex structure moduli space $\mathcal{M}_{CS}(X_4)$ of the four-fold X_4, which is the vacuum space of topological strings in the type IIA compactification on X_4, and open/closed mirror symmetry maps to closed-string mirror symmetry for four-folds. Adding a 4-form flux G_4 induces the Gukov-Vafa-Witten superpotential [58] on the moduli space $\mathcal{M}_{CS}(X_4)$, and this is the dual description of the off-shell deformation space $\mathcal{M}$ of the brane geometry and the obstruction superpotential $\mathcal{W}(C)$ on it. More precisely, the F-theory superpotential[8] on X_4 computes g_s corrections to the superpotential $\mathcal{W}(C)$ as captured by the relation [57]

$$\mathcal{W}_{\text{GVW}}(X_4) = \int_{X_4} G_4 \wedge \Omega^{(4,0)} = \sum_\Sigma \underline{N}_\Sigma(G)\underline{\Pi}^\Sigma(z,\hat{z}) + \mathcal{O}(g_s) + \mathcal{O}(e^{-1/g_s})\,, \tag{3.15}$$

[7] Objects defined in relative (co-)homology will be distinguished by an underline.

[8] See ref. [56] for the discussion from the M-theory perspective.

where the leading term on the right hand side is the result (3.14) for the B-type branes on the three-fold with the linear combination of relative periods determined by the flux G_4 on the four-fold. We will only consider the leading term in g_s in this chapter, which can be computed from the integral periods of a certain non-compact limit $X_4^\sharp$ of X_4, related to the three-fold Z by the open/closed duality of [48, 1, 56], as is discussed in section 3.8. The details of the compactification X_4 of $X_4^\sharp$ affect only the higher terms in g_s and can be computed similarly [57]. More details and many examples on the computation of the four-fold superpotential from the geometric period integrals can be found in [99, 100, 147].

In the following we turn to a more detailed discussion of the concepts and geometrical ideas outlined in the above section.

3.3. Geometry and deformation space of the B-model

We start with the definition of the geometrical structure that will be taken as a model for the open/closed deformation space $\mathcal{M}$, following [47, 46, 103, 1]. Let (Z, Z^*) be a mirror pair of Calabi-Yau three-folds and (L, E) a mirror pair of A/B-type branes on it. On-shell, the classical A-type brane geometry is perturbatively defined by a special Lagrangian submanifold $L \in H_3(Z)$ together with a flat bundle on it [51]. At the quantum level non-perturbative open worldsheet instantons may couple to the special Lagrangian submanifold L. Then an on-shell quantum A-type brane arises if the classical geometry is not destabilized by such instanton corrections [134, 135]. The mirror B-type geometry consists of a holomorphic sheaf E on Z^* describing a D-brane with holomorphic gauge bundle wrapped on an even-dimensional cycle. The concrete realization and application of open string mirror symmetry to this brane geometry, which will be central to all of the following, has been formulated in the pivotal work [142]. More details on the action of mirror symmetry on brane geometries can be found in [131, 35] or for toric branes in Appendix A.

The moduli space of the closed string B-model on Z^* is the space $\mathcal{M}_{CS}$ of complex structures, parametrizing the family $\mathcal{Z}^* \to \mathcal{M}_{CS}$ of three-folds with fiber $Z^*(z)$ at $z \in \mathcal{M}_{CS}$. Here $z = \{z_a\}$, $a = 1, ..., h^{2,1}(Z^*)$ denote some local coordinates on $\mathcal{M}_{CS}$. As discussed above the important concept in the Hodge-theoretic approach to open string mirror symmetry of [47, 46, 103] is the definition of an off-shell deformation space $\mathcal{M}$, which includes open string deformations. In order to study the obstruction superpotential on $\mathcal{M}$, one first defines $\mathcal{M}$ as an *unobstructed* deformation space for a relative homology problem and studies the functions $\underline{\Pi}^\Sigma : \mathcal{M} \to \mathbb{C}$ defined by integration over the dual cohomology space. In a second step, one then adds an obstruction, which can be shown to induce a superpotential on $\mathcal{M}$ proportional to a linear combination of these 'relative periods' $\underline{\Pi}^\Sigma$.

As outlined above the unobstructed moduli space for the family of relative cohomology groups can be defined as the moduli space of a *holomorphic* family of hypersurfaces $\mathcal{D}$ embedded into the family $\mathcal{Z}^*$ of Calabi-Yau three-folds [47, 46, 103]

$$
\begin{aligned}
i: \qquad & \mathcal{D} \hookrightarrow \mathcal{Z}^* \\
& D(z,\hat{z}) \hookrightarrow Z^*(z)
\end{aligned}
\tag{3.16}
$$

where $\hat{z} = \{\hat{z}_\alpha\}$ are local coordinates on the moduli space of the embeddings $i : D(z,\hat{z}) \hookrightarrow Z^*(z)$ for fixed complex structure z. The total moduli space $\mathcal{M}$ of this family is the fibration

$$
\begin{array}{ccc}
\hat{\mathcal{M}}(\hat{z}) & \longrightarrow & \mathcal{M} \\
 & & \downarrow \pi \\
 & & \mathcal{M}_{CS}(z)
\end{array}
\tag{3.17}
$$

where the point $z \in \mathcal{M}_{CS}$ on the base specifies the complex structure on the Calabi-Yau three-fold $Z^*(z)$ and the point $\hat{z} \in \hat{\mathcal{M}}$ on the fiber the embedding. In the context of string theory, the moduli z and $\hat{z}$ arise from states in the closed and open string sector, respectively. Note that the fields associated with the fiber and the base of $\mathcal{M}$ couple at a different order in string perturbation theory. This will be relevant when defining a Kähler metric on $T\mathcal{M}$ in section 3.8.

Following [47, 46, 148, 103], we consider functions on the unobstructed deformation space $\mathcal{M}$ given by 'period integrals' on the relative cohomology group defined by the brane geometry. The embedding $i : \mathcal{D} \hookrightarrow Z^*$ defines the space $\Omega^*(Z^*,\mathcal{D})$ of relative p-forms via the exact sequence

$$
0 \longrightarrow \Omega^*(Z^*,\mathcal{D}) \longrightarrow \Omega^*(Z^*) \xrightarrow{i^*} \Omega^*(\mathcal{D}) \longrightarrow 0 \; .
$$

The associated long exact sequence defines the relative three-form cohomology group

$$
H^3(Z^*,\mathcal{D}) \;\simeq\; \ker\,\big(H^3(Z^*) \to H^3(\mathcal{D})\big) \oplus \mathrm{coker}\,\big(H^2(Z^*) \to H^2(\mathcal{D})\big) \; , \tag{3.18}
$$

which provides the geometric model for the space of ground states of the open/closed topological B-model. In a generic[9] situation, the first summand equals $H^3(Z^*)$ and represents the closed string sector capturing the deformations of the complex structure of Z^*. The relation of the above sheaf cohomology groups considered in [47, 46, 103] and the Ext groups studied in [141] will be discussed in section 3.6.

By (3.18), a closed relative three-form $\underline{\Phi} \in \Omega^3(Z^*,\mathcal{D})$, representing an element of $H^3(Z^*,\mathcal{D})$, can be described by a pair (Φ,ϕ), where Φ is a 3-form on Z^* and ϕ a 2-form on $\mathcal{D}$. The differential is $d\underline{\Phi} = (d\Phi, i^*\Phi - d\phi)$ and the equivalence relation is given by $(\Phi,\phi) \sim (\Phi,\phi) + (d\alpha, i^*\alpha - d\beta)$ for $\alpha \in \Omega^2(Z^*)$, $\beta \in \Omega^1(\mathcal{D})$. The duality pairing between a 3-chain class $\gamma_\Sigma \in H_3(Z^*,\mathcal{D})$ and a relative p-form class $[\underline{\Phi}]$ is given by the integral

$$
\int_{\gamma_\Sigma} \underline{\Phi} = \int_{\mathrm{int}(\gamma_\Sigma)} \Phi - \int_{\partial\gamma_\Sigma} \phi \; . \tag{3.19}
$$

[9]That is $H^1(Z^*) \simeq 0$ and we made the simplifying assumption that D is ample, which is a reasonable condition on the divisor wrapped by a B-type brane. The Lefschetz hyperplane theorem then implies $H^1(D) \simeq 0$ and, by Poincaré duality, $H^3(D) \simeq 0$.

The fundamental holomorphic objects of the open/closed topological B-model are particular examples of (3.20), namely the relative period integrals of the holomorphic $(3,0)$ form $\underline{\Omega}$ on Z^*, viewed as the element $(\Omega, 0) \in H^3(Z^*, \mathcal{D})$, over a basis $\{\underline{\Gamma}_\Sigma\}$ of topological 3-chains:

$$\underline{\Pi}^\Sigma(z, \hat{z}) = \int_{\underline{\Gamma}_\Sigma} \underline{\Omega}, \qquad \underline{\Gamma}_\Sigma \in H_3(Z^*, \mathcal{D}) \ . \tag{3.20}$$

The cohomology group $H^3(Z^*, \mathcal{D})$ is constant over $\mathcal{M}$, but the Hodge decomposition given by the filtration $F^p H^3(Z^*, \mathcal{D})$ and the direction of the (3,0) form $\underline{\Omega}$ varies with the moduli. The period integrals $\underline{\Pi}^\Sigma(z, \hat{z})$ thus define a set of moduli dependent local functions on $\mathcal{M}$. Despite the fact, that there is not yet a superpotential on $\mathcal{M}$, these functions should have an important physical meaning in the unobstructed theory as well. In section 3.8 we will argue that they define a Kähler metric on $\mathcal{M}$ and thus determine the kinetic terms of the bulk and brane moduli in the effective action.

3.3.1. Obstructed deformation problem

The physical meaning of the period integrals is altered after adding an additional lower-dimensional brane charge on a 2-cycle, which induces an obstruction on $\mathcal{M}$. From a physics point of view this perturbation may be realized by either adding an additional brane on a 2-cycle in $\mathcal{D}$ or by switching on a 2-form gauge flux on the original brane on $\mathcal{D}$. A worldsheet derivation of the obstruction from the relevant Ext groups in the open string CFT will be given in section 3.6.

In the Hodge theoretic approach of [47, 46, 103, 1], the superpotential on $\mathcal{M}$ in the obstructed theory is given by a certain linear combination of the relative periods (3.20) of the unobstructed theory. This is similar to the case of closed string flux compactifications, where the flux superpotential on the space $\mathcal{M}_{CS}$ of complex structures can be computed in the unobstructed theory with $\mathcal{M}_{CS}$ as a true moduli space [149, 146, 150].

Let C_i denote the irreducible components of the 2-cycle carrying the additional brane charge and $C = \sum_i C_i$ their sum. If $[C] = 0$ as a class in $H_2(Z^*)$, there exists a 3-chain Γ in the sheaf cohomology group (3.18), with $\partial\Gamma = C$. In particular, the choice of the brane cycle C restricts the relevant co-homology to the subspace

$$H_3(Z^*, D) \ \longrightarrow \ H_3(Z^*, \sum_i C_i) \ . \tag{3.21}$$

The open/closed string superpotential $\mathcal{W}(z, \hat{z})$ on $\mathcal{M}$ for this brane configuration is computed by a relative period integral $\underline{\Pi}(z, \hat{z})$ on this subspace [47, 46, 103, 1].

As discussed above it was argued in [145], that a superpotential, that has the correct critical points to describe a supersymmetric brane on C, is given by the chain integral

$$\mathcal{T} = \int_{\gamma(C)} \Omega \qquad \partial\gamma(C) \neq 0 \ . \tag{3.22}$$

This expression was later derived from a dimensional reduction of the holomorphic Chern-Simons functional of [51] in [134, 135, 142].[10]

As it stands, (3.22) can be viewed either as a definition in absolute cohomology, or in relative cohomology, replacing $\Omega \to (\Omega, 0)$ and including the explicit boundary term in (3.20). The difference is important only off-shell and in this way the relative cohomology ansatz of [47, 46, 103, 1], building on the results of [142], can be viewed as a particular proposal for an off-shell definition of the superpotential.

In absolute cohomology, the integral (3.22) is a priori ill-defined because of non-vanishing boundary contributions from exact forms, which do not respect the equivalence relation $[\Omega] = [\Omega + d\omega]$. To obtain a well-defined pairing one may restrict homology to chains with boundary $\partial\gamma$ a holomorphic curve and cohomology to sections of the Hodge subspace $F^2H^3 = H^{3,0} \oplus H^{2,1}$ [93, 94].[11] This is the normal function point of view taken in [53, 54]. Since the curve $C = \partial\gamma$ being holomorphic corresponds to a critical point $d\mathcal{W}|_{\hat{z}_{\text{crit}}} = 0$ of the superpotential with respect to the open string moduli [145], continuous open-string deformations are excluded from the beginning and one obtains the critical value $W_{\text{crit}}(z) = \mathcal{W}(z, \hat{z}_{\text{crit}})$ of the superpotential as a function of the closed-string deformations z, only. The dependence of the critical superpotential $W_{\text{crit}}(z)$ on the closed string moduli z is still a highly interesting quantity and at the center of the works [53, 54] on open string mirror symmetry, which gave the first computation of disc instantons in compact Calabi-Yau three-folds from mirror symmetry. The dependence of the superpotential on open string deformations $\hat{z}$ is not captured by this definition.

In the relative cohomology ansatz of [47, 46, 103, 1], the pairing (3.22) is well-defined in cohomology also away from the critical points as a consequence of enlarging the co-homology spaces as in (3.18). The extra contribution to $H^3(Z^*, \mathcal{D})$ from the second factor in (3.18) describe additional degrees of freedom in the brane sector. According to this proposal, the relative periods $\underline{\Pi}(z, \hat{z})$ on the subspace $H^3(Z^*, C)$ describe the 'off-shell' superpotential $\mathcal{W}(z, \hat{z})$ depending on brane deformations $\hat{z}$. For consistency, $\mathcal{W}(z, \hat{z})$ should reduce to the critical superpotential $W_{\text{crit}}(z)$ at the critical points. We will verify this for particular examples in chapter 4.

Although we eventually end up with studying the periods on the restricted subspace $H^3(Z^*, C)$ in (3.21) for a fixed brane charge C, the introduction of the larger relative co-homology space $H_3(Z^*, D)$ was not redundant, even for fixed choice of obstruction brane C, as it was crucial for the definition of the finite-dimensional off-shell deformation space $\mathcal{M}$, on which the obstruction superpotential can be defined. As discussed the off-shell deformation space for a brane on C is generically infinite-dimensional, with most of the deformations

[10] More precisely, the chain integral gives the tension $\mathcal{T}$ of a domain wall realized by a brane wrapped on the 3-chain $\gamma(C)$.

[11] The potentially ambiguous boundary terms then vanish as $\int_\gamma \Omega + d\omega = \int_\gamma \Omega + \int_{\partial\gamma} \omega = \int_\gamma \Omega$ for ω a (2,0) form and $\partial\gamma$ a 2-cycle of type $(1,1)$.

representing heavy fields in space-time that should be integrated out. To define an effective superpotential we have to pick an appropriate set of 'light' fields and integrate out infinitely many others.

The ansatz of [47, 46, 103, 1] to define $\mathcal{M}$ by perturbing the unobstructed moduli space of a family $\mathcal{D}$ of hypersurfaces is thus not a circuitry, but rather a systematic way to define a finite-dimensional deformation space with parametric small obstruction, together with a local coordinate patch, on which a meaningful off-shell superpotential can be defined. As C can be embedded in different families of hyperplanes, the parametrization of the deformation space depends on the choice of the family $\mathcal{D}$ and this corresponds to a different choice of light fields for the effective superpotential.[12] Each choice covers only a certain patch of the off-shell deformation space and there will be many choices to parametrize the same physics and mathematics near a critical locus by slightly different relative cohomology groups. This choice of a set of light fields is inherent to the use of effective actions and should not be confused with an ambiguity in the definition.[13]

In the context of open string mirror symmetry, the most interesting aspect of the deformation spaces $\mathcal{M}$ constructed in this way is the presence of 'almost flat' directions in the open string sector, which lead to the characteristic A-model instanton expansion of the superpotential, as will be shown in the examples of chapter 4. The result passes some non-trivial consistency checks which provides some evidence in favor of this definition of off-shell string mirror symmetry.[14] On the other hand, for general massive deformations, one would not expect the simple notions of flatness and an integral instanton expansion observed in this work.

There are two important points missing in the above discussion, which will be further studied in the following. One is the selection of the proper homology element $\gamma(C)$ that computes the superpotential, given a 2-cycle C representing the lower-dimensional brane charge. The other one is the mirror map, which allows to extract a prediction for the disc and sphere instanton expansion for the A-model, starting from the result obtained from the relative periods of the B-model. The additional information needed to answer these questions comes from the variation of mixed Hodge structure on the Hodge bundle with fiber the relative cohomology group $H^3(Z^*, D)$. The Hodge filtration defines a grading by Hodge degree p of the cohomology space at each point $(z, \hat{z})$. In closed string mirror symmetry, restricting to $H^3(Z^*) \subset H^3(Z^*, D)$, this grading is identified with the $U(1)$-charge of the chiral ring elements in the conformal field theory on the string worldsheet. A similar interpretation in terms of an open/closed chiral ring has been proposed in [47, 46, 103]. The upshot of this

[12] One could always combine these 'different' families into a single larger family at the cost of increasing the dimension of the deformation space $\mathcal{M}$.

[13] An attempt to reformulate the relative cohomology approach of [47, 46, 103, 1] by using the excision theorem, as contemplated on in [151], is thus likely to produce just another parametrization corresponding to a slightly different choice of light fields, rather then a distinct description [123, 152, 153, 154].

[14] See also [1, 104, 3] for additional examples and arguments.

extra structure is, that there are *two* relevant relative period integrals associated with the brane charge C, distinguished by the grading, such that one gives the mirror map to the A-model, and the other one the superpotential [47, 46, 103, 1].

3.3.2. $\mathcal{N}=1$ special geometry of the open/closed deformation space

Using the above model for the open/closed deformation space of the B-model one can use the variation of mixed Hodge structure on the deformation family of relative cohomology groups $H^3(Z^*,\mathcal{D})$ of Z^* to capture the moduli dependence of the relative periods. The topological nature of the cohomology group $H^3(Z^*,\mathcal{D})$ leads, similar as in the closed string case, to an up to monodromies flat Gauss-Manin connection on the vacuum bundle over the open/closed deformation space. In the simplest case, $\mathcal{D}$ is a family of single hypersurfaces and the action of the closed and open string variations on the holomorphic $(3,0)$-form $\Omega^{(3,0)}$, written as a relative form $(\Omega^{(3,0)},0)$, is schematically given by[15]

$$\begin{array}{ccccccc} (\Omega^{(3,0)},0) & \xrightarrow{\delta_z} & (\Omega^{(2,1)},0) & \xrightarrow{\delta_z} & (\Omega^{(1,2)},0) & \xrightarrow{\delta_z} & (\Omega^{(0,3)},0) \\ & \searrow^{\delta_{\hat z}} & & \searrow^{\delta_{\hat z}} & & \searrow^{\delta_{\hat z}} & \\ & & (0,\omega^{(2,0)}) & \xrightarrow{\delta_z,\delta_{\hat z}} & (0,\omega^{(1,1)}) & \xrightarrow{\delta_z,\delta_{\hat z}} & (0,\omega^{(0,2)}) \end{array} \tag{3.23}$$

Here δ_z and $\delta_{\hat z}$ denote the closed and open string variations, respectively and $\omega^{(2,0)} = i_{\delta_{\hat z}}\Omega^{(3,0)}$ is the contraction of the holomorphic $(3,0)$-form with a vector field along the open string deformation. The variations δ can be identified with the flat Gauss-Manin connection ∇, which captures the variation of mixed Hodge structure on the bundle with fibers the relative cohomology groups. The mathematical background is described in [93, 94, 95, 96, 97, 98].

The flatness of the Gauss-Manin connection leads to a non-trivial "$\mathcal{N}=1$ special geometry" of the combined open/closed field space, that governs the open/closed chiral ring of the topological string theory [46, 47] similar as in the closed string case. This geometric structure leads to a Picard-Fuchs system of differential equations satisfied by the relative period integrals

$$\mathcal{L}_a \underline{\Pi}^\Sigma = 0, \qquad \underline{\Pi}^\Sigma(z,\hat z) = \int_{\underline{\Gamma}_\Sigma} \Omega, \qquad \underline{\Gamma}_\Sigma \in H_3(Z^*,\mathcal{D})\ . \tag{3.24}$$

Here $\{\mathcal{L}_a\}$ is a system of linear differential operators, $z(\hat z)$ stands collectively for the closed (open) string parameters and the holomorphic $(3,0)$-form Ω and its period integrals are defined in relative cohomology. The relative periods $\underline{\Pi}^\Sigma(z,\hat z)$ determine the mirror map and the combined open/closed string superpotential, which can be written in a unified way as above

$$\mathcal{W}_{\mathcal{N}=1}(z,\hat z) = \mathcal{W}_{\text{flux}}(z) + \mathcal{W}_{\text{brane}}(z,\hat z) = \sum_{\underline{\Gamma}_\Sigma \in H^3(Z^*,\mathcal{D})} \underline{N}_\Sigma \underline{\Pi}^\Sigma(z,\hat z)\ . \tag{3.25}$$

[15] Here we display the label which gives the Hodge type of the form. In the following we denote the holomorphic $(3,0)$-form again by Ω.

Here $\mathcal{W}_{\text{flux}}(z)$ is again the closed string superpotential proportional to the periods over cycles $\gamma_\Sigma \in H^3(Z^*)$ and $\mathcal{W}_{\text{brane}}(z, \hat{z})$ the brane superpotential proportional to periods over chains γ_Σ with non-empty boundary $\partial\gamma_\Sigma$. The coefficients $\underline{N}_\Sigma$ are the corresponding 'flux' and brane numbers.[16]

In the following we are going to implement this general structure for the class of toric branes on compact Calabi-Yau manifolds.[17] In this context, the relative cohomology problem is naturally defined by a family $\mathcal{D}$ of toric hypersurfaces in the B-model. In [46, 47] this identification was used to set up the appropriate problem of mixed Hodge structure for branes in non-compact Calabi-Yau manifolds and to compute the Picard-Fuchs system of the $\mathcal{N} = 1$ special geometry. This approach was extended to the compact case in [103] by relating $\mathcal{D}$ to the algebraic Chern class $c_2(E)$ of a B-brane as obtained from a matrix factorization. These two definitions of $\mathcal{D}$ are closely related and it is straightforward to check that they coincide in concrete examples; in particular the hypersurfaces defined in [103] fit into the definition of $\mathcal{D}$ in [46, 47].[18]

3.3.3. Relative periods and normal functions

As discussed above, a preferred parametrization adapted to topological string states and open-string mirror symmetry is to parametrize the off-shell deformations of the D-brane on a 2-cycle C by the deformations of a holomorphic family of 4-cycles $\mathcal{D}$ that embed $C \in H_2(D)$. The relative periods capturing the superpotential for the brane on C are obtained by restriction to the subspace $H_3(Z^*, C) \subset H_3(Z^*, D)$. Mathematically, this class of parametrizations derives directly from the concept of rational equivalence and the on-shell meaning of the superpotential as an Abel-Jacobi invariant, as will be discussed now.[19]

To this end, consider a Calabi-Yau three-fold Z_0^* together with an ample divisor D_0. We assume that $H^{1,0}(Z_0^*) = H^{2,0}(Z_0^*) = 0$, such that the complex structure deformations of the pair (Z_0^*, D_0) are unobstructed. Then this pair (Z_0^*, D_0) extends to a family $\mathcal{X}$ of Calabi-Yau three-folds together with a family of ample divisors $\pi : (\mathcal{X}, \mathcal{D}) \to \Delta$ fibered over the disc Δ, which parametrizes a local patch of the combined moduli space $\mathcal{M}$ of the family obtained by deforming the central fiber $\pi^{-1}(0) = (Z_0^*, D_0)$. $\mathcal{M}$ is a fibration $\hat{\mathcal{M}} \to \mathcal{M} \to \mathcal{M}_{CS}$, where the base $\mathcal{M}_{CS}$ corresponds to complex structure deformations z of the family of Calabi-Yau three-folds $\mathcal{X}$, while the fiber $\hat{\mathcal{M}}$, parametrized by the coordinates $\hat{z}$, corresponds to the

[16]To obtain the physical superpotential, an appropriate choice of reference brane has to be made for the chain integrals, since a relative period more precisely computes the brane tension of a domain wall [145, 134, 135, 142]. This should be kept in mind in the following discussion where we simply refer to 'the superpotential'.

[17]See Appendix A for details on toric branes.

[18]As was stressed in section 3.6 of [54], the chain integrals, which define the *normal functions* associated with the superpotential, do not depend on the details of the infinite complexes constructed in [155]. Our results suggest that the relevant information for the superpotential is captured by the linear σ-model to be defined in section 3.7.

[19]For a related mathematical discussion, see [98].

deformations of the family of divisors $\mathcal{D}$.

Since the holomorphic three-form $\Omega(z)$ of the Calabi-Yau three-fold Z_z^* vanishes on the divisor $D_{(z,\hat{z})}$, the three form $\Omega(z)$, which is an element of $H^3(Z_z^*)$, lifts to an element $\underline{\Omega}(z,\hat{z})$ of the relative cohomology group $H^3(Z_z^*, D_{(z,\hat{z})})$. We define the integral relative periods as above

$$\underline{\Pi}(\underline{\Gamma};z,\hat{z}) \;=\; \int_{\underline{\Gamma}_{(z,\hat{z})}} \underline{\Omega}(z,\hat{z}) \;, \tag{3.26}$$

where $\underline{\Gamma}_{(z,\hat{z})}$ is an integral relative cycle in $H_3(Z_z^*, D_{(z,\hat{z})}, \mathbb{Z})$ whose boundary $\partial\underline{\Gamma}_{(z,\hat{z})}$ is trivial as a class in $H_2(Z^*, \mathbb{Z})$. For concreteness we often assume that the boundary is the difference $\partial\underline{\Gamma}_{(z,\hat{z})} = C^+_{(z,\hat{z})} - C^-_{(z,\hat{z})}$ of two 2-cycles in $H_2(D_{z,\hat{z}})$ with $[C^+_{(z,\hat{z})}] = [C^-_{(z,\hat{z})}]$ in $H_2(Z_z^*, \mathbb{Z})$.

As discussed above, following the fundamental works [142, 145], it was proposed in [46, 47, 103] that the relative period (3.26) defines the off-shell tension $\mathcal{T}(z,\hat{z})$ of a physical D-brane wrapped on the chain $\underline{\Gamma}_{(z,\hat{z})}$, that is $\mathcal{T}(z,\hat{z}) = \underline{\Pi}(\underline{\Gamma};z,\hat{z})$. This D-brane represents a domain wall interpolating between the two configurations obtained by wrapping a D-brane on $C^+_{(z,\hat{z})}$ or on $C^-_{(z,\hat{z})}$ and its tension measures the difference of the value of the superpotentials for the two D-brane configurations

$$\mathcal{T}(z,\hat{z}) = \mathcal{W}(C^+_{(z,\hat{z})}) - \mathcal{W}(C^-_{(z,\hat{z})}) \;. \tag{3.27}$$

The vacuum condition in the open-string direction is $\frac{d}{d\hat{z}}\mathcal{W}(C^{\pm})|_{\hat{z}=\hat{z}_{crit}} = 0$ and it holds if $C^{\pm}_z := C^{\pm}_{(z,\hat{z}_{\text{crit}})}$ is a holomorphic curve [145]. Imposing this condition on both branes implies $\frac{d}{d\hat{z}}\mathcal{T}(z,\hat{z}) = 0$ as well.

Mathematically speaking, the vacuum configurations hence lie within the so-called Noether-Lefschetz locus, defined as [156]

$$\mathcal{N} \;=\; \left\{ (z,\hat{z}) \in \Delta \;\middle|\; 0 \equiv \frac{d\underline{\Pi}(z,\hat{z})}{d\hat{z}} \right\} \;. \tag{3.28}$$

Equivalently the locus $\mathcal{N}$ can be specified by the vanishing condition

$$\mathcal{N} \;=\; \left\{ (z,\hat{z}) \in \Delta \;\middle|\; 0 \equiv \vec{\pi}(z,\hat{z};\partial\underline{\Gamma}_{(z,\hat{z})}) \right\} \;, \tag{3.29}$$

for the period vector of the divisor $D_{(z,\hat{z})}$

$$\vec{\pi}(z,\hat{z};\partial\underline{\Gamma}_{(z,\hat{z})}) \;=\; \left(\int_{\partial\underline{\Gamma}_{(z,\hat{z})}} \omega_{\hat{a}}^{(2,0)}(z,\hat{z}) \right) \;, \quad \hat{a} = 1,\ldots,\dim H^{2,0}(D_{(z,\hat{z})}) \;. \tag{3.30}$$

Here $\omega_{\hat{a}}^{(2,0)}(z,\hat{z})$ is a basis of two forms for $H^{2,0}(D_{(z,\hat{z})})$. Hence the critical locus of D-brane vacua is mapped to the subslice of complex structures on the surface $D_{(z,\hat{z})}$, where certain linear combinations of period vectors on the surface vanish. At such points in the complex structure the Picard lattice of the surface $D_{(z,\hat{z})}$ is enhanced due to the appearance of an additional integral $(1,1)$-form.

At the Noether-Lefschetz locus $(z, \hat{z}_{\text{crit}}) \in \mathcal{N}$ there is an interesting connection between the relative periods and another mathematical quantity studied in [53, 54]. By the result of [156], the relative period $\underline{\Pi}(z, \hat{z})$ evaluated at the Noether-Lefschetz locus $(z, \hat{z}_{\text{crit}}) \in \mathcal{N}$ gives (modulo bulk periods) the Abel-Jacobi invariant associated to the normal function of the algebraic curve $\partial\underline{\Gamma}_{(z,\hat{z}_{\text{crit}})}$:

$$\underline{\Pi}(z, \hat{z}_{\text{crit}}) = \nu_{c_2^{\text{alg}}(\partial\underline{\Gamma}_{(z,\hat{z}_{\text{crit}})})}(z) \quad \text{mod} \quad (\text{bulk periods}) \, . \tag{3.31}$$

Specifically, the Abel-Jacobi invariant is defined via the normal function $\nu_{c_2^{\text{alg}}(\alpha)}(z)$ as

$$AJ \, : \, CH^2(Z_z^*) \to J^3(Z_z^*) \simeq \frac{F^2H^3(Z_z^*)^*}{H_3(Z_z^*, \mathbb{Z})} \, ; \qquad \alpha \mapsto \nu_{c_2^{\text{alg}}(\alpha)}(z) \, , \tag{3.32}$$

where, in the concrete setting, the normal function is defined as the chain integral

$$T(z) = \int_{\Gamma_z^{\pm}} \Omega(z) = \nu_{c_2^{\text{alg}}(C_z^+ - C_z^-)}(z) \quad \text{mod} \quad (\text{bulk periods}) \, . \tag{3.33}$$

Here $\partial\Gamma_z^{\pm} = C_z^+ - C_z^-$, with $C_z^{\pm}$ the holomorphic curves at fixed $\hat{z} = \hat{z}_{\text{crit}}$. As already discussed, (only) at the critical locus, the above integral is well-defined in absolute cohomology, because the potentially dangerous boundary terms vanish by holomorphicity of the boundary $\partial\Gamma_z^{\pm}$ and the Hodge type of Ω. The normal functions (3.33) have been introduced in [53, 54] to study the on-shell values of the superpotentials

$$T(z) = W(C_z^+) - W(C_z^-) \, .$$

By the above argument, these are the restrictions of the relative period integrals (3.26) to the critical locus $\mathcal{N}$.

There is also a partial inverse of this relation, which recovers the relative periods for the family of divisors starting from the normal functions. To this end, recall the meaning of rational equivalence and the Abel-Jacobi invariant. The second algebraic Chern class c_2^{alg} takes values in the second Chow group $CH^2(Z_z^*)$, which consists of equivalence classes of algebraic cycles of co-dimension two modulo rational equivalence [157].[20] Two algebraic cycles α and β of co-dimension two are rationally equivalent, if we can find a subvariety V of co-dimension one, in which α and β are rationally equivalent as co-dimension one cycles. This is the case if α and β are given by two linearly equivalent divisors on V, that is $[\alpha - \beta] = 0 \in CH^1(V)$.[21] Moreover, rational equivalence implies that the Abel-Jacobi invariant vanishes.

Starting from an algebraic cycle α of co-dimension two with $c_2^{\text{top}}(\alpha) = 0$ we can find a three chain Γ^α such that $\alpha = \partial\Gamma^\alpha$, and associate a normal function $\nu_{c_2^{\text{alg}}(\alpha)}$ to it via the

[20]The second algebraic Chern class is a refined invariant of the topological second Chern class [157].

[21]If the subvariety V is not normal the cycles α and β are rationally equivalent, if their Weil divisors D_α and D_β are linearly equivalent in the normalization $\tilde{V}$ of V, namely $\alpha \sim \beta$ if $D_\alpha \sim D_\beta$ with $f : \tilde{V} \to V$ and $\alpha = f_* D_\alpha$ and $\beta = f_* D_\beta$.

integral (3.33). By (3.32), the normal function vanishes for algebraic two cycles $C_z^{\pm}$ that are rationally equivalent [54]. On the contrary, if C_z^+ and C_z^- are not rationally equivalent, we obtain an element in the relative cohomology of each family $\mathcal{D}(z,\hat{z})$ of divisors that contains the two holomorphic curves $C_z^{\pm}$ at a 'critical' value $\hat{z}=\hat{z}_{\text{crit}}$. Indeed, since C_z^+ and C_z^- are *not* rationally equivalent, $C_z^+ - C_z^-$ defines by Poincaré duality a non-trivial Element $\omega \in \text{Pic}(D_{(z,\hat{z}_{\text{crit}})}) \simeq H^{1,1}(D_{(z,\hat{z}_{\text{crit}})}) \cap H^2(D_{(z,\hat{z}_{\text{crit}})},\mathbb{Z})$. Since the algebraic cycle α is topologically trivial on Z_z^*, the associated two form ω is *not* induced from the hypersurface X_z and lifts to a relative three form $\underline{\Theta}_{(z,\hat{z}_{\text{crit}})}$ by the relation

$$H^3(Z_z^*, D_{(z,\hat{z})}) \simeq \ker\left(i^*: H^3(Z_z^*) \to H^3(D_{(z,\hat{z})})\right) \oplus \text{coker}\left(i^*: H^2(Z_z^*) \to H^2(D_{(z,\hat{z})})\right) ,$$

with $i: D_{(z,\hat{z})} \hookrightarrow Z_z^*$. By construction, the three-chain $\Gamma^\alpha \simeq \underline{\Gamma}_{(z,\hat{z}_{\text{crit}})}$ is a representative of the relative homology class in $H_3(Z_z^*, D_{(z,\hat{z}_{\text{crit}})})$ dual to $\underline{\Theta}_{(z,\hat{z}_{\text{crit}})}$. Surjectivity of the boundary map of homology then asserts that the above construction assigns to each normal function a relative period on $H^3(Z_z^*, D_{(z,\hat{z})})$, which measures the superpotential of the off-shell deformation parametrized by the family $\mathcal{D}(z,\hat{z})$.

Using this connection between normal functions (3.33), that is to say domain walls between critical points $C_z^{\pm}$, and the off-shell tensions represented by the integral relative periods (3.26) ending on $C_{(z,\hat{z})}^{\pm}$, we may calculate the critical tensions as follows. First determine the possible critical points as the vanishing locus (3.29) of the periods of the surface $D_{(z,\hat{z})}$. The critical domain wall tension is then given by the relative period associated with the vanishing period on the surface, evaluated at the critical point z_{crit}

$$T(z) = \mathcal{T}(z,\hat{z}_{\text{crit}})\,. \tag{3.34}$$

This determines the critical tension up to a possible addition of a bulk period $\Pi_{\text{bulk}}(z)$.

The vanishing condition (3.29), classifying the critical points, can be studied very explicitly for off-shell deformations in a single open-string parameter $\hat{z}$, which is sufficient to determine the on-shell tensions. In this case the surface $D_{(z,\hat{z})}$ has geometric genus one and it is isogenic to a K3 surface [158], that is the integral Hodge structures of the surface $D_{(z,\hat{z})}$ can be mapped to the equivalent Hodge structure of its isogenic K3 surface.

One particular type of solutions to the vanishing condition arises at the discriminant locus of the isogenic K3 surface, where the period vector, associated with a geometrically vanishing cycle in the K3 surface, develops a zero. However, this type of solution is non-generic in the sense that it is often related to points in the deformation space with a domain wall with zero tension. The generic critical points arise instead from a zero of the period vector, which is a linear combination of volumes of geometric cycles in the K3 surface rather then the volume of an irreducible cycle. The typical example is a point where the volumes of two different cycles coincide, such that the period vector associated with the difference vanishes. At these particular symmetric points there is an 'accidental' global symmetry of the K3 lattice, exchanging the two cycles. More generally the generic critical points should be classified by special symmetric points in the K3 moduli studied in [159].

3.4. Generalized hypergeometric systems for the B-model

As described already above the variation of mixed Hodge structure on the family of relative cohomology groups can be translated into to set of differential operators, which annihilate the relative periods (3.20). In this section we are going to derive a generalized hypergeometric system of differential operators for the deformation problem at hand, in the concrete framework of toric branes on toric Calabi-Yau hypersurfaces first defined in [142] and further scrutinized in [1, 104, 3]. The result is a system of differential equations acting on the relative cohomology space and its periods, whose associated Gauss-Manin system and solutions will be studied in subsequent sections. For a mathematical treatment of the GKZ systems in this context, see [160].

In section 3.7 it will be shown how the same set of differential operators can be obtained from a dual non-compact four-fold. This reflects the duality of B-type branes on the three-fold Z^* to an M/F-theory compactification on a four-fold $X_4^{\sharp *}$ determined by the open/closed duality of [48, 1, 56]. Specifically, the set of differential operators for the relative periods on Z^* and for the four-fold periods on $X_4^{\sharp *}$ have the superpotential periods in (3.14) and (3.15) as common solutions, and the superpotential can be equivalently computed on the three-fold or on the four-fold. This gives also a complementary view on the variation of Hodge structures on the relative cohomology.

For clarity we relegate the definitions of the family of toric branes in compact toric hypersurfaces used in this section to the Appendix A.5 and refer to [142, 1] for further details. A brief review of toric geometry is given in the Appendix A. Further details about generalized hypergeometric systems for the closed-string case can be found in section 2.4.4 and [107, 108, 109].

3.4.1. Extended hypergeometric systems for relative periods

We proceed with the derivation of a GKZ hypergeometric system which annihilates the relative period integrals (3.20) on the relative cohomology group for a class of toric branes. The definition of the (union of) hypersurfaces $\mathcal{D}$ cannot preserve all torus symmetries of the toric ambient space. Instead (some of) the torus actions move the position of the branes.[22] As a consequence, the relative periods are no longer annihilated by all the operators $\mathcal{Z}_k$ and depend on additional parameters specifying the geometry of $\mathcal{D}$. The differential equations (2.81) for the period integrals imply on the level of forms

$$\mathcal{L}(l)\ \Omega = d\omega(l)\ ,$$
$$\mathcal{Z}_k\ \Omega = d\omega_k\ .$$

The exact terms on the r.h.s. contribute only to integrals over 3-chains $\hat{\gamma} \in H_3(Z^*, \mathcal{D})$ with non-trivial boundaries $\partial\hat{\gamma}$. The modification of the differential equations for the relative

[22] This is what one would expect intuitively from the formulation of mirror symmetry as T-duality [36].

periods can be computed from these boundary terms.

To keep the discussion simple we derive the differential operators for the relative periods on the mirror of the quintic and present general formulae at the end of this section. The integral points of the polyhedron $\Delta(Z)$ are

$$\begin{array}{|c|cccrrrrc|}\hline \Delta(Z) & \tilde{\nu}_0 & = & (& 0 & 0 & 0 & 0 &) \\ & \tilde{\nu}_1 & = & (& 1 & 0 & 0 & 0 &) \\ & \tilde{\nu}_2 & = & (& 0 & 1 & 0 & 0 &) \\ & \tilde{\nu}_3 & = & (& 0 & 0 & 1 & 0 &) \\ & \tilde{\nu}_4 & = & (& 0 & 0 & 0 & 1 &) \\ & \tilde{\nu}_5 & = & (& -1 & -1 & -1 & -1 &) \\ \hline \end{array} \tag{3.35}$$

leading to the defining polynomial for the mirror quintic:

$$P(Z^*) = a_0 + a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5(X_1X_2X_3X_4)^{-1} \ .$$

The holomorphic (3,0) form on the mirror quintic can be explicitly represented as the residuum

$$\Omega = \mathrm{Res}_P \, \frac{a_0}{P} \prod_k \frac{dX_k}{X_k} \ , \tag{3.36}$$

at $P = 0$ [107]. Here we have changed the normalization with respect to (2.80) to the standard convention $\Pi(a_i) \to a_0\Pi(a_i)$. The GLSM for the quintic is specified by one charge vector $l^1 = (-5, 1, 1, 1, 1, 1)$ which defines one differential operator $\mathcal{L}_1 = \mathcal{L}(l^1)$ annihilating the periods on the mirror. For this operator we do not get an exact term, and we find

$$\left(\prod_{i=1}^{5} \vartheta_i + z \prod_{i=1}^{5} (\vartheta_0 - i) \right) \Omega = 0 \ . \tag{3.37}$$

Furthermore the operators $\mathcal{Z}_k$ give rise to the relations

$$\sum_{i=0}^{5} \vartheta_i \, \Omega = 0, \qquad (\vartheta_i - \vartheta_5) \, \Omega = d\omega_i \ , \quad i = 1, ..., 4, \tag{3.38}$$

with

$$\omega_i = (-)^{i+1} \mathrm{Res}_P \, \frac{a_0}{P} \prod_{\substack{j=1 \\ j \neq i}}^{4} \frac{dX_j}{X_j} \ . \tag{3.39}$$

Equation (3.38) expresses the torus invariance of the period integrals in absolute cohomology and implies that the integrals depend only on the single invariant complex modulus z_1 defined as in (2.82). In relative cohomology, the exact terms on the r.h.s. descend to non-trivial 2-forms on $\mathcal{D}$ by the equivalence relation

$$H^3(Z^*, \mathcal{D}) \ni (\Xi, \xi) \sim (\Xi + d\alpha, \xi + i^*\alpha - d\beta) \ , \tag{3.40}$$

where $i:\mathcal{D}\hookrightarrow Z^*$ is the embedding. The exact pieces in (3.38) may give rise to boundary terms that break the torus symmetry and introduce an additional dependence on moduli $\hat{z}_\alpha$ associated with the geometry of the embedding of $\mathcal{D}$.

To proceed we need to specify the family of hypersurfaces $\mathcal{D}$. We consider a simple one-parameter family $\mathcal{D}$ of hypersurfaces defined by a linear equation, which can be put into the standard form

$$\mathcal{D}:\ Q=1+X_1=0\ , \tag{3.41}$$

by a coordinate transformation on X_1.[23] In order to determine the preserved torus symmetries we examine the boundary contributions (3.38) with respect to the hypersurfaces $\mathcal{D}$ by evaluating the pullbacks of the two forms (3.39). One finds $i^*\omega_k=0$ for $k=2,3,4$ and

$$i^*\omega_1=\mathrm{Res}_{P_D}\frac{a_0}{P_D}\prod_{i=2}^{4}\frac{dX_i}{X_i}\ =\mathrm{Res}_{P,Q}\ \frac{a_0X_1}{P\,Q}\prod_{k=1}^{4}\frac{dX_k}{X_k}\ , \tag{3.42}$$

where $P_{\mathcal{D}}=P(Z^*)|_{Q=0}=(a_0-a_1)+a_2X_2+a_3X_3+a_4X_4-a_5(X_2X_3X_4)^{-1}$. In the second equation above, we represented the pull-back as a double residue in $P=0$ and $Q=0$ in the ambient space as in [103], which allows for a direct evaluation of periods as in [104] and is convenient also for higher degree hypersurfaces.

In the presence of the hypersurface $\mathcal{D}$ the torus action $X_1\to\lambda X_1$ with $\lambda\in\mathbb{C}^*$ generated by the operator $\mathcal{Z}_1$ is broken, whereas the remaining torus symmetries, associated to the operators $\mathcal{Z}_0,\mathcal{Z}_2,\mathcal{Z}_3,\mathcal{Z}_4$, are preserved.[24] From these four differential operators for the six parameters a_i it follows that the family $\mathcal{D}\subset Z^*$ depends precisely on the two moduli

$$z=-\frac{a_1a_2a_3a_4a_5}{a_0^5},\qquad \hat{z}=-\frac{a_1}{a_0}\ , \tag{3.43}$$

together with the logarithmic derivatives

$$\theta:=z\,\partial_z=\vartheta_5\ ,\qquad \hat{\theta}:=\hat{z}\,\partial_{\hat{z}}=\mathcal{Z}_1=\vartheta_1-\vartheta_5\ . \tag{3.44}$$

Here z is the complex structure modulus of Z^* and $\hat{z}$ is the open-string position modulus parametrizing the position of the brane. This analysis justifies in retrospect that the defined family of hypersurfaces (3.41) indeed depends on a single open-string modulus.

The next task is to determine the differential operators $\mathcal{L}$ of the extended GKZ hypergeometric system associated to the periods of the relative (3,0) form representative

$$\underline{\Omega}:=(\Omega,0)=\left(\mathrm{Res}_P\frac{a_0}{P}\prod_k\frac{dX_k}{X_k}\ ,0\right)\ . \tag{3.45}$$

[23] At first glance it seems that we have chosen a rigid hypersurface $\mathcal{D}$. However, as we vary the embedding of the Calabi-Yau manifold Z^* in the ambient toric space, we effectively change the hypersurface $\mathcal{D}$ in in the Calabi-Yau manifold Z^*.

[24] Alternatively one may express the independence of the periods on extra parameters in (3.41) by a modified differential operator Z_1, see also the discussion below (3.49).

Due to (3.37) the operator $\mathcal{L}_1$ annihilates the three form Ω even on the level of forms (and not just on the level of the absolute three-form cohomology), and therefore the operator $\mathcal{L}_1$ annihilates also the relative three form Ω. In the extended relative cohomology setting, however, there is an additional differential operator $\mathcal{L}_2$ governing the functional dependence of the exact piece $d\omega_1$. It is straightforward to check that the (2,0) form $i^*\omega_1$ of the hypersurface $\mathcal{D}$ defined in (3.42) obeys

$$(\vartheta_1 - \hat{z}(\vartheta_0 - 1))\, i^*\omega_1 = 0 \ .$$

Together with the relation

$$\theta_2\underline{\Omega} = \mathcal{Z}_1(\Omega, 0) = (d\omega_1, 0) \sim (0, -i^*\omega_1) \tag{3.46}$$

associated to the broken torus action, we determine the second operator $\mathcal{L}_2$ to be

$$\mathcal{L}_2 = (\vartheta_1 - \hat{z}(\vartheta_0 - 1))(\vartheta_1 - \vartheta_5) \ .$$

Thus in summary we find that the relative (3,0)-form cohomology class, represented by the relative three-form $\underline{\Omega}$, is annihilated by the six differential operators

$$\begin{aligned} &\mathcal{L}_a\underline{\Omega} \sim 0 \ , \quad a = 1, 2 \ , \\ &\mathcal{Z}_k\underline{\Omega} \sim 0 \ , \quad k = 0, 2, 3, 4 \ . \end{aligned} \tag{3.47}$$

With the help of the differential operators $\mathcal{Z}_k, k = 0, 2, 3, 4$, it is straightforward to derive the two extended GKZ operators $\mathcal{L}_a, a = 1, 2$, in terms of the moduli z and $\hat{z}$ and their logarithmic derivatives θ and $\hat{\theta}$

$$\begin{aligned} \mathcal{L}_1 &= (\theta + \hat{\theta})\theta^4 + z\prod_{i=1}^{5}(-5\theta - \hat{\theta} - i) =: \mathcal{L}_1^{\text{bulk}} + \mathcal{L}_1^{\text{bdry}}\hat{\theta} \ , \\ \mathcal{L}_2 &= \big((\theta + \hat{\theta}) - \hat{z}(-5\theta - \hat{\theta} - 1)\big)\hat{\theta} =: \mathcal{L}_2^{\text{bdry}}\hat{\theta} \ . \end{aligned} \tag{3.48}$$

Here $\mathcal{L}_1^{\text{bulk}} = \theta^5 + z\prod_{i=1}^{5}(-5\theta - i)$ is the $\hat{\theta}$-independent GKZ operator of the quintic and the operators $\mathcal{L}_a^{bdry}$ are always accompanied by at least one derivative $\hat{\theta}$ and thus are only sensitive to boundary contributions

$$0 = \mathcal{L}_a \int_{\gamma_\Sigma} \underline{\Omega} = \mathcal{L}_a^{\text{bulk}} \int_{\text{int}(\gamma_\Sigma)} \Omega - \mathcal{L}_a^{\text{bdry}} \int_{\partial\gamma_\Sigma} i^*\omega_1 \ , \quad a = 1, 2 \ , \tag{3.49}$$

with $\mathcal{L}_2^{\text{bulk}} \equiv 0$.

The significance of the interplay of the operator $\hat{\theta}$ with the operator $\mathcal{L}_a^{\text{bdry}}$ is reflected in (3.46). We observe that the (linear combinations) of boundary operators $\mathcal{L}_a^{\text{bdry}}$, which are not accompanied with a non-vanishing bulk operator $\mathcal{L}_a^{\text{bulk}}$, become the GKZ operators of the periods localizing on the hypersurface $\mathcal{D}$. As noted above, the Hodge variation on the hypersurface $P_{\mathcal{D}} = 0$ is isomorphic to that of the mirror of the quartic K3 surface and $i^*\omega_1$ is a

$\Delta(Z)$	$\tilde{\nu}_0$	$=$	(	0	0	0	0	0	)
	$\tilde{\nu}_1$	$=$	(	1	0	0	0	0	)
	$\tilde{\nu}_2$	$=$	(	0	1	0	0	0	)
	$\tilde{\nu}_3$	$=$	(	0	0	1	0	0	)
	$\tilde{\nu}_4$	$=$	(	0	0	0	1	0	)
	$\tilde{\nu}_5$	$=$	(	-1	-1	-1	-1	0	)
$\Delta_\flat(Z, L) = \Delta\cup$	$\tilde{\nu}_6$	$=$	(	1	0	0	0	1	)
	$\tilde{\nu}_7$	$=$	(	0	0	0	0	1	)

Table 3.1.: Vertices of the enhanced polyhedron $\Delta_\flat(Z, L)$ for the quintic example.

representative for the holomorphic (2,0)-form. The associated K3 periods $\int i^*\omega_1$ are precisely annihilated by these GKZ operators arising form the boundary sector.

Thus we have obtained two differential operators $\mathcal{L}_a,\ a = 1, 2$, in (3.48) annihilating the relative periods. These operators can be rewritten in a concise form by realizing that they represent the differential operators $\mathcal{L}(l)$ for a different GKZ system of the type (2.81) specified by the two linear relations

$$\tilde{l}^1 = (-5; 1, 1, 1, 1, 1, 0, 0), \qquad \tilde{l}^2 = (-1; 1, 0, 0, 0, 0, 1, -1)\ , \tag{3.50}$$

together with the five-dimensional enhanced toric polyhedron $\Delta_\flat$ with the integral vertices in Table 3.1. This polyhedron defines a Calabi-Yau four-fold $X_4^\sharp$, which is dual to the brane compactification in the sense of [48]. A systematic construction, which associates a (compact) dual F-theory four-fold to a mirror pair of toric branes defined as in [142], is given in sections 3.7 and 3.8.

Note that the enhanced polyhedron $\Delta_\flat$ gives rise to six operators $\tilde{\mathcal{Z}}_k, k = 0, \ldots, 5$, with $\tilde{\mathcal{Z}}_k = \mathcal{Z}_k$ for $k = 0, 2, 3, 4$. The two additional operators $\tilde{\mathcal{Z}}_1$ and $\tilde{\mathcal{Z}}_5$ guarantee that the functional dependence on the local coordinates $\tilde{z}_a(\tilde{l}^a)$ defined by the general relation (2.82) also coincide with the moduli of the relative cohomology problem defined in (3.43). Moreover, the GKZ operators $\tilde{\mathcal{L}}_a = \mathcal{L}(\tilde{l}^a)$ obtained from the general expression[25]

$$\mathcal{L}(l) = \prod_{k=1}^{l_0}(\vartheta_0 - k)\prod_{l_i>0}\prod_{k=0}^{l_i-1}(\vartheta_i - k) - (-1)^{l_0} z_a \prod_{k=1}^{-l_0}(\vartheta_0 - k)\prod_{l_i<0}\prod_{k=0}^{-l_i-1}(\vartheta_i - k) \tag{3.51}$$

coincide with the two operators $\mathcal{L}_a$ of the relative cohomology problem in the local coordinates

[25] The same formula describes also the generalized hypergeometric operators of GKZ type for the closed-string compactification [109, 108] and this will be used in the examples to determine the periods of the three-fold Z^* and the surface $\mathcal{D}$ below. The distinction between the three different cases arises only from the different generators l^a, which encode the action of the gauge symmetry of the GLSM associated with the surface $\mathcal{D}$, the three-fold Z^*, the brane geometry $(Z^*, \mathcal{D})$ and the dual F-theory four-fold $X_4^\sharp$, respectively, with the latter two cases having the identical generators in the decoupling limit.

defined by (2.82).

$$\mathcal{L}_a(z,\hat{z}) = \tilde{\mathcal{L}}_a(\tilde{z}_1 = z, \tilde{z}_2 = \hat{z}) \ .$$

In fact one can show that all differential operators $\mathcal{L}(l)$ for l a linear combination of l^1, l^2 also annihilate the relative periods.[26]

3.4.2. Picard-Fuchs and GKZ equations for domain wall tensions

As discussed above, the flat Gauss-Manin connection on the relative cohomology bundle leads to Picard-Fuchs type of differential operators for the relative periods, which provide an effective method to determine and to study the tensions $\mathcal{T}(z,\hat{z})$. Therefore there should be a relation of the derived differential operators to the inhomogeneous Picard-Fuchs equations of on-shell domain walls as analyzed in [53, 54].

For concreteness, we assume that the holomorphic curves $C_z^{\pm}$ which lie at the ends of the domain wall are contained in the intersection of the hypersurface $Z^* : \ P = 0$ with two hyperplanes $D_{1,2}$ defined in a certain toric ambient space. Choose coordinates such that the equation for D_1 does not depend on the closed-string moduli z, typically of the form[27]

$$D_1 : \ x_1^a + \eta\, x_2^b = 0 \,,$$

where x_i are some homogeneous coordinates on the ambient space, a, b some constants that depend on the details and η a fixed constant, which is a phase factor in appropriate coordinates. This hyperplane can be deformed into a family $\mathcal{D}_1 : \ x_1^a + \hat{z}\, x_2^b = 0$ by replacing the constant η by a complex parameter $\hat{z}$. The relative three-form $\underline{\Omega}$ and the relative period integrals on the family of cohomology groups $H^3(Z^*, \mathcal{D}_1)$, satisfy a set of Picard-Fuchs equations

$$\mathcal{L}_a(\theta,\hat{\theta})\, \underline{\Omega} = \underline{d\omega}^{(2,0)} \quad \Rightarrow \quad \mathcal{L}_a(\theta,\hat{\theta})\ \mathcal{T}(z,\hat{z}) = 0\,, \qquad a = 1, ..., A\,,$$

where a is some label for the operators. As above the differential operators can be split into two pieces

$$\mathcal{L}_a(\theta,\hat{\theta}) =: \mathcal{L}_a^{\mathrm{bulk}} - \mathcal{L}_a^{\mathrm{bdry}}\,\hat{\theta}\,, \tag{3.52}$$

where the bulk part $\mathcal{L}_a^{\mathrm{bulk}}(\theta)$ acts only on the closed-string moduli z and the boundary part $\mathcal{L}_a^{\mathrm{bdry}}(\theta,\hat{\theta})\,\hat{\theta}$ contains at least one derivative in the parameters $\hat{z}$. Since the dependence on $\hat{z}$ localizes on D_1, the derivatives $2\pi i\,\hat{\theta}\,\mathcal{T}(z,\hat{z})$ are proportional to the periods (3.30) on the surface D_1

$$2\pi i\,\hat{\theta}\,\mathcal{T}(z,\hat{z}) = \pi(z,\hat{z}) \ . \tag{3.53}$$

[26]For the quintic the additional operators are of the form $\mathcal{L}^{bdry}\theta_2$, where $\mathcal{L}^{bdry} i^*\omega_1 = 0$ modulo exact 2-forms on D.

[27]Note that the equation for D_1 is a priori defined in the ambient space. However, by restriction to the hypersurface Z^* we also identify D_1 with a divisor on the hypersurface Z^*. For ease of notation we denote both the divisor of the ambient space and of the hypersurface with the same symbol D_1.

Rearranging eq.(3.52) and restricting to the critical point $\hat{z} = \eta$ one obtains an inhomogeneous Picard-Fuchs equation

$$\mathcal{L}_a^{\text{bulk}}\, T(z) = f_a(z) \tag{3.54}$$

with $T(z) = \mathcal{T}(z, \eta)$ and

$$2\pi i\, f_a(z) = \mathcal{L}_a^{\text{bdry}}\, \pi(z, \hat{z})\Big|_{\hat{z}=\eta} \ . \tag{3.55}$$

In absolute cohomology the inhomogeneous term $f_a(z)$ is due to the fact that the bulk operators $\mathcal{L}_a^{\text{bulk}}$ satisfy

$$\mathcal{L}_a^{\text{bulk}}\, \Omega = d\beta \quad \Rightarrow \quad \mathcal{L}_a^{\text{bulk}} \int_{\Gamma\in H_3(X,\mathbb{Z})} \Omega = 0 \ , \tag{3.56}$$

where d is the differential in the absolute setting. This is sufficient to annihilate the period integrals over cycles, as indicated on the right hand side of the above equation, but leads to boundary terms in the chain integral (3.33). In the absolute setting and based on Griffiths-Dwork reduction the inhomogeneous term $f_a(z)$ has been determined by a residue computation in [54]. Here we see that the functions $f_a(z)$ are different derivatives of the surface period $\pi(z, \hat{z})$, restricted to the critical point. Hence, together with the bulk Picard-Fuchs operators, the surface period determine both the critical locus (3.29) and the critical tension.

In the examples we are going to find that the inhomogeneous terms $f_a(z)$ satisfy a hypergeometric differential equation as well:

$$\mathcal{L}_a^{\text{inh}}\, f_a(z) = 0 \ . \tag{3.57}$$

The hypergeometric operators $\mathcal{L}_a^{\text{inh}}$ descend from the Picard-Fuchs operators $\mathcal{L}^{\mathcal{D}}$ of the surface, which annihilate the surface periods $\mathcal{L}^{\mathcal{D}}\pi(z, \hat{z}) = 0$.[28] Specifically, if $f_a(z)$ is non-zero, the operator $\mathcal{L}_a^{\text{inh}}$ can be defined as

$$\mathcal{L}_a^{\text{inh}} = \Big(\mathcal{L}^{\mathcal{D}} + [\mathcal{L}_a^{\text{bdry}}, \mathcal{L}^{\mathcal{D}}]\, \mathcal{L}_a^{\text{bdry}\,-1}\Big)_{\hat{z}=\eta} \ , \tag{3.58}$$

where the operators on the right hand side are restricted to the critical point as indicated.

It follows from the above that the inhomogeneous terms $f_a(z)$ can be written as an infinite hypergeometric series in the closed-string moduli. However, on general grounds the $f_a(z)$ need to be well-defined over the open/closed moduli space, which simplifies *on-shell* to a finite cover of the complex structure moduli space $\mathcal{M}_{CS}(Z^*)$ of the three-fold [73]. This implies that the hypergeometric series $f_a(z)$ can be written as rational functions in the closed string moduli and the roots of the extra equations defining the curves C.[29]

In the examples of chapter 4 we observe that already the leading terms of the surface periods $\pi(z, \hat{z})$ become rational functions at the special symmetric points on the Noether-Lefschetz

[28] For simplicity we suppress an index for distinguishing several Picard-Fuchs operators $\mathcal{L}^{\mathcal{D}}$.

[29] We are grateful to Johannes Walcher for explaining to us this property of the inhomogeneous terms and for pointing out the results of [161] on this issue.

locus $\mathcal{N}$ in this sense. Hence there appears to be a connection between the enhancement of the Picard-lattice of the surface at these points, rationality of its periods and D-brane vacua. The rationality property is preserved when acting with $\mathcal{L}^{\text{bdry}}$ in (3.55) to obtain the inhomogeneous term f_a. In the examples we verify, that the contribution $f_a(C_{\alpha_\ell})$ of a particular boundary curve C_{α_ℓ} to the inhomogeneous term can be written in closed form as follows:

$$f_a(C_{\alpha_\ell}) \;=\; \frac{p_a(\psi,\alpha)}{q_a(\psi,\alpha)}|_{\alpha=\alpha_\ell(\psi)} \;=\; \frac{g_a(\psi,\alpha)}{\prod_i \Delta_l(C)^{\gamma_i^a}}|_{\alpha=\alpha_\ell(\psi)}\,, \tag{3.59}$$

where p_a, q_a are polynomials in the variables (ψ,α). Here $\psi=\psi(z)$ is a short-hand for the fractional power of the closed string moduli z appearing in the defining equation of the hypersurface Z^* and $\{\alpha_\ell\}$ are the roots of the extra equations defining the curves, with the root α_ℓ corresponding to the component C_{α_ℓ}. Moreover, the zeros of the denominator appear only at the zeros of the components $\Delta_i(C)$ of the open-string discriminant, where different roots/curves coincide for special values of the moduli ψ. The exponents γ_i^a are some constants and $g_a(\psi,\alpha)$ some functions without singularities in the interior of the moduli space.

3.5. Gauss-Manin connection and integrability conditions

The Picard-Fuchs system for the relative periods derived in the previous section captures the variation of mixed Hodge structure on the relative cohomology group $H^3(Z^*,\mathcal{D})$. In this section we work out some relations and predictions of mirror symmetry for a family of A- and B-branes from the associated Gauss-Manin connection.

3.5.1. Gauss-Manin connection on the open/closed deformation space $\mathcal{M}$

Geometrically we can view $H^3(Z^*,\mathcal{D})$ as the fiber of a complex vector bundle over the open/closed deformation space $\mathcal{M}$. As the relative cohomology group[30] H^3 depends only on the topological data, the fiber is up to monodromy constant over $\mathcal{M}$, and there is a trivially flat connection, ∇, called the Gauss-Manin connection. The Hodge decomposition $H^3=\oplus_{p=0}^3 H^{3-p,p}$ varies over $\mathcal{M}$, as the definition of the Hodge degree depends on the complex structure. The Hodge filtrations F^p

$$H^3=F^0\supset F^1\supset F^2\supset F^3\supset F^4=0\,,\qquad F^p=\oplus_{q\geq p}H^{q,3-q}\subset H^3\,,$$

define holomorphic subbundles $\mathcal{F}^p$ whose fibers are the subspaces $F^p\subset H^3$. The action of the Gauss-Manin connection ∇ on these subbundles has the property $\nabla(\mathcal{F}^p)\subset\mathcal{F}^{p-1}\otimes T^*_{\mathcal{M}}$, known as Griffiths transversality.

Concretely, the mixed Hodge structure on the relative cohomology space $H^3(Z^*,\mathcal{D})$ looks as follows. The Hodge filtrations are

$$F^3= \qquad H^{3,0}(Z^*,\mathcal{D}) \qquad = H^{3,0}(Z^*)\,,$$

[30]Letters without arguments refer to relative cohomology over $\mathbb{C}$, e.g. $H^3=H^3(Z^*,\mathcal{D};\mathbb{C})$.

$$\begin{aligned} F^2 = & \ F^3 \oplus H^{2,1}(Z^*,\mathcal{D}) & = F^3 \oplus H^{2,1}(Z^*) \oplus H^{2,0}_{\text{var}}(\mathcal{D}) \ , \\ F^1 = & \ F^2 \oplus H^{1,2}(Z^*,\mathcal{D}) & = F^2 \oplus H^{1,2}(Z^*) \oplus H^{1,1}_{\text{var}}(\mathcal{D}) \ , \\ F^0 = & \ F^1 \oplus H^{0,3}(Z^*,\mathcal{D}) & = F^1 \oplus H^{0,3}(Z^*) \oplus H^{0,2}_{\text{var}}(\mathcal{D}) \ , \end{aligned} \tag{3.60}$$

where the equations to the right display the split $H^3(Z^*,\mathcal{D}) \simeq \ker\big(H^3(Z^*) \to H^3(\mathcal{D})\big) \oplus \text{coker}\,\big(H^2(Z^*) \to H^2(\mathcal{D})\big)$. The weight filtration is defined as

$$W_2 = 0 \ , \qquad W_3 = H^3(Z^*) \ , \qquad W_4 = H^3(Z^*,\mathcal{D}) \ ,$$

such that the quotient spaces $W_3/W_2 \simeq H^3(Z^*)$ and $W_4/W_3 \simeq H^2(\mathcal{D})$ define pure Hodge structures. Variations in the closed (δ_z) and open ($\delta_{\hat{z}}$) string sector act schematically as

$$\begin{array}{ccccccc} F^3 \cap W_3 & \xrightarrow{\delta_z} & F^2 \cap W_3 & \xrightarrow{\delta_z} & F^1 \cap W_3 & \xrightarrow{\delta_z} & F^0 \cap W_3 \\ & \searrow^{\delta_{\hat{z}}} & & \searrow^{\delta_{\hat{z}}} & & \searrow^{\delta_{\hat{z}}} & \\ & & F^2 \cap (W_4/W_3) & \xrightarrow{\delta_z,\delta_{\hat{z}}} & F^1 \cap (W_4/W_3) & \xrightarrow{\delta_z,\delta_{\hat{z}}} & F^0 \cap (W_4/W_3) \end{array} \tag{3.61}$$

The variation of the Hodge structure over $\mathcal{M}$ can be measured by the period matrix

$$\underline{\Pi}^{\Sigma}_A = \int_{\gamma_\Sigma} \alpha_A \, , \qquad \alpha_A \in H^3 \ ,$$

where γ_Σ is a fixed topological basis for $H_3(Z^*,\mathcal{D})$ and $\{\alpha_A\}$ with $A = 1, ..., \dim(H^3)$ denotes a basis of relative 3-forms. One may choose an ordered basis $\{\alpha_A^{(q)}\}$ adapted to the Hodge filtration, such that the subsets $\{\alpha_A^{(q')}\}$, $q' \leq q$ span the spaces F^{3-q} for $q = 0, ..., 3$.

To make contact between the Hodge variation and the B-model defined at a point $m \in \mathcal{M}$, the Gauss-Manin connection has to be put into a form compatible with the chiral ring properties of a SCFT. Chiral operators of definite $U(1)$ charge are identified with forms of definite Hodge degree, which requires a projection onto the quotient spaces F^p/F^{p+1} at the point m. Moreover, the canonical CFT coordinates t_a, centered at $m \in \mathcal{M}$, should flatten the connection ∇ and we require

$$\nabla_a \alpha_A^{(q)}(m) = \partial_{t_a} \alpha_A^{(q)}(m) \stackrel{!}{=} (C_a(t) \cdot \alpha_A^{(q)})(m) \ \in F^{3-q-1}/F^{3-q}|_m \ . \tag{3.62}$$

The second equation is an important input as it expresses the non-trivial fact, that in the CFT, an infinitesimal deformation in the direction t_a is generated by an insertion of (the descendant) of a chiral operator $\phi_a^{(1)}$ in the path integral, which in turn can be described by a naive multiplication by the operator $\phi_a^{(1)}$ represented by the connection matrix $C_a(t)$. The above condition assumes that such a simple relation holds on the full open/closed deformation space for all deformations in F^2/F^3. Thus $\phi_a^{(1)}$ can be either a bulk field of left-right $U(1)$ charge $(1,1)$ or a boundary operator of total $U(1)$ charge 1. The consistency of the results obtained below with this ansatz and the correct matching with the CFT deformation space, discussed in section 3.6, provides evidence in favor of a proper CFT realization of this structure.

Phrased differently, we consider the $\alpha_A^{(q)}$ as flat sections of an "improved" flat connection D_a in the sense of [87, 22]

$$D_a\alpha_A^{(q)}=0, \qquad D_a=\partial_{z_a}-\Gamma_a(z)-C_a(z), \qquad [D_a,D_b]\;=\;0\;,$$

where z_a are local coordinates on $\mathcal{M}$, the connection terms $\Gamma_a(z)$ and $C_a(z)$ are maps from $\mathcal{F}^{3-q}$ to $\mathcal{F}^{3-q}$ and $\mathcal{F}^{3-q-1}$, respectively, and $\Gamma_a(z)$ vanishes in the canonical CFT coordinates t_a.[31]

Instead of working in generality, we study the Gauss-Manin connection again for the relative cohomology group on the two parameter family of branes on the quintic defined by (3.41). We consider a large volume phase with moduli (2.82) defined by the following linear combination of charge vectors:

$$\tilde{l}^1=(-4;0,1,1,1,1,-1,1), \qquad \tilde{l}^2=(-1;1,0,0,0,0,1,-1)\;. \tag{3.63}$$

A complete set of differential operators derived from (3.51) is given by

$$\begin{aligned}
\mathcal{L}_1&=\theta_1^4+(4z_1(\theta_1-\theta_2)-5z_1z_2(4\theta_1+\theta_2+4))\prod_{i=1}^{3}(4\theta_1+\theta_2+i)\;,\\
\mathcal{L}_2&=\theta_2(\theta_1-\theta_2)+z_2(\theta_1-\theta_2)(4\theta_1+\theta_2+1)\;,\\
\mathcal{L}_3&=\theta_1^3(\theta_1-\theta_2)+(4z_1\prod_{i=1}^{3}(4\theta_1+\theta_2+i)+z_2\theta_1^3)(\theta_1-\theta_2)\;.
\end{aligned} \tag{3.64}$$

Computing the ideal generated by the $\mathcal{L}_k$ acting on $\underline{\Omega}$ shows that H^3 is a seven-dimensional space spanned by the multiderivatives $(1,\theta_1,\theta_2,\theta_1^2,\theta_1\theta_2,\theta_1^3,\theta_1^2\theta_2)$ of $\underline{\Omega}$. The dimensions $d_q=\dim(F^{3-q}/F^{3-q+1})$ are $1,2,2,2$ for $q=0,1,2,3$, respectively. The $d_1=2$ directions tangent to $\mathcal{M}$ represent the single complex structure deformation $z=z_1z_2$ of the mirror quintic Z^* and the parameter $\hat{z}=z_2$ parametrizing the family of hypersurfaces.

To implement a CFT like structure at a point $m\in\mathcal{M}$, one may take linear combinations of the multiderivatives acting on $\underline{\Omega}$ to obtain ordered bases $\{\alpha_A^{(q)}\}$ and $\{\gamma_\Sigma\}$ which bring the period matrix into a block upper triangular form[32]

$$\underline{\Pi}_A^\Sigma\;=\;\begin{pmatrix}1 & * & * & *\\ 0 & 1\!\!1_{d_1\times d_1} & * & *\\ 0 & 0 & 1\!\!1_{d_2\times d_2} & *\\ 0 & 0 & 0 & 1\!\!1_{d_3\times d_3}\end{pmatrix}\;. \tag{3.65}$$

[31]See section 2.6 of [22] for the definition of canonical coordinates from the (closed-string) CFT point of view.

[32]It is understood, that all entries in the following matrices are block matrices operating on the respective subspaces of definite $U(1)$ charge, with dimensions determined by the numbers d_q.

Griffiths transversality then implies that in the local coordinates at a point of maximal unipotent monodromy

$$\begin{pmatrix}\nabla\alpha_1^{(0)}\\ \nabla\alpha_2^{(1)}\\ \nabla\alpha_3^{(1)}\\ \nabla\alpha_4^{(2)}\\ \nabla\alpha_5^{(2)}\\ \nabla\alpha_6^{(3)}\\ \nabla\alpha_7^{(3)}\end{pmatrix} = \begin{pmatrix} 0 & \sum_{a=1}^2 \frac{dz_a}{z_a} M_a^{(1)} & 0 & 0 \\ 0 & 0 & \sum_{a=1}^2 \frac{dz_a}{z_a} M_a^{(2)} & 0 \\ 0 & 0 & 0 & \sum_{a=1}^2 \frac{dz_a}{z_a} M_a^{(3)} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix}\alpha_1^{(0)}\\ \alpha_2^{(1)}\\ \alpha_3^{(1)}\\ \alpha_4^{(2)}\\ \alpha_5^{(2)}\\ \alpha_6^{(3)}\\ \alpha_7^{(3)}\end{pmatrix} , \qquad (3.66)$$

where the moduli-dependent matrices $M_a^{(q)}$ of dimension $d_{q-1} \times d_q$ are derivatives of the entries of $\underline{\Pi}$ in (3.65). The above expression is written in logarithmic variables $\log(z_a)$, anticipating the logarithmic behavior of the periods at the point of maximal unipotent monodromy centered at $z_a = 0$. In local coordinates x_a centered at a generic point $m \in \mathcal{M}$, the periods are analytic in x_a and dz_a/z_a should be replaced with dx_a. The left upper block can be brought into the form

$$\sum_{a=1}^{2} \frac{dz_a}{z_a} M_a^{(1)} = (\frac{dq_1}{2\pi i\, q_1}, \frac{dq_2}{2\pi i\, q_2})$$

by the variable transformation

$$q_a(z) = \exp(2\pi i\, \underline{\Pi}_1^{a+1}(z)) \ . \qquad (3.67)$$

It has been proposed in [47, 46] that (3.67) represents the mirror map between the A-model Kähler coordinates $t_a = \frac{1}{2\pi i} \ln(q_a)$ on the open/closed deformation space of an A-type compactification (Z, L) and the coordinates z_a on the complex structure moduli space of an B-type compactification (Z^*, E) near a large complex structure point. We propose that the above flatness conditions defines more generally the mirror map between the open/closed deformation spaces for any point $m \in \mathcal{M}$. It is worth stressing that the mirror map defined by the above flatness argument coincides with the mirror map obtained earlier in [142, 162] for non-compact examples by a physical argument, using domain wall tensions and the Ooguri-Vafa expansion at a large complex structure point. This coincidence can be viewed as experimental evidence for the existence of a more fundamental explanation of the observed flat structure from the underlying topological string theory, as advocated for in this chapter.

Identifying $\alpha_1^{(0)}$ with the unique operator $\phi^{(0)} = 1$ and the $\alpha_{a+1}^{(1)}$ with the charge one operators $\phi_a^{(1)}$ associated with the flows parametrized by $\log(q_a)$, (3.66) implements the CFT relation

$$\nabla_{q_a} \phi^{(0)} = \phi_a^{(1)} = \phi_a^{(1)} \cdot \phi^{(0)} \ ,$$

discussed below (3.62). The above series of arguments and manipulations is standard material in closed string mirror symmetry and led to the deep connection between the geometric Hodge variations of Calabi-Yau three-folds in the B-model and A-model quantum cohomology on the mirror.[33] After the variable transformation (3.67) and restricting to the subspace $H^3(Z^*)$ describing the complex moduli space $\mathcal{M}_{CS}$ of the mirror quintic, (3.66) becomes

$$\begin{pmatrix} \nabla\alpha_{cl}^{(0)} \\ \nabla\alpha_{cl}^{(1)} \\ \nabla\alpha_{cl}^{(2)} \\ \nabla\alpha_{cl}^{(3)} \end{pmatrix} = \begin{pmatrix} 0 & \frac{dq}{2\pi i\, q} & 0 & 0 \\ 0 & 0 & C(q)\frac{dq}{2\pi i\, q} & 0 \\ 0 & 0 & 0 & -\frac{dq}{2\pi i\, q} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{cl}^{(0)} \\ \alpha_{cl}^{(1)} \\ \alpha_{cl}^{(2)} \\ \alpha_{cl}^{(3)} \end{pmatrix} \tag{3.68}$$

with $\alpha_{cl}^{(p)} \in H^{3-p,p}(Z^*,\mathbb{C})$ and $q = q_1 q_2 = e^{2\pi i t}$. Under mirror symmetry these data get mapped to the Kähler volume t and the so-called Yukawa coupling $C(q) = 5 + \mathcal{O}(q)$, which describes the classical intersection and the Gromov-Witten invariants on the quintic. In the CFT, the quantities $C(q)$ represent the moduli-dependent structure constants of the ring of chiral primaries defined in [21].

The point which we are stressing here is that at least part of these concepts continue to make sense for the Hodge variation (3.66) on the full relative cohomology space $H^3(Z^*,\mathcal{D})$ over the open/closed deformation space $\mathcal{M}$ fibered over $\mathcal{M}_{CS}$. More importantly, the Hodge theoretic definition of mirror symmetry described above gives correct results for the open string analogues of the Gromov-Witten invariants in those cases, where results have been obtained by different methods, such as space-time arguments involving domain walls [142, 162].[34]

In this sense, the existence of a flat structure observed above, and the agreement of the Hodge theoretic results with other methods, if available, urges for a proper CFT description of the deformation families defined over $\mathcal{M}$ and an appropriate open-string extension of A-model quantum cohomology. In the following we collect further evidence in favor of an interesting integrable structure on the open/closed deformation space, working in the B-model.

3.5.2. Integrability conditions

The correlation functions of the topological family of closed-string CFTs satisfy the famous WDVV integrability condition [82, 83]. In the context of the B-model on a Calabi-Yau three-fold, this condition becomes part of the $\mathcal{N}=2$ special Kähler geometry of the complex structure moduli space, which implies, amongst others, the existence of a single holomorphic prepotential $\mathcal{F}$ that determines all entries of the period matrix in the canonical CFT coordinates t_a.

There exists no prepotential for the period matrix (3.65) on the relative cohomology group $H^3(Z^*,\mathcal{D})$, but certain aspects of the $\mathcal{N}=2$ special geometry of the closed-string sector $H^3(Z^*) \subset H^3(Z^*,\mathcal{D})$ generalize to the larger cohomology space, justifying the term $\mathcal{N}=1$

[33] See [35, 93, 94] for background material and a comprehensive list of references.
[34] See [49, 47, 46, 163, 103, 1, 104] for various examples.

special geometry [48, 47, 46].[35] Some aspects of this $\mathcal{N} = 1$ special geometry have been worked out for non-compact Z^* in [47, 46, 163] and we add here some missing pieces for the compact case. In the following we work at a "large complex structure point" $m_0 \in \mathcal{M}$ of maximal unipotent monodromy. The existence of such points m_0 follows from the general property of the GKZ systems.

We start from the following general ansatz for the 7-dimensional period vector of the holomorphic 3-form

$$\underline{\Pi}_0^\Sigma = \big(1,\ t,\ \hat{t},\ F_t(t),\ W(t,\hat{t}),\ F_0(t),\ T(t,\hat{t})\ \big)\ , \tag{3.69}$$

where t is the closed- and $\hat{t}$ the open-string deformation, related to the flat normal crossing divisor coordinates $(t_1(z_1,z_2), t_2(z_1,z_2))$ of (3.67) by the linear transformation $t = t_1 + t_2$, $\hat{t} = t_2$. The subset of periods in the closed-string sector is determined by the prepotential $\mathcal{F}$ as $(1, t, F_t = \partial_t\mathcal{F}, F_0(t) = 2\mathcal{F}(t) - t\partial_t\mathcal{F})$ and depends only on t. The additional periods $(\hat{t}, W(t,\hat{t}), T(t,\hat{t}))$ are so far arbitrary functions, except that the leading behavior at $z_a = 0$ is, schematically,

$$t,\hat{t} \sim \log(z_.), \qquad F_t, W \sim \log^2(z_.), \qquad F_0, T \sim \log^3(z_.)\ .$$

The function W is in some sense the closest analogue of the closed string prepotential and indeed has been conjectured to be a generating function for the open-string disc invariants in [47, 46, 103].

For an appropriate choice of basis $\{\alpha_A^{(q)}\}$, the period matrix takes the upper triangular form (3.65) with entries

$$(\underline{\Pi}) = \begin{pmatrix} 1 & t & \hat{t} & F_t & W & F_0 & T \\ 0 & 1 & 0 & F_{t,t} & W_{,t} & F_{0,t} & T_{,t} \\ 0 & 0 & 1 & 0 & W_{,\hat{t}} & 0 & T_{,\hat{t}} \\ 0 & 0 & 0 & 1 & 0 & -t & \mu \\ 0 & 0 & 0 & 0 & 1 & 0 & \rho \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\ , \tag{3.70}$$

where the derivatives w.r.t. t and $\hat{t}$ are denoted by subscripts and the functions μ and ρ are defined by

$$\mu = \frac{W_{,t\hat{t}}T_{,tt} - W_{,tt}T_{,t\hat{t}}}{CW_{,t\hat{t}}}, \qquad \rho = \frac{T_{,t\hat{t}}}{W_{,t\hat{t}}}, \qquad C = \mathcal{F}_{,ttt} = F_{t,tt}\ .$$

[35] $\mathcal{N} = 1, 2$ denotes the number of 4d space-time supersymmetries of the Calabi-Yau compactification of the physical type II string to four dimensions with and without branes.

The connection matrices read

$$M_t = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & W_{,tt} & 0 & 0 \\ 0 & 0 & 0 & 0 & W_{,t\hat{t}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & \mu_t \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_{,t} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; M_{\hat{t}} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & W_{,t\hat{t}} & 0 & 0 \\ 0 & 0 & 0 & 0 & W_{,\hat{t}\hat{t}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_{,\hat{t}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_{,\hat{t}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{3.71}$$

The integrability condition $\partial_t M_{\hat{t}} - \partial_{\hat{t}} M_t + [M_{\hat{t}}, M_t] = 0$ implies that

$$\rho(t,\hat{t}) = aW_{,\hat{t}} + b \; , \qquad \mu(t,\hat{t}) = aC^{-1}(t)\left(\int (W^2_{,t\hat{t}} - W_{,tt}W_{,\hat{t}\hat{t}})\, d\hat{t} + g(t)\right) \; , \tag{3.72}$$

with a, b some complex constants and $g(t)$ an undetermined function. The relation (3.62) then implies that the period T is of the form

$$T(t,\hat{t}) = \int \left(\frac{a}{2} W^2_{,\hat{t}} + bW_{,\hat{t}}\right) d\hat{t} + f(t) \; , \tag{3.73}$$

with $\partial_t^2 f(t) = g(t)$. The integrability condition (3.73) determines the top period in the open-string sector in terms of the other periods, up to the function $f(t)$. In this sense it is similar to the relation in the closed-string sector, that determines the top period F_0 in terms of the other periods. The integration constants can be fixed by determining the leading behavior of the periods in the large volume limit, as we will do in Appendix D for the quintic example.

The above argument and the integrability relation (3.73) applies to any two parameter family with one closed- and one open-string modulus and can be straightforwardly generalized to more parameter cases. For a given geometry, such as the quintic family described by the operators $\mathcal{L}_k$ in (3.64), one can of course reach the same conclusion by studying the explicit solutions and also determine the function $f(t)$. As noticed below (3.49), the relative forms in the open-string sector of this family can be associated with the Hodge variation on a quartic K3 surface. The period vector $\vec{\pi}$ of the K3 surface is spanned by the solutions $\partial_{\hat{t}}(\hat{t}, W, T) = (1, W_{,\hat{t}}, T_{,\hat{t}})$ and the integrability condition (3.73) represents an algebraic relation $\vec{\pi}^T\, \hat{\eta}\, \vec{\pi} = 0$ amongst the K3 periods, where $\hat{\eta}$ is the intersection matrix. We will come back to the intersection form $\hat{\eta}$ when we discuss the topological metric in section 3.8.

One has to note that the above discussion was essentially independent of the choice of the large complex structure point m_0 and a similar argument for other m shows that the integrability condition (3.73) holds for any $m \in \mathcal{M}$ in the local coordinates defined by (3.67).

A cautious remark

The similarities of the above arguments with those in the case of closed string mirror symmetry may have obscured the fact that one crucial datum is still incomplete: the topological metric η on the open/closed state space. In the closed-string sector, the topological metric $\eta_{\rm cl}$ is given by the classical intersection matrix on $H_3(Z^*)$ and its knowledge permits, amongst others, the determination of the true geometric periods as a particular linear combinations of the solutions to the Picard-Fuchs equations. More importantly, the topological metric is needed to complete the argument that identifies $C(q)$ in (3.68) with the structure constants of the chiral ring of [21], as well as to access the non-holomorphic and higher genus sector of the theory using the tt^*-equations [87] and the holomorphic anomaly equation [22].

3.6. Relation to CFT correlators

As discussed in the last section the structure of the Gauss-Manin connection, which is associated to the variation of the mixed Hodge structure, hints at a worldsheet interpretation. In this section we elaborate further on this aspect and point out the relation between relative periods and CFT correlators. In particular we motivate the obstruction of the D7-brane moduli space by D5-branes from the worldsheet point of view.

3.6.1. Open/closed observables

The relevant closed-string observables in the BRST cohomology of the topological B-model of a Calabi-Yau manifold are locally given by [20]

$$\phi^{(p)} = \phi^{(p)}{}_{\bar\imath_1\cdots\bar\imath_p}^{j_1\cdots j_p}\eta^{\bar\imath_1}\cdots\eta^{\bar\imath_p}\theta_{j_1}\cdots\theta_{j_p}\ , \tag{3.74}$$

where the worldsheet fermions, $\eta^{\bar\imath} = \psi_+^{\bar\imath} + \psi_-^{\bar\imath}$ and $\theta_i = g_{i\bar\jmath}\left(\psi_+^{\bar\jmath} - \psi_-^{\bar\jmath}\right)$, are sections of the pullbacks of the anti-holomorphic tangent bundle and the holomorphic cotangent bundle of target-space Calabi-Yau manifold. For the Calabi-Yau three-fold Z^* these observables $\phi^{(p)}$ are identified geometrically with representatives in the sheaf cohomology groups

$$\phi^{(p)} \in H^p(Z^*, \Lambda^p TZ^*) \simeq H^{(3-p,p)}(Z^*)\ , \qquad p = 0,1,2,3\ . \tag{3.75}$$

The last identification is due to the contraction with the unique holomorphic (3,0) form of the Calabi-Yau three-fold Z^*. The integer p represents the left and right $U(1)$ charge of the bulk observable $\phi^{(p)}$.

The local open-string observables for a worldsheet with B-type boundary are analogously given by

$$\hat\phi^{(p+q)} = \hat\phi^{(p+q)}{}_{\bar\imath_1\cdots\bar\imath_p}^{j_1\cdots j_q}\eta^{\bar\imath_1}\cdots\eta^{\bar\imath_p}\theta_{j_1}\cdots\theta_{j_q}\ . \tag{3.76}$$

In the absence of a background gauge field on the worldvolume of the brane the fermionic modes θ_j vanish along Neumann directions whereas the fermionic modes $\eta^{\bar\imath}$ vanish along Dirichlet directions on the boundary of the worldsheet [164]. Hence, locally we view the fermionic modes θ_j as sections of the normal bundle and the fermionic modes $\eta^{\bar\imath}$ as sections of the anti-holomorphic cotangent bundle of the brane. With background fluxes on the brane worldvolume the boundary conditions become twisted and obey [51]

$$\theta_i = F_{i\bar{\jmath}}\eta^{\bar{\jmath}} \ . \tag{3.77}$$

In [141] it is explicitly demonstrated that the observables (3.76) in the BRST cohomology of the open-string sector for a brane E arise geometrically as elements of the extension groups

$$\hat{\phi}^{(p+q)} \in \mathrm{Ext}^{p+q}(E,E) \ , \qquad p+q = 0,1,2,3 \ . \tag{3.78}$$

In the present context, the integer $p+q$ is equal to the total $U(1)$-charge of the open-string observable $\hat{\phi}^{(p+q)}$.

Deformations of the topological B-model are generated by the marginal operators, which correspond to BRST observables with $U(1)$-charge one, and hence they appear in the cohomology groups $H^{(2,1)}(Z^*)$ and $\mathrm{Ext}^1(E,E)$ for the closed and open deformations, respectively.

In order to make contact with the Hodge filtration of $H^3(Z^*,\mathcal{D})$ we interpret the divisor $\mathcal{D}$ of the Calabi-Yau three-fold Z^* as the internal worldvolume of a B-type brane. For a divisor the extension groups (3.78) simplify [141], and in particular $\mathrm{Ext}^1(\mathcal{D},\mathcal{D})$ reduces to $H^0(\mathcal{D},N\mathcal{D}) \simeq H^{(2,0)}(\mathcal{D})$, where the last identification results again from the contraction with the holomorphic (3,0)-form. For our particular example the cohomology groups $H^{(2,1)}(Z^*)$ and $H^{(2,0)}(\mathcal{D}_1)$ are both one-dimensional and therefore are generated by the closed- and open-string marginal operators ϕ and $\hat{\phi}$

$$\phi^{(1)} \in H^{(2,1)}(Z^*) \subset F^2/F^3 \ , \qquad \hat{\phi}^{(1)} \in H^{(2,0)}(\mathcal{D}_1) \subset F^2/F^3 \ .$$

Due to the identification $F^2/F^3 = H^{(2,1)}(Z^*,\mathcal{D}_1) \simeq H^{(2,1)}(Z^*) \oplus H^{(2,0)}(\mathcal{D}_1)$ we observe that the infinitesimal deformations $\nabla_t \alpha_1^{(0)} \sim \phi^{(1)}$ and $\nabla_{\hat{t}} \alpha_1^{(0)} \sim \hat{\phi}^{(1)}$ in (3.62) precisely agree with the closed and open marginal operators $\phi^{(1)}$ and $\hat{\phi}^{(1)}$. As a consequence the discussed Picard-Fuchs equations, governing the Hodge filtration F^p, describe indeed the deformation space associated to the closed and open marginal operators $\phi^{(1)}$ and $\hat{\phi}^{(1)}$.

3.6.2. Obstructions and CFT correlators

In the presence of B-type boundaries infinitesimal deformations are generically obstructed at higher order. These obstructions are encoded in the moduli-dependent superpotential generated by disc correlators with insertions of bulk and boundary marginal operators [165, 166, 167, 140]. The relevant disc correlators arise from non-trivial ring relations involving marginal operators and the (unique) boundary top element $\hat{\phi}^{(3)} \in \mathrm{Ext}^3(\mathcal{D},\mathcal{D})$. Hence the superpotential is extracted by identifying the element $\hat{\phi}^{(3)}$ in the relative cohomology group $H^3(Z^*,\mathcal{D})$. For the family of hypersurfaces $\mathcal{D}$ the extension group $\mathrm{Ext}^3(\mathcal{D},\mathcal{D})$ becomes [141]

$$\hat{\phi}^{(3)} \in \mathrm{Ext}^3(\mathcal{D},\mathcal{D}) \simeq H^2(\mathcal{D},N\mathcal{D}) \simeq H^{(2,2)}(\mathcal{D}) \ ,$$

where locally $\hat{\phi}^{(3)} = \hat{\phi}^{(3)}{}_{\bar{\imath}\bar{\jmath}}^{k}\eta^{\bar{\imath}}\eta^{\bar{\jmath}}\theta_k$. It is obvious that the cohomology group $H^{(2,2)}(\mathcal{D})$ does not appear in the filtration F^p of the relative cohomology group $H^3(Z^*, \mathcal{D})$. On the other hand the variation of mixed Hodge structure encodes by construction the ring relations of the observables generated by the marginal operators $\phi^{(1)}$ and $\hat{\phi}^{(1)}$. Therefore we conclude that these marginal operators do not generate the boundary-boundary top element $\hat{\phi}^{(3)}$. Thus the analyzed deformation problem is unobstructed and does not give rise to a non-vanishing superpotential.

From a physics point of view the family of divisors $\mathcal{D}$ describes a family of holomorphic hypersurfaces, which all give rise to supersymmetric B-brane configurations, and hence we should not expect any obstructions resulting in a superpotential.

However, the result of the above analysis drastically changes as we add a D5-brane charge on 2-cycles in $\mathcal{D}$, e.g. by adding non-trivial background fluxes on the worldvolume of the B-type brane. From a space-time perspective [143, 168], we expect the appearance of F-terms precisely for those two-form background fluxes, whose field strength takes values in the variable cohomology of the hypersurface $\mathcal{D}$

$$F \in \text{coker}\,\left(H^2(Z^*, \mathbb{Z}) \to H^2(\mathcal{D}, \mathbb{Z})\right)\,. \tag{3.79}$$

These fluxes induce a macroscopic superpotential [143, 168, 169, 144]

$$W = \int_D F \wedge \omega = \int_\Gamma F \wedge \Omega\,, \tag{3.80}$$

where $\omega \in H^{2,0}(D)$ is obtained by contracting the bulk (3,0)-form Ω with a section of the normal bundle to D. The second expression, derived in a more general context in [144], is equivalent to the first one for an appropriate choice of 5-chain with boundary D.

In the microscopic worldsheet description the worldvolume flux $F_{i\bar{\jmath}}$ yields twisted boundary conditions (3.77), and the fermionic modes θ_i of the open-string observables (3.76) are in general no longer sections of the normal bundle $N\mathcal{D}$. Instead they should be viewed as appropriate section in the restricted tangent bundle, $TZ^*|_{\mathcal{D}}$ [141]. As a consequence we can trade (without changing the $U(1)$ charge) fermionic modes $\eta^{\bar{\jmath}}$ with appropriate fermionic modes θ_i. As a result the boundary top element $\hat{\phi}^{(3)}$ can now be associated with an element in the variable two-form cohomology

$$\text{Ext}^3(\mathcal{D}, \mathcal{D}) \ni \hat{\phi}^{(3)k}_{\bar{\imath}\bar{\jmath}}\eta^{\bar{\imath}}\eta^{\bar{\jmath}}\theta_k \overset{F_{i\bar{\jmath}}}{\longleftrightarrow} \hat{\phi}^{(3)jk}_{\bar{\imath}}\eta^{\bar{\imath}}\theta_j\theta_k \overset{\Omega_{ijk}}{\longleftrightarrow} \hat{\phi}^{(3)}_{i\bar{\jmath}}\, dx^i \wedge dx^{\bar{\jmath}} \in \text{coker}\,\left(H^2(Z^*) \to H^2(\mathcal{D})\right)\,. \tag{3.81}$$

Thus in the presence of worldvolume background fluxes the boundary top element $\hat{\phi}^{(3)}$ *does* correspond to an element in the Hodge structure filtration F^p, and the superpotential is described by a solution of the Picard-Fuchs equations. In this way the a priori unobstructed deformation problem of divisors $\mathcal{D}$ in the Calabi-Yau three-fold is capable to describe superpotentials associated to D5-brane charges in $H_2(\mathcal{D})$ [47, 46, 103, 1].

On the other hand, since the discussed F-term fluxes (3.79) are elements of the variable cohomology of the hypersurface $\mathcal{D}$, *i.e.* the field strength of the fluxes can be extended to exact two forms in the ambient Calabi-Yau space, they do not modify the D5-brane K-theory charges. Therefore if a suitable D5-brane interpretation is applicable the flux-induced superpotentials describe domain wall tensions between pairs of D5-branes, which wrap homologically equivalent two cycles.

When written in the flat coordinates $t_a = \frac{1}{2\pi i}\ln(q_a)$ in (3.67), the Gauss-Manin connection on the total cohomology space takes the form:

$$\begin{pmatrix} \nabla\alpha_1^{(0)} \\ \nabla\alpha_2^{(1)} \\ \nabla\alpha_3^{(1)} \\ \nabla\alpha_4^{(2)} \\ \nabla\alpha_5^{(2)} \\ \nabla\alpha_6^{(3)} \\ \nabla\alpha_7^{(3)} \end{pmatrix} = \sum_b \begin{pmatrix} 0 & C_b^{(0)}(q_a)\frac{dq_b}{q_b} & 0 & 0 \\ 0 & 0 & C_b^{(1)}(q_a)\frac{dq_b}{q_b} & 0 \\ 0 & 0 & 0 & C_b^{(2)}(q_a)\frac{dq_b}{q_b} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1^{(0)} \\ \alpha_2^{(1)} \\ \alpha_3^{(1)} \\ \alpha_4^{(2)} \\ \alpha_5^{(2)} \\ \alpha_6^{(3)} \\ \alpha_7^{(3)} \end{pmatrix} . \tag{3.82}$$

The most notable difference to the closed-string case (cf. (3.68)) is that, whereas the matrix $C_b^{(0)}$ still is of the canonical form $(C_b^{(0)})_1{}^l = \delta_{bl}$, the matrices $C_b^{(q)}$ are now both moduli dependent for $q = 1, 2$:

$$(C_t^{(1)}) = \begin{pmatrix} C & W_{,tt} \\ 0 & W_{,t\hat{t}} \end{pmatrix},\ (C_{\hat{t}}^{(1)}) = \begin{pmatrix} 0 & W_{,t\hat{t}} \\ 0 & W_{,\hat{t}\hat{t}} \end{pmatrix},\ (C_t^{(2)}) = \begin{pmatrix} -1 & \mu_{,t} \\ 0 & \rho_{,t} \end{pmatrix},\ (C_{\hat{t}}^{(2)}) = \begin{pmatrix} 0 & \mu_{,\hat{t}} \\ 0 & \rho_{,\hat{t}} \end{pmatrix}.$$

In correspondence with the closed-string sector it is tempting to interprete the $d_{q-1} \times d_q$ matrices $C_b^{(q)}$ as the structure constants of a ring of open and closed chiral operators

$$\phi_b^{(1)} \cdot \phi_k^{(q)} \stackrel{?}{=} (C_b^{(q)})_k{}^l \phi_l^{(q+1)} ,$$

as described in [47, 46, 170]. A rigorous CFT derivation of such a relation is non-trivial, as the Hodge variation mixes bulk and boundary operators and describes the bulk-boundary ring in the sense of [141], about which little is known in the context of topological strings (see however [85, 84]). A related complication is the need of a topological metric on the space of closed and open BRST states that mixes contributions at different order of the string coupling. The most direct way to connect the closed-string periods with a CFT quantity is the interpretation as overlap functions between boundary states and chiral operators [131], and it is likely that a similar idea can be applied to the entries of the relative period matrix. It would be interesting to make this precise. It would also be interesting to understand more generally the relation of the above concepts to the CFT results obtained from matrix factorizations in [171, 172, 173, 174].

3.7. GLSM and enhanced toric polyhedra

We observed in section 3.4 that the differential operators we derived from the variation of mixed Hodge structure on the Calabi-Yau three-fold for the case of toric branes can be interpreted as the GKZ-system of a non-compact Calabi-Yau four-fold. This reflects the open/closed duality of B-branes and Calabi-Yau four-folds of [48, 56]. In this section we are going to show how one can systematically associate a gauged linear σ-model (GLSM) to a given toric B-brane and how to extract the GKZ-system from it.

The GLSM puts the Calabi-Yau and the brane geometry on equal footing and allows to study the phases of the combined system by standard methods of toric geometry. The GLSM thus provides valuable information on the global structure of the combined open/closed deformation space which will be important for identifying and investigating the various phases of the brane geometry, in particular large volume phases as well as transitions among different phases [49, 73].

3.7.1. Construction of the open/closed GLSM

We will use the concept of toric polyhedra to define the GLSM for the mirror pairs of toric brane geometries. This approach has the advantage of giving a canonical construction of the B-model mirror to a certain A-brane geometry and provides a short-cut to derive the generalized hypergeometric system for the relative periods as given in (3.51).

As discussed in Appendix A, Batyrev's correspondence describes a mirror pair of toric hypersurfaces (Z, Z^*) by a pair of dual polyhedra $(\Delta, \Delta^\star)$. What we are proposing here is that there is a similar correspondence between "enhanced polyhedra" $(\Delta_\flat(Z, L), \Delta_\flat^\star(Z^*, E))$ and the pair (Z, Z^*) of mirror manifolds *together* with the pair of mirror branes (L, E) as defined before.

The enhanced polyhedron $\Delta_\flat(Z, L)$ has the following simple structure: The points $\nu_i(Z)$ of $\Delta(Z)$ defining the manifold Z are a subset of the points of $\Delta_\flat(Z, L)$ that lie on a hypersurface H in a five-dimensional lattice Λ_5. We choose an ordering of the points $\mu_i \in \Delta_\flat(Z, L)$ and coordinates on Λ_5 such that the points in H are given by

$$(\mu_i) = (\nu_i, 0),\ i = 1, ..., k\ ,$$

where k is the number of points of $\Delta(Z)$. The brane geometry is described by k' extra points ρ_i with $(\rho_i)_5 < 0$, where k' is related to the number $\hat{n}$ of (obstructed) moduli of the brane by $k' = \hat{n} + 1$. Thus $\Delta_\flat(Z, L)$ is defined as the convex hull of the points

$$\Delta_\flat(Z, L) = \text{conv}\left(\{\mu_i(\Delta(Z))\} \cup \{\rho_i(L)\}\right), \qquad \{\mu_i(\Delta(Z))\} \subset \Delta_\flat(Z, L) \cap H\ . \tag{3.83}$$

For simplicity we assume that the polyhedron $\Delta_\flat^\star$ can be naively defined as the dual of $\Delta_\flat$ in the sense of [109].

To make contact between the definition of the toric branes (as e.g. in Appendix A) and the extra points ρ_i, consider the linear dependences between the points of $\Delta_\flat(Z,L)$

$$\sum_i \underline{l}_i^a(\Delta_\flat)\mu_i = 0 \ . \tag{3.84}$$

These relations may be split into two sets in an obvious way. There are $h^{1,1}(Z)$ relations, say

$$(\underline{l}^a(\Delta_\flat)) = (l^a(\Delta), 0^{k'}), \ a = 1, ..., h^{1,1}(Z) \ ,$$

which involve only the first k points and reflect the original relations $l^a(\Delta)$ between the points $\nu_i(Z)$ of $\Delta(Z)$; they correspond to Kähler classes of the manifold Z. The remaining relations $\underline{l}^a(\Delta_\flat)$, $a > h^{1,1}(Z)$ involve also the extra points ρ_i. To describe a brane as defined by the charge vectors $\hat{l}^a(L)$ we choose the points ρ_i such that the remaining relations are of the form

$$(\underline{l}^a(\Delta_\flat)) = (\hat{l}^a(L), ...), \ a > h^{1,1}(Z) \ .$$

The above prescription for the construction of the enhanced polyhedron $\Delta_\flat(Z,L)$ from the polyhedron $\Delta(Z)$ for a given manifold Z is well-defined if we require a minimal extension by $k' = \hat{n}+1$ points.

3.7.2. Differential equations on the moduli space from the GLSM

The combined open/closed string deformation space of the brane geometries (Z,L) or (Z^*,E) can now be studied by standard methods of toric geometry. Let[36] $\{l_i^a\}$ denote a *specific* choice of basis for the generators of the relations (3.84) in the GLSM and a_i the coefficients of the hypersurface equation $P = \sum_i a_i y_i$ of the mirror B-model. From the homogeneous coordinates a_i on the complex moduli space one may define local coordinates associated with the choice of a basis $\{l_i^a\}$ as above by[37]

$$z_a = (-)^{l_0^a} \prod_i a_i^{l_i^a} \ , \qquad a = 1, ..., M+N \ . \tag{3.85}$$

Given the toric polyhedra we define the Picard-Fuchs system as the canonical GKZ system associated with $\Delta_\flat$.[38] By the results of [115, 109], the generalized hypergeometric system associated to $(\Delta_\flat, \Delta_\flat^\star)$ leads to the following differential operators for $a = 1, ..., M+N$:

$$\mathcal{L}_a = \prod_{k=1}^{l_0^a}(\vartheta_{a_0} - k) \prod_{l_i^a>0} \prod_{k=0}^{l_i^a-1}(\vartheta_{a_i} - k) - (-1)^{l_0^a} z_a \prod_{k=1}^{-l_0^a}(\vartheta_{a_0} - k) \prod_{l_i^a<0} \prod_{k=0}^{-l_i^a-1}(\vartheta_{a_i} - k) \ . \tag{3.86}$$

[36]The underscore on $\underline{l}^a(\Delta_\flat)$ will be dropped again to simplify notation.

[37]The sign is a priori convention but receives a meaning if the classical limit of the mirror map is fixed as in [108].

[38]We are tacitly assuming that the GKZ system $\{\mathcal{L}_a\}$ is already a complete Picard-Fuchs system, which is possibly only true after a slight modification of the GKZ system.

Here ϑ_x denotes a logarithmic derivative $\vartheta_x = x\frac{\partial}{\partial x}$ and the derivatives of the homogeneous coordinates a_i on the complex structure moduli and the local coordinates (2.82),(3.85) are related by $\vartheta_{a_i} = \sum_a l_i^a \vartheta_{z_a}$. The products are defined to run over non-negative k only so that the derivatives ϑ_{a_0} appear only in one of the two terms for given a. Therefore we obtained the GKZ-equations in (3.51) as claimed. The solutions of the Picard-Fuchs system in (3.51) have a nice expansion around $z_a = 0$; expansions around other points in the moduli space can be obtained from a change of variables.

For an appropriate choice of basis vectors l^a, given by the mori vectors of a triangulation (see below), the solutions to the GKZ system can be written down explicitly by the Frobenius method [108, 110], as already discussed in section 2.4.4. This will be important for explicit computations in chapter 4. In appropriate variables the generating functions are given by

$$B_{\{l^a\}}(z_a;\rho_a) = \sum_{n_1,\ldots,n_N\in\mathbb{Z}_0^+} \frac{\Gamma\left(1-\sum_a l_0^a(n_a+\rho_a)\right)}{\prod_{i>0}\Gamma\left(1+\sum_a l_i^a(n_a+\rho_a)\right)} \prod_a z_a^{n_a+\rho_a}\,. \tag{3.87}$$

By differentiation with respect to ρ_i and evaluation at $\rho_i = 0$ one can construct the solutions to (3.51),(3.86).

3.7.3. Phases of the GLSM and structure of the solutions

In the previous definitions we have used a specific choice of basis $\{l_i^a\}$ to define the local coordinates (2.82) and the differential operators (3.51). Different choices of coordinates correspond to different phases of the GLSM [5]. The extreme cases are on the one hand a large volume phase in all the Kähler parameters, where the GLSM describes a smooth classical geometry and on the other hand a pure Landau-Ginzburg phase. In between there are mixed phases, where only some of the moduli are at large volume and other moduli are fixed in a stringy regime of small volume. A nice instanton expansion can be expected a priori only for moduli at large volume. Representing the GLSM by the toric polyhedron $\Delta_\flat$, the different phases of the GLSM may be studied by considering different triangulations of the polyhedron [175, 109] (see figure 3.1 for an example). Without going into the technical details of this procedure, let us outline the relevance of this phase structure in the present context. A given B-brane configuration corresponds to a critical point of the superpotential which lies in a certain local patch of the parameter space. To study the critical points in a given patch and to give a nice local expansion of the superpotential it is necessary to work in the appropriate local coordinates. The different triangulations of $\Delta_\flat$ define different regimes in the parameter space, where the relative periods $\underline{\Pi}^\Sigma$ have a certain characteristic behavior depending on whether the brane moduli are at large or at small volume. To find an interesting instanton expansion we look for triangulations that correspond to patches where at least some of the moduli are at large volume.

From the interpretation of the system $\{\mathcal{L}_a\}$ of differential operators as the Picard-Fuchs system for the relative periods on Z^* we expect the solutions of the equations (**??**) to have

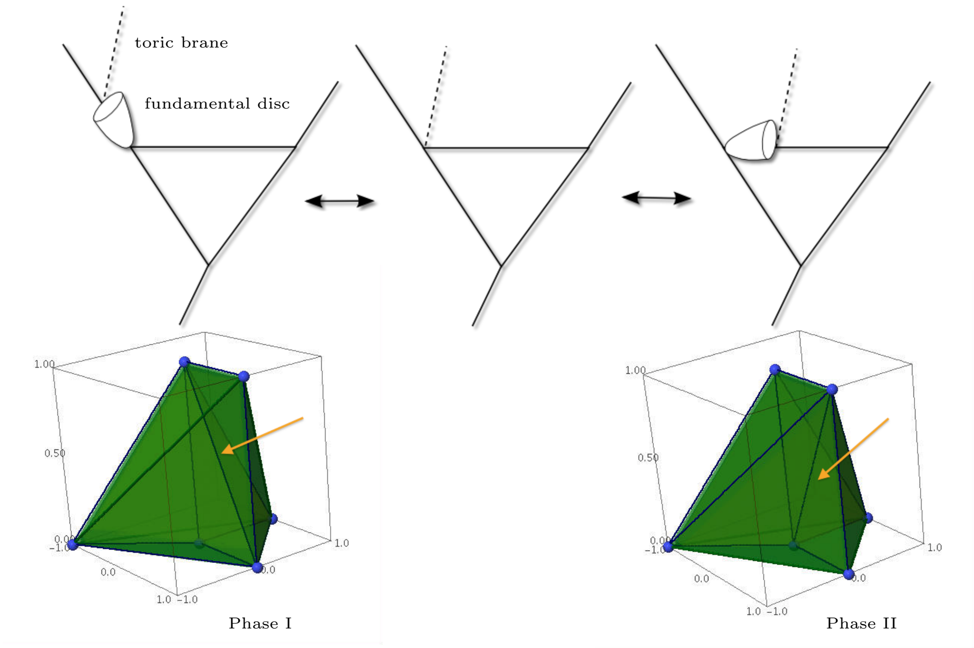

Figure 3.1.: Two different large volume triangulations of $\Delta_\flat(Z, L)$ for a toric brane on local $\mathbb{P}^2$ [49]. The two phases are connected by an open string flop transition at the marked edge [49, 73].

the following structure:

1. There are $2M + 2$ solutions $\underline{\Pi}(z)$ that represent the periods of Z^* up to linear combination and depend only on the complex structure moduli z_a, $a = 1, ..., h^{1,1}(Z)$ of Z^*.

2. There are $2N$ further solutions $\hat{\underline{\Pi}}(z, \hat{z})$ that do depend on all deformations and define the mirror map for the open string deformations and the superpotential (more precisely: brane tensions).

3.[39]For a maximal triangulation corresponding to a large complex structure point centered at $z_a = 0\ \forall a$, there will be a series solution $\omega_0(z_a) = 1 + \mathcal{O}(z_a)$ and $M + N$ solutions $\omega_c(z_a)$ with a single log behavior that define the open/closed mirror maps as (c is fixed in the following equation)

$$t_c(z_a) = \frac{\omega_c(z_a)}{\omega_0(z_a)} = \frac{1}{2\pi i} \ln(z_c) + S_c(z_a) \ ,$$

where $S_c(z_a)$ is a series in the coordinates z_a.

[39]The following holds for appropriate choices of normalization and the sign in (2.82) that have been made in (3.51), explaining the special appearance of the entry $i = 0$ corresponding to the fiber of the anti-canonical bundle.

It follows from a) that the mirror map $t^{\rm cl}(z)$ in the closed string sector does not involve the open string deformations, similarly as has been observed in [142, 162, 46, 47] in the non-compact case.[40] However the open string mirror map $t^{\rm op}(z,\hat{z})$ depends on both types of moduli. For explicit computations of the mirror maps at various points in the moduli we refer to the examples in chapter 4.

The special solution $\Pi = \mathcal{W}_{\rm brane}(z,\hat{z})$ has the further property that its instanton expansion near a large volume/large complex structure point encodes the Ooguri-Vafa invariants of the brane geometry:

$$\mathcal{W}_{\rm inst}(q_a) = \sum_\beta G_\beta q^\beta = \sum_\beta \sum_{k=1}^{\infty} N_\beta \frac{q^{k.\beta}}{k^2} \, . \tag{3.88}$$

Here β is the non-trivial homology class of a disc, $\beta \in H^2(Z,L)$, q^β a weight factor related to its appropriately defined Kähler volume, G_β the fractional Gromov-Witten type coefficients in the instanton expansion and N_β the integral Ooguri-Vafa invariants [52].

3.8. F-theory four-folds and effective $\mathcal{N}=1$ supergravity

The four-dimensional effective action for the brane compactification (Z,L), or its mirror (Z^*,E), should be described by a general $\mathcal{N}=1$ supergravity theory, which depends on the Kähler potential and the superpotential through the function [176, 177]

$$\mathcal{G} = K(z,\hat{z};\bar{z},\bar{\hat{z}}) + \ln \mathcal{W}(z,\hat{z}) + \ln \bar{\mathcal{W}}(\bar{z},\bar{\hat{z}}) \, . \tag{3.89}$$

Here $(z,\hat{z})$ denote again local complex coordinates on the bundle $\mathcal{M} \to \mathcal{M}_{CS}$. Similarly the construction of the las section associates to the mirror pair of brane compactifications (Z,L) and (Z^*,E) a dual four-fold compactification with the same supersymmetry. Some details of this duality, including a lift to a full F-theory compactification, will be explained below. Subsequently we compare the effective couplings in these two descriptions and show that they lead to a consistent proposal for the $\mathcal{N}=1$ supergravity function $\mathcal{G}(z,\hat{z};\bar{z},\bar{\hat{z}})$.

3.8.1. Four-fold dualities and F-theory

The construction of the last section associates a toric four-fold polyhedron $\Delta_\flat$ to a toric brane configuration defined as in [142]. For the quintic example, $\Delta_\flat$ is defined in (3.1) and has the property that the toric GKZ system associated with $\Delta_\flat$ reproduces the GKZ system for the relative cohomology group for the brane compactification (Z^*,E) via (3.51). In the following we describe some details of the duality map on the target and moduli spaces.

First note that the polyhedron $\Delta_\flat$ and its dual polyhedron $\Delta_\flat^\star$ actually define a mirror pair of Calabi-Yau four-folds $X_4^\sharp$ and $X_4^{\sharp *}$. As is clear from the derivation of the Picard-Fuchs equations in section 3.4, the relative periods of the brane compactification (Z^*,E) are

[40] This statement holds at zero string coupling.

identified with the periods of holomorphic $(4, 0)$-form on the four-fold $X_4^{\sharp *}$. It follows that the complex structure deformations of the brane compactification (Z^*, E) map to the complex structure of the four-fold $X_4^{\sharp *}$. Adding mirror symmetry, one obtains the following relation between the different compactifications:

$$\begin{array}{ccc} \begin{array}{c}(Z, L)\\ (\mathrm{A}-\mathrm{branes})\end{array} & \xleftrightarrow[\text{symmetry}]{\text{mirror}} & \begin{array}{c}(Z^*, E)\\ (\mathrm{B}-\mathrm{branes})\end{array} \\ {\scriptstyle 4f}\big\downarrow{\scriptstyle dual} & & {\scriptstyle 4f}\big\downarrow{\scriptstyle dual} \\ X_4^{\sharp} & \xleftrightarrow[\text{symmetry}]{\text{mirror}} & X_4^{\sharp *} \end{array} \tag{3.90}$$

The vertical maps in this diagram preserve, whereas the horizontal maps exchange, the notion of complex and Kähler moduli.

The mirror pair $(X_4^{\sharp}, X_4^{\sharp *})$ of four-folds constructed in this way has a very special geometric structure that reflects the mirror symmetry between A-type and B-type branes on the mirror pair (Z, Z^*) of Calabi-Yau three-folds. Whereas the correspondence between *moduli spaces* is manifest on the B-type side as the relation between the periods of (Z^*, E) and $X_4^{\sharp *}$,[41] there is a simple correspondence between the *target spaces* on the A-type side. Namely, the mirror four-fold $X_4^{\sharp}$ is a fibration over the complex plane

$$\begin{array}{ccc} Z & \longrightarrow & X_4^{\sharp} \\ & & \big\downarrow{\scriptstyle \pi} \\ & & \mathbb{C} \end{array} \tag{3.91}$$

with generic fiber a Calabi-Yau three-fold of type Z and a degenerate central fiber at the origin specified by the toric polyhedron constructed in [48]. Thus the dual four-fold $X_4^{\sharp *}$ that captures the relative periods of the brane compactification (Z^*, E) is effectively constructed by fibering the Calabi-Yau three-fold Z for the A-branes over $\mathbb{C}$ and then taking the (four-fold) mirror of the fibration $X_4^{\sharp}$ obtained in this way.

The mirror pair $(X_4^{\sharp}, X_4^{\sharp *})$ of non-compact Calabi-Yau four-folds can be related to a honest four-dimensional F-theory compactification by a simple $\mathbb{P}^1$ compactification of the non-compact base of $X_4^{\sharp}$. In this way one obtains a mirror pair of compact Calabi-Yau four-folds (X_4, X_4^*), where X_4^* is the four-fold for F-theory compactification.

$$\begin{array}{ccccc} Z & \longrightarrow & X_4 & \xleftrightarrow[\text{symmetry}]{\text{mirror}} & X_4^* \quad (\mathrm{F}-\mathrm{theory}) \\ & & \big\downarrow{\scriptstyle \pi} & & \\ & & \mathbb{P}^1 & & \end{array} \tag{3.92}$$

An important point is to identify the image of the large base limit of X_4 in the moduli space of the F-theory compactification on X_4^*, which can be deduced from the mirror map and the

[41] An explicit match of period integrals for a class of examples can be found in [48] and the Appendix E.

methods of [178, 179]. The result is that the large volume limit $\mathrm{Im}\, S = \mathrm{Vol}(\mathbb{P}^1) \to \infty$ maps under mirror symmetry to a weak coupling limit $g_s \to 0$

$$\mathrm{Vol}(\mathbb{P}^1) \sim 1/g_s \ \to \infty \ . \tag{3.93}$$

Thus the pair of four-folds $(X_4^\sharp, X_4^{\sharp *})$ is recovered in the decompactification/weak-coupling limit and the diagram (3.90) is completed downwards to

$$\begin{array}{ccc} X_4^\sharp & \overset{\text{mirror}}{\underset{\text{symmetry}}{\longleftrightarrow}} & X_4^{\sharp *} \\ {\scriptstyle \mathrm{Vol}(\mathbb{P}^1)\to\infty}\uparrow & & \uparrow{\scriptstyle g_s\to 0} \\ X_4 & \overset{\text{mirror}}{\underset{\text{symmetry}}{\longleftrightarrow}} & X_4^* \end{array} \tag{3.94}$$

On the one hand, the details of the $\mathbb{P}^1$ compactification determine the subleading corrections in g_s but become irrelevant in the decompactification/decoupling limit. On the other hand it is worth stressing, that the "duality" between non-compact Calabi-Yau four-folds $(X_4^\sharp, X_4^{\sharp *})$ and the mirror pair of brane compactifications (Z, L) and (Z^*, E), which underlies the superpotential computation, represents the *strict* limit $g_s = 0$, where most of the degrees of freedom decouple from the superpotential sector. A true duality can exist only at the level of the lower row of the above diagram [57].

In the example of the quintic in section 3.4, the $\mathbb{P}^1$ compactification can be obtained by adding the extra vertex

$$\tilde{\nu}_8 = (1, 0, 0, 0, -1) \tag{3.95}$$

to the toric polyhedron in Table 3.1 of the non-compact Calabi-Yau four-fold $X_4^\sharp$. This defines a compact Calabi-Yau four-fold X_4. The Kähler modulus S of the compact $\mathbb{P}^1$ base is described by the charge vector

$$\tilde{l}^3 = (0; -2, 0, 0, 0, 0, 0, 1, 1) \ .$$

The mirror manifold X_4^* is elliptically fibered and defines an F-theory compactification that will be used to compute the effective couplings in the effective four-dimensional $\mathcal{N} = 1$ supergravity theory in the following section.

3.8.2. Effective $\mathcal{N} = 1$ supergravity

According to the above discussion the $\mathcal{N} = 1$ superpotential appearing in the $\mathcal{N} = 1$ supergravity function (3.89) is

$$\mathcal{W}(z, \hat{z}) = \sum_\Sigma \underline{N}_\Sigma \int_{\gamma_\Sigma} \underline{\Omega}^{(3,0)}, \qquad \gamma_\Sigma \in H_3(Z^*, \mathcal{D}) \ , \tag{3.96}$$

if we consider the brane compactification (Z^*, E) as in [47, 46, 103, 1], or, alternatively

$$\mathcal{W}(z,\hat{z},S) = \sum_{\Sigma} N_\Sigma \int_{\gamma_\Sigma} \Omega^{(4,0)}, \qquad \gamma_\Sigma \in H_4(X^*) \ , \tag{3.97}$$

for the F-theory compactification on the dual four-fold X_4^*.[42] As discussed above, the difference between the four-fold periods and the (relative) three-fold periods are subleading corrections in small g_s.[43]

As for the Kähler potential, consider first the $\mathcal{N}=2$ Kähler potential on the base of the fibration $\mathcal{M}$, that is on the complex structure moduli space $\mathcal{M}_{CS}$ for the string compactification on Z^* without branes. This is given by [180, 50, 181]

$$K_{CS}(z;\bar{z}) = -\ln Y_{CS} \ , \qquad Y_{CS} = -i\int_{Z^*} \Omega\wedge\bar{\Omega} = -i \sum_{\gamma_\Sigma\in H_3(Z^*)} \Pi^\Sigma(z)\, \eta_{\Sigma\Lambda}\, \bar{\Pi}^\Lambda(\bar{z}) \ .$$

Here $\Sigma,\Lambda = 1,...,h^3(Z^*)$ and $\eta_{\Sigma\Lambda}$ is the symplectic intersection matrix on $H_3(Z^*,\mathbb{Z})$, which represents the constant, topological metric on the space of ground states in the SCFT [22]. Restricting the sum in (3.96) to the "flux" superpotential, that is to the absolute cohomology $H_3(Z^*)$, one obtains a function

$$\mathcal{G}_{CS} = K_{CS}(z;\bar{z}) + \ln \mathcal{W}_{CS}(z) + \ln \bar{\mathcal{W}}_{CS}(\bar{z}), \qquad \mathcal{W}_{CS}(z) = \sum_{\gamma_\Sigma\in H^3(Z^*)} N_\Sigma \Pi_\Sigma(z) \ , \tag{3.98}$$

that depends only on the closed string moduli and is invariant under Kähler transformations generated by rescalings of the holomorphic (3,0) form, $\Omega\to e^f\Omega$.

We will now give two independent arguments, that the $\mathcal{N}=1$ Kähler potential on the full $\mathcal{N}=1$ deformation space $\mathcal{M}$ can be written, to leading order in g_s, as

$$K(z,\hat{z};\bar{z},\bar{\hat{z}}) = -\ln Y \ , \qquad Y = -i \sum_{\gamma_\Sigma\in H_3(Z^*,\mathcal{D})} \underline{\Pi}^\Sigma(z,\hat{z})\, \underline{\eta}_{\Sigma\Lambda}\, \underline{\bar{\Pi}}^\Lambda(\bar{z},\bar{\hat{z}}) \tag{3.99}$$

with a pairing matrix $(\underline{\eta})$ defined below. Indeed this ansatz is a natural guess in view of the extension of the summation from $H^3(Z^*)$ to $H^3(Z^*,\mathcal{D})$ in the $\mathcal{N}=1$ superpotential (3.96) and defines an $\mathcal{N}=1$ supergravity function $\mathcal{G}(z,\hat{z};\bar{z},\bar{\hat{z}})$ which is invariant under Kähler transformations generated by rescalings of the relative (3,0) form

$$\underline{\Omega}\to e^f\underline{\Omega}, \qquad \Pi_\Sigma \to e^f \underline{\Pi}_\Sigma \ .$$

Note that since the Kähler metric for the closed and open string deformations arises from different worldsheet topologies, the pairing $(\underline{\eta})$ on $H_3(Z^*,\mathcal{D})$ necessarily mixes terms of different order in the string coupling g_s.

[42]See also [147, 99] for an early discussion of four-fold periods and superpotentials in lower dimensions.

[43]See the Appendix D for some details of the computation and a precise match between the periods in the quintic example.

The first argument comes from the results of [148] on the effective space-time action for orientifold compactifications of D7-branes. It has been shown there, that the Kähler metric obtained by dimensional reduction of a D7-brane worldvolume wrapping the orientifold plane $\mathcal{D}_\star$ in an orientifold $Z_\star$ is consistent, at first order in the brane deformation, with a Kähler potential $K = -\ln Y_{OF}$ with

$$Y_{OF} = -i\int_{Z_\star} (P^{(3)}\underline{\Omega}) \wedge (P^{(3)}\underline{\bar{\Omega}}) + \tilde{g}\int_{\mathcal{D}_\star} (P^{(2)}\underline{\Omega}) \wedge (P^{(2)}\underline{\bar{\Omega}}) \ . \tag{3.100}$$

Here $P^{(3)}$ and $P^{(2)}$ are projection operators onto the two summands in (3.18), and $\tilde{g}$ is g_s times a constant. This is of the form (3.99) with the pairing matrix

$$(\underline{\eta}) = \begin{pmatrix} \eta_{Z_\star} & 0 \\ 0 & i\tilde{g}\,\tilde{\eta}_{\mathcal{D}_\star} \end{pmatrix} \ , \tag{3.101}$$

where $\tilde{\eta}_{\mathcal{D}_\star}$ is the (symmetric) intersection matrix on $H^2_{var}(\mathcal{D}_\star)$.

The second argument is obtained by computing the Kähler potential of the dual four-fold X_4^*, which is of a similar form as (3.99) [75]:

$$K(z,\hat{z},S;\bar{z},\bar{\hat{z}},\bar{S}) = -\ln Y \ , \qquad Y = \sum_{\gamma_\Sigma \in H_4(X^*)} \Pi^\Sigma(z,\hat{z},S)\,\eta_{\Sigma\Lambda}\,\bar{\Pi}^\Lambda(\bar{z},\bar{\hat{z}},\bar{S}) \ . \tag{3.102}$$

Here η denotes the topological intersection matrix on $H_4(X_4^*)$ and S is the afore mentioned extra modulus in the compact manifold. An explicit computation[44] in the weak coupling limit $\mathrm{Im}\,S \to \infty$ then shows that the four-fold Kähler potential can be rewritten, to leading order in g_s, as the sum of two terms, corresponding to a split (3.101) with the two blocks given by the symplectic form η_{Z^*} on $H_3(Z^*,\mathbb{Z})$ and $\tilde{\eta}_{\mathcal{D}}$ the intersection matrix on the hypersurface $\mathcal{D}$. Thus, to leading order in g_s, the F-theory result is in perfect agreement with the local orientifold result of [148].

The role of the intersection matrix $\tilde{\eta}_{\mathcal{D}}$ as defining a topological metric in the open-string sector is natural in view of the localization of the open-string degrees of freedom on $\mathcal{D}$. In the quintic example, $\mathcal{D}$ describes the K3 geometry that captures the variation of the chain integrals with the open-string deformations, as described below (3.49), and $\tilde{\eta}_{\mathcal{D}}$ is simply the K3 intersection matrix $\hat{\eta}$ discussed below (3.73). Explicitly, the resulting Kähler potential for the four-fold X_4 defined in Table 3.1 together with (3.95), reads, to leading order in small g_s and in an expansion near the large complex structure point,

$$K = -\ln(\tilde{g}^{-1}\,\tilde{Y}), \qquad \tilde{Y} = \frac{5i}{6}(t-\bar{t})^3 + \tilde{g}\left(-\frac{1}{6}(t_1 - 1t_1)^4 + \frac{5}{12}(t-\bar{t})^4\right) + \mathcal{O}(|t|^2) \ , \tag{3.103}$$

where (t, t_1) are the flat coordinates of (3.63), (3.69) and $\tilde{g}^{-1} = 2\,\mathrm{Im}\,S$. The second summand in the order $\tilde{g}^1$ term is a correction predicted by the dual F-theory four-fold, which is not

[44] See the Appendix D for details.

captured by the dimensional reduction, (3.100). By mirror symmetry, (3.103) then represents a prediction for the Kähler metric on the deformation space of A-type branes on the quintic Z. It would be interesting to understand the relation of the above proposal to the metric described by Hitchin in [133].

4

Chapter 4

Examples and applications of off-shell mirror symmetry

Contents

In this chapter we apply the techniques of the preceding chapter to the examples of type II/F-theory superpotentials for brane geometries on toric hypersurfaces with several open/closed string deformations. These results can be found in the original publications [1, 2, 3]. Using the small Hodge variation associated with the surface operators and the GKZ system on the relative cohomology group we are able to efficiently compute the integral relative periods and the mirror map. Using this we obtain enumerative predictions for the number of discs in the corresponding open/closed A-model geometry.

The application of the off-shell techniques to multi-parameter models is also interesting for another reason. These models often allow for extremal transitions through points with enhanced gauge symmetry. In the considered models we are able to track the fate of certain domain wall tensions through these transitions and find that they fall into representations of the non-perturbative gauge group near the transition point. For some models we furthermore find tensionless domain walls at the point of the enhanced gauge symmetry.

The models we are going to consider are given by:

- We start in section 4.1 by considering the example of branes on the quintic in detail. In particular we show how the results of [54, 53] for the rigid involution brane can be obtained by off-shell techniques.

- Afterwards we consider two different divisors on the hypersurface $\mathbf{X}_{12}^{(1,2,2,3,4)}$ in section 4.2. This example will show that two different off-shell deformations of a brane lead to the same on-shell potential. Furthermore it will turn out that the Hodge problem on the second divisor is equivalent to that of a particular two-parameter family of K3 surfaces. The relevant zero of the period vector will be at an orbifold point of the K3, which has been interpreted as a point with half-integral B-field for the closed-string compactification on the local geometry [127] and seems to be a generic feature.

- Our third and fourth example are the two elliptic fibrations $\mathbf{X}_{18}^{(1,1,1,6,9)}$ and $\mathbf{X}_{9}^{(1,1,1,3,3)}$ in sections 4.3 and 4.4 respectively. These are both compactifications of the non-compact Calabi-Yau manifold $\mathcal{O}(-3)_{\mathbb{P}^2}$ and we are able to reproduce the results of [142, 162] in the decompactification limit. In addition we study the deformation of the involution brane on $\mathcal{O}(-3)_{\mathbb{P}^2}$.

- The last model in section 4.5 is given by a certain toric brane on the hypersurface $\mathbf{X}_{8}^{(1,1,2,2,2)}$. Here the zero of the derivative of the relevant superpotential period will turn out to correspond to a point where the geometric volumes of two 2-cycles in the subsystem K3 coincide. By our Hodge theoretic methods we are able to compute the inhomogeneous Picard-Fuchs equations and obtain, via the mirror map, the disc invariants of the domain wall tensions of the corresponding A-model geometry. We furthermore study the extremal transition to the one-parameter model $\mathbf{X}_{2,4}^{(1,1,1,1,1,1)}$.

Further examples of type II/F-theory superpotentials with several open/closed deformations can be found in the original publication [3].

4.1. Branes on the quintic $\mathbf{X}_{5}^{(1,1,1,1,1)}$

We first study a family of toric branes on the quintic that includes branes that have been studied before in [53, 54, 103] by different means. We recover these results for special choice of boundary conditions and study connected configurations.

4.1.1. Bulk geometry

As discussed earlier the toric ambient space of quintic hypersurface $\mathbf{X}_{5}^{(1,1,1,1,1)}$ is given by the polyhedron $\Delta(Z)$ in Table 4.1

$\Delta(Z)$	$\tilde{\nu}_0$	=	(	0	0	0	0	)
	$\tilde{\nu}_1$	=	(	1	0	0	0	)
	$\tilde{\nu}_2$	=	(	0	1	0	0	)
	$\tilde{\nu}_3$	=	(	0	0	1	0	)
	$\tilde{\nu}_4$	=	(	0	0	0	1	)
	$\tilde{\nu}_5$	=	(	−1	−1	−1	−1	)

Table 4.1.: Vertices of the bulk geometry $\Delta(Z)$ for the quintic example $\mathbf{X}_{5}^{(1,1,1,1,1)}$.

A maximal triangulation of $\Delta(Z)$ leads to the following relation

$$\begin{array}{c|c|ccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline \tilde{l}^1 & -5 & 1 & 1 & 1 & 1 & 1 \end{array} . \qquad (4.1)$$

The mirror geometry can be specified by a hypersurface equation in $\mathbb{P}^4/\mathbb{Z}_5^3$:

$$P(Z^*): \; x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - x_1x_2x_3x_4x_5\, z^{-\frac{1}{5}} = 0 \;, \qquad (4.2)$$

where $z = -\frac{a_1a_2a_3a_4a_5}{a_0^5}$ denotes the local coordinate, which is adapted to the point of maximal unipotent monodromy. One has also to take into account the $\mathbb{Z}_5^3$-orbifold action which acts as $x_i \to \lambda_k^{g_{k,i}}\, x_i$ with $\lambda_k^5 = 1$ and the weights

$$\mathbb{Z}_5: \; g_1 = (1,-1,0,0,0), \quad \mathbb{Z}_5: \; g_2 = (1,0,-1,0,0)\,, \quad \mathbb{Z}_5: \; g_3 = (1,0,0,-1,0)\;. \qquad (4.3)$$

4.1.2. Brane geometry

As in section 3.4.1. we consider a one parameter family of A-branes defined by an additional charge vector

$$\begin{array}{c|c|ccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline \hat{l} & 1 & -1 & 0 & 0 & 0 & 0 \end{array} . \qquad (4.4)$$

As discussed in the last chapter we may associate with this brane geometry a five-dimensional toric polyhedron $\Delta_\flat(Z,L)$ that contains the points of the polyhedron $\Delta(Z)$ of the quintic as a subset on the hypersurface $y_5 = 0$ (see Table 4.2). Choosing a maximal triangulation of

$\Delta(Z)$	$\tilde{\nu}_0$	=	(	0	0	0	0	0	)
	$\tilde{\nu}_1$	=	(	1	0	0	0	0	)
	$\tilde{\nu}_2$	=	(	0	1	0	0	0	)
	$\tilde{\nu}_3$	=	(	0	0	1	0	0	)
	$\tilde{\nu}_4$	=	(	0	0	0	1	0	)
	$\tilde{\nu}_5$	=	(	−1	−1	−1	−1	0	)
$\Delta_\flat(Z,L) = \Delta\cup$	$\tilde{\nu}_6$	=	(	1	0	0	0	1	)
	$\tilde{\nu}_7$	=	(	0	0	0	0	1	)

Table 4.2.: Vertices of the enhanced polyhedron $\Delta_\flat(Z,L)$ for the quintic example.

$\Delta_\flat(Z,L)$ determines the following basis of generators[1]

$$\begin{array}{c|c|ccccc|cc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline l^1 & -4 & 0 & 1 & 1 & 1 & 1 & 1 & -1 \\ l^2 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \end{array} , \qquad (4.5)$$

[1]The following computations have been performed using parts of existing computer code [124].

where the last two entries correspond to the extra points describing the brane geometry. In the local variables[2]

$$z_1 = -\frac{a_2a_3a_4a_5a_6}{a_0^4a_7}, \qquad z_2 = -\frac{a_1a_7}{a_0a_6} \ , \tag{4.6}$$

the hypersurface equations for the B-brane geometry read

$$\begin{aligned} P(Z^*):\ & x_1^5+x_2^5+x_3^5+x_4^5+x_5^5 - x_1x_2x_3x_4x_5\, z^{-\frac{1}{5}} = 0 \ , \\ Q(\mathcal{D}):\ & x_1^5 + x_1x_2x_3x_4x_5\ z_2 z^{-\frac{1}{5}} = 0 \ . \end{aligned} \tag{4.7}$$

Here $z = -z_1z_2$ denotes the complex structure modulus of the Calabi-Yau geometry Z^*.

From (4.5) one can immediately proceed and solve the toric Picard-Fuchs system (3.51) to derive the mirror maps and the superpotentials and we will do so momentarily. However it is instructive to take a closer look at the geometry of the problem of mixed Hodge variations on the relative cohomology groups (3.61) and to explicitly recover the subsystem K3-system. Rewriting the superpotential $P(Z^*)$ in the original variables y_i of the toric ambient space and restricting to the hypersurface $Q(\mathcal{D}):\ y_1 = y_0$ defines the following boundary superpotential $P_{\mathcal{D}} = P(Z^*)|_{y_1=y_0}$ for the relative cohomology problem on $\mathcal{D}$:

$$P_{\mathcal{D}} = (a_0+a_1)y_0 + a_2y_2 + a_3y_3 + a_4y_4 + a_5y_5 \ .$$

The boundary superpotential $P_{\mathcal{D}}$ describes a K3 surface defined as a quartic polynomial in $\mathbb{P}^3$ after the transformation of variables $y_i = x_i^4$, $i = 1, ..., 4$:

$$P_{\mathcal{D}} = x_1^4 + x_2^4 + x_3^4 + x_4^4 + z_{\mathcal{D}}^{-1/4}\ x_1x_2x_3x_4 \ . \tag{4.8}$$

Thus the part of the Hodge variation associated with the lower row in (3.61), which can be properly defined as a subspace through the weight filtration [47, 46, 103], is the usual Hodge variation associated with the complex structure of the family of K3 manifolds defined by $P_{\mathcal{D}}$. The complex structure determined by the (2,0) form $\omega^{(2,0)}$ on the K3 is parametrized by the modulus

$$z_{\mathcal{D}} = \frac{z_1}{(1+z_2)^4} \xrightarrow{a_6/a_7=-1} \frac{a_2a_3a_4a_5}{(a_0+a_1)^4} \ ,$$

which is a special combination of the closed and open string moduli. Since the dependence of the Hodge variation on the brane modulus z_2 localizes on $\mathcal{D}$, the open string mirror map and the brane tension will be directly related to periods on the K3 surface (4.8) as discussed before.

The differential operators (3.51) in the local variables z_1, z_2 defined by (4.5) read

$$\begin{aligned} \mathcal{L}_1 &= (\theta_1^4 - z_1\prod_{i=1}^{4}(4\theta_1+\theta_2+i))(\theta_1-\theta_2) \ , \\ \mathcal{L}_2 &= (\theta_2 + z_2(4\theta_1+\theta_2+1))(\theta_1-\theta_2) \ . \end{aligned} \tag{4.9}$$

[2] We equipped z_1 with an additional minus sign compared to (2.82) for later convenience.

The above operators $\mathcal{L}_1$ and $\mathcal{L}_2$ reveal the relation of the variation of mixed Hodge structure to the family of K3 manifolds defined in (4.8). Indeed the combination $(\theta_1-\theta_2)$ is the direction of the open string parameter that localizes on $\mathcal{D}$. The split

$$\mathcal{L}_a = \tilde{\mathcal{L}}_a(\theta_1 - \theta_2)$$

shows that the solutions π_σ of the equations $\tilde{\mathcal{L}}_a \pi_\sigma = 0$ are just the K3 periods. The operator $\tilde{\mathcal{L}}_2$ imposes that the periods depend non-trivially only on the variable $z_\mathcal{D}$[3]

$$\tilde{\mathcal{L}}_2\left((z_2+1)^{-1} f(z_\mathcal{D})\right) = 0\,,$$

whereas the operator $\tilde{\mathcal{L}}_1$ reduces to the Picard-Fuchs operator of the K3 surface in the new variable $z_\mathcal{D}$. It follows that the solutions of the K3 system are the first variations of the relative periods w.r.t. the open string deformation and a critical point $\delta_{\hat{z}}\mathcal{W} = 0$ corresponds to a particular solution π of the K3 system that vanishes at that point. The solution that describes the involution brane is determined by requiring the right transformation property under the discrete symmetry of the moduli space as in [103].

Further differential operators can be obtained from linear combinations of the basis vectors l^a. E.g. the linear combination $l = l^1 + l^2$ defines the differential operator

$$\mathcal{L}_1' = \theta_2\theta_1^4 + z_1 z_2 \prod_{i=1}^{5}(4\theta_1 + \theta_2 + i)\ ,$$

which also annihilates the relative periods.[4] The mirror maps can be computed to be

$$\begin{aligned} -z_1(t_1,t_2) &= q_1 + (24\,q_1^2 - q_1\,q_2) + (-396\,q_1^3 - 640\,q_1^2\,q_2) + \dots\ , \\ z_2(t_1,t_2) &= q_2 + (-24\,q_1\,q_2 + q_2^2) + (972\,q_1^2\,q_2 - 178\,q_1\,q_2^2 + q_2^3) + \dots \end{aligned} \tag{4.10}$$

with $q_a = \exp(2\pi i\, t_a)$. The deformation parameters t_1 and t_2 are the flat coordinates near the large complex structure point $z_1 = z_2 = 0$ associated with open string deformations [47, 46]. Their physical interpretation is the quantum volume of two homologically distinct discs as measured by the tension of D4 domain walls on the A-model side [142, 162]. The other solutions of the differential operators (3.51) describe the brane tensions (3.25) of the domain walls in the family. We proceed with a study of various critical points of the superpotential.

4.1.3. Near the involution brane

Large volume in the A-model: $z_1 \sim 0$

To study brane configurations mirror to the involution brane of [53] we consider the following D5-brane locus

$$x_2^5 + x_3^5 = 0, \qquad x_4^5 + x_5^5 = 0, \qquad x_1^5 - x_1x_2x_3x_4x_5\, z^{-\frac{1}{5}} = 0\ .$$

[3]The z_2 dependent prefactor arises from the normalization of the holomorphic form.

[4]One can further factorize the above operators to a degree four differential operator which together with $\mathcal{L}_2$ represents a complete Picard-Fuchs system.

Comparing with (4.7) we search for a superpotential with critical locus near $z_2 = -1$ and arbitrary z_1. Let us first look at the large volume phase $z_1 \sim 0$ of the mirror A-brane, where one expects an instanton expansion with integral coefficients. The local variables (4.6) are centered at $z_1 = z_2 = 0$, not $z_2 = -1$, however. To get a nice expansion of the superpotential near the locus $z_2 + 1 = 0$ we change variables to

$$u = z_1^{-1/4}(1+z_2), \qquad v = z_1^{1/4} \ .$$

Examining the z_2-dependent solution of the GKZ system in these variables, we find the superpotential

$$c\,\mathcal{W}(u,v) = \frac{u^2}{8} + 15v^2 + \frac{5u^3v}{48} - \frac{15uv^3}{2} + \frac{u^6}{46080} + \frac{35v^2u^4}{384} - \frac{15v^4u^2}{8} + \frac{25025v^6}{3} + \dots\ , \quad (4.11)$$

which has the expected critical locus $\delta_{\hat{z}}\mathcal{W} = 0$ at $u = 0$ for all values of v. Here c is a constant that cannot be fixed from the consideration of the differential equations (4.9) alone.[5] At the critical locus $u = 0$ the above expression yields the critical value $W_{\text{crit}}(z) = \mathcal{W}(u = 0, v = z^{1/4})$

$$W_{\text{crit}}(z) = 15\,\sqrt{z} + \frac{25025}{3}\,z^{3/2} + \frac{52055003}{5}\,z^{5/2} + \dots\ . \quad (4.12)$$

Here the constant can be fixed to $c = 1$ by an analytic continuation argument given in Appendix F or by comparing (4.12) with the result of [53] for $W_{\text{crit}}(z)$.

As alluded to earlier, the differential operators (4.9) have the special property that the periods of Z^* are amongst their solutions. One may check that the open string mirror maps (4.10) conspire such that the mirror map for the remaining modulus $z = -z_1 z_2$ at the critical point coincides with the closed string mirror map for the quintic. Using the multi-cover prescription of [52, 53] and expressing (4.12) in terms of the exponentials $q(z) = \exp(2\pi i\,t(z)) = z + \mathcal{O}(z^2)$ one obtains the integral instanton expansion of the A-model

$$\begin{aligned}\frac{W_{\text{crit}}(z(q))}{\omega_0(q)} &= 15\,\sqrt{q} + \frac{2300}{3}\,q^{3/2} + \frac{2720628}{5}\,q^{5/2} + \dots\ ,\\ &= \sum_{k\,\text{odd}} \left(\frac{15}{k^2} q^{k/2} + \frac{765}{k^2} q^{3k/2} + \frac{544125}{k^2} q^{5k/2} + \dots \right) .\end{aligned}$$

To make contact with the inhomogeneous Picard-Fuchs equation of [54], we rewrite the differential operators above in terms of the bulk modulus z and the open string deformation z_2 and split off the z_2 dependent terms as in (3.52). In particular the operator $\mathcal{L}'_1$ leads to a

[5] The precise linear combination of the solutions of the Picard-Fuchs system that corresponds to a given geometric cycle can be determined by an intersection argument and possibly analytic continuation, similarly as in the closed string case [37]. Such an argument has been made in the present example already in [53] and is re-derived in Appendix F.

non-trivial equation of the form $\theta\mathcal{L}_{\text{bulk}}\Pi = -\mathcal{L}_{\text{open}}\Pi$, where

$$\begin{aligned}\mathcal{L}_{\text{bulk}} &= \theta^4 - 5z\prod_{i=1}^{4}(5\theta+i)\ , \qquad \mathcal{L}_{\text{open}} = \mathcal{L}_1' - \theta\mathcal{L}_{\text{bulk}}\ ,\\ \mathcal{L}_1' &= (\theta+\theta_2)\theta^4 - z\prod_{i=1}^{5}(5\theta+\theta_2+i)\end{aligned} \tag{4.13}$$

and $\theta = \theta_z$. Setting $\Pi = \mathcal{W}(u,v)$ and restricting to the critical locus $z_2 = -1$ one obtains

$$\mathcal{L}_{\text{bulk}}W_{\text{crit}}(z) = \frac{15}{16}\sqrt{z}\ . \tag{4.14}$$

This identifies the inhomogeneous Picard-Fuchs equation of [53, 54] as the restriction of (3.51) to the critical locus.

While the result (4.12) had been obtained in [53], the above derivation gives some extra information. Since the definition of the toric branes holds off the involution locus, the superpotential $\mathcal{W}(u,v)$ describes more generally any member of the family of toric A-branes defined by (4.4), not just the involution brane. It describes also the deformation of the large volume superpotential away from $z_2 = -1$. It should be noted that the use of the closed string mirror map in [53] was strictly speaking an assumption, as the closed string mirror map measures the quantum volume of fundamental sphere instantons, not the quantum tension of D4 domain walls wrapping discs, which is the appropriate coordinate for the integral expansion of [52]. It is neither obvious nor true in general that this D4 tension agrees with half the sphere volume of the fundamental string, in particular off the involution locus. In the present case it is not hard to justify this choice and to check it from the computation of the mirror map, but more generally there will be corrections to the D4 quantum volume that are not determined by the closed string mirror map.

Small volume in the A-model: $1/z_1 \sim 0$

Another interesting point in the moduli space is the Landau-Ginzburg point of the B-model. This case has been studied previously in [103], so we will be very brief. The only non-trivial thing left to check is that the system of differential equations obtained in [103] from Griffiths-Dwork reduction is equivalent to the toric GKZ system (3.51) transformed to the local variables near the LG point. Choosing local variables

$$x_1 = \frac{a_0}{(a_2a_3a_4a_5)^{1/4}}\left(\frac{-a_7}{a_6}\right)^{1/4}, \qquad x_2 = \frac{a_1}{(a_2a_3a_4a_5)^{1/4}}\left(\frac{-a_7}{a_6}\right)^{5/4},$$

one obtains by a transformation of variables the differential operators

$$\begin{aligned}\mathcal{L}_1 &= (x_1^4(\theta_1+\theta_2)^4 - 4^4\prod_{i=1}^{4}(\theta_1-i))(\theta_1+5\theta_2)\,,\\ \mathcal{L}_2 &= (x_2(\theta_1-1) - x_1\theta_2)\,(\theta_1+5\theta_2)\,,\\ \mathcal{L}_1' &= x_1^5(\theta_1+\theta_2)^4\theta_2 - 4^4x_2\prod_{i=1}^{5}(\theta_1-i)\,,\end{aligned} \tag{4.15}$$

where θ_i denotes the logarithmic derivatives θ_{x_i}. The superpotential is

$$\mathcal{W}(x_1,x_2) = -\frac{x_1^2}{2} - \frac{x_2x_1}{6} - \frac{x_1^6}{11520} - \frac{x_2x_1^5}{3840} - \frac{x_2^2x_1^4}{2688} - \frac{x_2^3x_1^3}{3456} - \frac{x_2^4x_1^2}{8448} - \frac{x_2^5x_1}{49920} + \dots,$$

which has its critical locus at $x_2 = -x_1$, which corresponds to $u = 0$ in these coordinates. In terms of the closed string variable $x = -x_1x_2^{-1/5}$ at the Landau Ginzburg point, the expansion at the critical locus reads

$$W_{\text{crit}}(x) = -\frac{x^{5/2}}{3} - \frac{x^{15/2}}{135135} - \frac{x^{25/2}}{1301375075} + \dots,$$

which satisfies a similar equation as (4.14)

$$\mathcal{L}_{\text{bulk}}W_{\text{crit}} = \frac{15}{16}\,x^{5/2}\,,$$

where $\mathcal{L}_{\text{bulk}} = 5^{-4}x^5\theta_x^4 - 5\prod_{i=1}^4(\theta_x - i)\,.$

4.1.4. Large radius invariants for the A-model

The geometry of A-branes is notorically more difficult to study than that of the B-branes. For the type of B-branes studied above, the A-model mirror geometry can be in principle obtained from the toric framework of [142]. The conjectural family of A-branes mirror to the B-brane family studied above, is defined on the quintic hypersurface Z in the toric ambient space $W = \mathcal{O}(-5)_{\mathbb{P}^4}$ with homogeneous coordinates

$$\begin{array}{c|ccccc} \tilde{x}_0 & \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 & \tilde{x}_4 & \tilde{x}_5 \\ \hline -5 & 1 & 1 & 1 & 1 & 1 \end{array}\,. \qquad (4.16)$$

The Kähler moduli (t_1, t_2) mirror to the complex structure coordinates (z_1, z_2) are defined by the equations

$$\textstyle\sum_{i=1}^5 |\tilde{x}_i|^2 - 5|\tilde{x}_0|^2 = \operatorname{Im} t = \operatorname{Im} t_1 + \operatorname{Im} t_2, \qquad |\tilde{x}_1|^2 - |\tilde{x}_0|^2 = \operatorname{Im} t_2 \qquad (4.17)$$

with $\operatorname{Im} t_1, \operatorname{Im} t_2 > 0$. The first constraint holds on all of Z and the Kähler modulus t describes the closed string deformation, the overall Kähler volume of Z. The second constraints holds only on the Lagrangian submanifold L and describes the open-string deformation.

The toric framework of [142] gives an explicit description of the geometry of Lagrangian subspaces in the ambient space W, which has been used to study an interesting class of non-compact branes for Calabi-Yau ambient spaces, see e.g. [142, 162, 47, 46]. The clear geometric picture of the toric description is lost for hypersurfaces and the searched for subspace $L \subset Z$ carrying the mirror A-brane has no simple description, at least at general points in the (full) moduli space and to our knowledge. However, by the homological mirror symmetry conjecture [45], we expect that a corresponding A-brane, which is mirror to the B-brane given in terms of the discussed divisor together with its curvature 2-form, should be present in the A-model geometry. Clearly, in order to complete our picture a constructive recipe of mapping B-branes to the corresponding A-branes for compact Calabi-Yau geometries is desirable.

Since the A-model geometry is naively independent of the complex structure moduli, one is tempted to choose a very special form of the hypersurface constraint to simplify the geometry. In [182] it is shown how the number 2875 of lines in the generic quintic can be determined from the number of lines in a highly degenerate quintic, defined by the hypersurface constraint

$$P(Z_\alpha) = p_1.p_4 + \alpha\, p_5, \qquad \alpha \in \Delta\ . \tag{4.18}$$

Here α is a parameter on the complex disc Δ and the p_k are degree k polynomials in the homogeneous coordinates of $\mathbb{P}^4$. At the point $\alpha = 0$ the quintic splits into two components of degree one and four. Katz shows, that there are 1600+1275=2875 holomorphic maps to lines in the two components of the central fiber that deform to the fiber at $\alpha \neq 0$.

The $N_1 = 2875$ curves of degree one contribute to the tension $\mathcal{T}$ of a D4-brane wrapping the 4-cycle $\Gamma = H \cap Z$ as

$$\mathcal{T} = -\frac{5}{2}t^2 + \frac{1}{4\pi^2}\left(2875\sum_k \frac{q^k}{k^2} + 2\cdot 609250\sum_k \frac{q^{2k}}{k^2} + ...\right)\ ,$$

where $q = \exp(2\pi i t)$, H is the hyperplane class and the dots denote linear and constant terms in t as well as instanton corrections from maps of degree $d > 2$. In the singular Calabi-Yau the generic 4-cycle splits into two components and one expects two separate contributions

$$\mathcal{T}^{(1)} = c^{(1)}\, t^2 + \frac{N_d^{(1)}}{4\pi^2}\sum_k \frac{q^{dk}}{k^2} + ..., \qquad \mathcal{T}^{(2)} = \mathcal{T} - \mathcal{T}_1$$

with $N_1^{(1)} = 1600$ and $N_1^{(2)} = 1275$.

As explained in [182], there are other genus zero maps to the two components, that develop nodes at the intersection locus $p_1 = p_4 = 0$ upon deformation, and they do not continue to exist as maps from S^2 to S^2. The idea is that in the presence of the Lagrangian A-brane on the degenerate quintic, the nodes of the spheres can open up to become the boundary of holomorphic disc instantons ending on L. Indeed the two independent double logarithmic solutions of the Picard-Fuchs system (4.9) can be written in the flat coordinates as

$$\begin{aligned}\mathcal{T}^{(1)} &= -2t^2 + \frac{1}{4\pi^2}\sum_k \frac{1}{k^2}\left(1600q^k + 2\cdot 339800q^{2k} + ...\right) + \mathcal{T}^{(o)}(t_1,t_2)\ ,\\ \mathcal{T}^{(2)} &= -\frac{1}{2}t^2 + \frac{1}{4\pi^2}\sum_k \frac{1}{k^2}\left(1275q^k + 2\cdot 269450q^{2k} + ...\right) - \mathcal{T}^{(o)}(t_1,t_2)\ ,\end{aligned} \tag{4.19}$$

showing the expected behavior and adding up to the closed-string period $\mathcal{T}$. The split of the degree two curves, $N_2 = 609250 = 339800{+}269450 = (258200{+}\frac{1}{2}163200){+}(187850{+}\frac{1}{2}163200)$ is compatible with the results of [183].

$n_2 \backslash n_1$	0	1	2	3	4	5
0	0	−320	13280	−1088960	119783040	−15440622400
1	20	1600	−116560	12805120	−1766329640	274446919680
2	0	2040	679600	−85115360	13829775520	−2525156504560
3	0	−1460	1064180	530848000	−83363259240	16655092486480
4	0	520	−1497840	887761280	541074408000	−95968626498800
5	0	−80	1561100	−1582620980	931836819440	639660032468000
6	0	0	−1152600	2396807000	−1864913831600	1118938442641400
7	0	0	580500	−2923203580	3412016521660	−2393966418927980
8	0	0	−190760	2799233200	−5381605498560	4899971282565360
9	0	0	37180	−2078012020	7127102031000	−9026682030832180
10	0	0	−3280	1179935280	−7837064629760	14557931269209000
11	0	0	0	−502743680	7104809591780	−20307910970428360
12	0	0	0	155860160	−5277064316000	24340277955510560
13	0	0	0	−33298600	3187587322380	−24957649473175420
14	0	0	0	4400680	−1549998228000	21814546476229120
15	0	0	0	−272240	597782974040	−16191876966658500
16	0	0	0	0	−178806134240	10157784412551120
17	0	0	0	0	40049955420	−5351974901676280

Table 4.3.: Predictions for Ooguri–Vafa invariants for the brane geometry on the quintic three-fold specified in Table 4.2.

The extra contribution $\mathcal{T}^{(o)}(t_1, t_2)$ can be written as

$$\mathcal{T}^{(o)}(t_1, t_2) = 4tt_2 - 2t_2^2 + \frac{1}{4\pi^2} \sum_{\substack{k, n_1, n_2 \\ n_1 \neq n_2}} \frac{1}{k^2} \; N_{n_1,n_2} (q_1^{n_1} q_2^{n_2})^k \; .$$

The first few coefficients N_{n_1,n_2} for small n_i, including the contributions from $n_1 = n_2$, are listed in Table 4.3.

According to the general philosophy of the Hodge theoretic mirror map described in the previous sections, the double logarithmic solutions represent the generating function of holomorphic discs ending on the A-brane L. In the above basis we find

$$F_t = \mathcal{T}^{(1)} + \mathcal{T}^{(2)} = \mathcal{T}, \qquad W = \mathcal{T}^{(1)} \; .$$

Assuming that the normalization argument leading to (4.19) is correct, the numbers N_{n_1,n_2} of Table 4.3 are genuine Ooguri-Vafa invariants for the A-brane geometry predicted by mirror symmetry. It would be interesting to justify the above arguments and the prediction for the disc invariants in Table 4.3 by an independent computation. Further evidence for the above results will be given in the appendix D, by deriving the same result from the afore mentioned duality to Calabi-Yau four-folds.

4.2. Domain wall tensions on $\mathbf{X}_{12}^{(1,2,2,3,4)}$

As a second example we study branes on the two moduli Calabi-Yau manifold $Z = \mathbf{X}_{12}^{(1,2,2,3,4)}$. This example in particular shows that different off-shell deformations of the same on-shell geometry lead to the same on-shell domain wall tensions.

4.2.1. Bulk geometry

The toric ambient geometry of the hypersurface $\mathbf{X}_{12}^{(1,2,2,3,4)}$ is given by the polyhedron $\Delta(Z)$ in Table 4.4.

$\Delta(Z)$	$\tilde{\nu}_0$	=	(	0	0	0	0	)
	$\tilde{\nu}_1$	=	(	2	2	3	4	)
	$\tilde{\nu}_2$	=	(	−1	0	0	0	)
	$\tilde{\nu}_3$	=	(	0	−1	0	0	)
	$\tilde{\nu}_4$	=	(	0	0	−1	0	)
	$\tilde{\nu}_5$	=	(	0	0	0	−1	)
	$\tilde{\nu}_6$	=	(	1	1	1	2	)

Table 4.4.: Vertices of the bulk geometry $\Delta(Z)$ for $\mathbf{X}_{12}^{(1,2,2,3,4)}$.

A maximal triangulation of $\Delta(Z)$ leads to the following relations [108]

$$\begin{array}{c|c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline l^1 & -6 & -1 & 1 & 1 & 0 & 2 & 3 \\ l^2 & 0 & 1 & 0 & 0 & 1 & 0 & -2 \end{array} . \tag{4.20}$$

Written in homogeneous coordinates of $\mathbb{P}^4_{1,2,2,3,4}/\mathbb{Z}_6^2$ the hypersurface constraint for the mirror manifold reads

$$\begin{aligned} P(Z^*) &= a_1 x_1^{12} + a_2 x_2^6 + a_3 x_3^6 + a_4 x_4^4 + a_5 x_5^3 + a_0 x_1 x_2 x_3 x_4 x_5 + a_6 x_1^6 x_4^2 \\ &= x_1^{12} + x_2^6 + x_3^6 + x_4^4 + x_5^3 + \psi\, x_1 x_2 x_3 x_4 x_5 + \phi\, x_1^6 x_4^2 \,. \end{aligned} \tag{4.21}$$

In the second equation the variables x_i have been rescaled to display the dependence on the torus invariant parameters $\psi = z_1^{-1/6} z_2^{-1/4}$ and $\phi = z_2^{-1/2}$, with the z_a given by (2.82). On the mirror manifold, the Greene-Plesser orbifold group acts as $x_i \to \lambda_k^{g_{k,i}}\, x_i$ with weights[6]

$$\mathbb{Z}_6 :\, g_1 = (1, -1, 0, 0, 0), \quad \mathbb{Z}_6 :\, g_2 = (1, 0, -1, 0, 0)\,, \tag{4.22}$$

where we denote the generators by λ_k with $\lambda_{1,2}^6 = 1$. The closed-string periods near the large complex structure point can be generated by evaluating the functions $B_{\{l^a\}}(z_a; \rho_a)$ in (3.87) and its derivatives with respect to ρ_i at $\rho_1 = \rho_2 = 0$ [108].

[6]The other factors of the Greene-Plesser group give nothing new, using a homogeneous rescaling of the projective coordinates, e.g. for the factor generated by $g_3 = (1, 0, 0, -1, 0)$ with $\lambda_3^4 = 1$ one finds $g_3 \sim g_1^3 g_2^3$.

In this geometry we consider the set of curves defined by the equations

$$C_{\alpha,\kappa} = \{x_2 = \eta x_3\ ,\ x_4 = \alpha x_1^3\ ,\ x_5 = \kappa\sqrt{\alpha\eta\psi}x_3x_1^2\}\,,$$
$$\eta^6 = -1\,, \qquad \kappa^2 = -1\,, \qquad \alpha^4 + \phi\alpha^2 + 1 = 0\,. \tag{4.23}$$

The labels (η, α, κ) are identified as $(\eta, \alpha, \kappa) \sim (\eta\lambda_1\lambda_2^{-1}, \alpha\lambda_1^3\lambda_2^3, \kappa)$ under the orbifold group. In the following we choose to label each orbit of curves by $(\alpha, \kappa) := (e^{i\pi/6}, \alpha, \kappa)$. Note that a rotation of η corresponds to a change of sign for α in this notation, $(e^{3i\pi/6}, \alpha, \kappa) = (-\alpha, \kappa)$. Instead of choosing a fixed η we can also fix the sign of α and keep two choices for η^3.

We are going to calculate the domain wall tensions and the superpotentials for the vacua $C_{\alpha_1,\kappa}$ and $C_{\alpha_2,\kappa}$ and for this we will study two families of divisors. The family $Q(\mathcal{D}_1) = x_2^6 + \hat{z}x_3^6$ interpolates between vacua related by a sign flip of η^3 or of the root α of the quartic equation. The family $Q(\mathcal{D}_2) = x_4^4 + \hat{z}x_1^6x_4^2$ interpolates between any two different roots α.

4.2.2. First divisor

We start with the analysis of the divisor

$$Q(\mathcal{D}_1) = x_2^6 + z_3x_3^6\,. \tag{4.24}$$

To obtain some geometrical understanding of the surface defined by the intersection $P = 0 = Q(\mathcal{D}_1)$ we explicitly solve for $x_3 = (-z_3)^{-1/6}x_2$ and rescale x_2 to find

$$P_{\mathcal{D}_1} = x_1^{12} + x_2^6 + x_4^4 + x_5^3 + \tilde{\psi}x_1x_2^2x_4x_5 + \phi x_1^6x_4^2\,. \tag{4.25}$$

Here $\tilde{\psi} = u_1^{-1/6}u_2^{-1/4}$, $\phi = u_2^{-1/2}$ are expressed in terms of the previous parameters as

$$u_1 = -\frac{z_1}{z_3}(1 - z_3)^2, \qquad u_2 = z_2\,. \tag{4.26}$$

Changing coordinates to $\tilde{x}_2 = x_2^2$ displays the family $\mathcal{D}_1$ as a double cover of a family of toric K3 surfaces associated to a GLSM with charges

$\mathcal{D}_1$: (4.27)

	0	1	2	4	5	6
$\tilde{l}^1$	-6	-1	2	0	2	3
$\tilde{l}^2$	0	1	0	1	0	-2

and with the two algebraic K3 moduli (4.26). The two covers are distinguished by a choice of sign for x_2.

The family of algebraic K3 manifolds obtained from (4.25) by the variable change $\tilde{x}_2 = x_2^2$ generically has four parameters with the two extra moduli multiplying the monomials $x_1^3x_4^3$ and $x_1^9x_4$. Since these terms are forbidden by the Greene-Plesser group of the Calabi-Yau three-fold, the embedded surface is at a special symmetric point with the coefficients of these monomials set to zero. The periods on the K3 surface at this point can be computed from the GKZ system for the two parameter family, obtained from (3.51) with the charge vectors $\{\tilde{l}\}$ in eq. (4.27):

$$\begin{aligned}\mathcal{L}_1^{\mathcal{D}_1} &= \tilde{\theta}_1(2\tilde{\theta}_1-1)\prod_{k=0}^{2}(-3\tilde{\theta}_1+2\tilde{\theta}_2+k)-\frac{9}{2}u_1(\tilde{\theta}_1-\tilde{\theta}_2)\prod_{k=1,2,4,5}(6\tilde{\theta}_1+k)\,,\\ \mathcal{L}_2^{\mathcal{D}_1} &= \tilde{\theta}_2(\tilde{\theta}_2-\tilde{\theta}_1)-u_2(2\tilde{\theta}_2-3\tilde{\theta}_1)(2\tilde{\theta}_2-3\tilde{\theta}_1+1)\,,\end{aligned}\tag{4.28}$$

where $\tilde{\theta}_a = u_a\frac{d}{du_a}$. Apart from the regular solutions this system has two extra solutions depending on fractional powers in the u_i:

$$\begin{aligned}\pi_1(u_1,u_2) &= \frac{c_1}{2}\,B_{\{\tilde{l}\}}(u_1,u_2;\tfrac{1}{2},0)=\frac{4c_1}{\pi}\sqrt{u_1}\;{}_2F_1(-\tfrac{1}{4},-\tfrac{3}{4},\tfrac{1}{2},4u_2)+\mathcal{O}(u_1^{3/2})\,,\\ \pi_2(u_1,u_2) &= \frac{c_2}{2}\,B_{\{\tilde{l}\}}(u_1,u_2;\tfrac{1}{2},\tfrac{1}{2})=\frac{12c_2}{\pi}\sqrt{u_1u_2}\;{}_2F_1(-\tfrac{1}{4},\tfrac{1}{4},\tfrac{3}{2},4u_2)+\mathcal{O}(u_1^{3/2})\,.\end{aligned}\tag{4.29}$$

Here c_a are some normalization constants not determined by the differential operators. Later they will be fixed to one by studying the geometric periods on the surface.

As indicated, the exceptional solutions vanish at the critical point $u_1=0$ as $\sim\sqrt{u_1}$, with the coefficient a hypergeometric series in the modulus $u_2=z_2$. These solutions arise as the specialization of the standard solutions of the four parameter family of K3 manifolds to the special symmetric point.[7] Since $u_1=0$ is not at the discriminant locus of the K3 family for general u_2, there is no geometric vanishing cycle associated with the zero of $\pi_{1,2}$. Instead the zero at $u_1=0$ arises from the 'accidental' cancellation between the volumes of different classes at the symmetric point.[8] The periods (4.29) have the special property that their leading terms $\sim\sqrt{u_1}$ near the critical point $u_1=0$ can be written in closed form as

$$\lim_{z_3\to 1}\frac{\pi_a(u_1,u_2)}{(1-z_3)}=\frac{4c_a}{\pi}\cdot\left.\frac{(i\alpha)(2\alpha^2-\phi)(\alpha^2+\phi)}{\psi^3}\right|_{\alpha=\alpha_{a,+}}\,,\tag{4.30}$$

where

$$\alpha_{1,\pm}=\pm\sqrt{\frac{-\phi+\sqrt{\phi^2-4}}{2}}\,,\qquad \alpha_{2,\pm}=\pm\sqrt{\frac{-\phi-\sqrt{\phi^2-4}}{2}}\,,\tag{4.31}$$

denote the roots of the quartic equation $\alpha^4+\phi\alpha^2+1=0$ appearing in the definition (4.23). Hence the leading part of the two K3 periods near the symmetric point is proportional to a rational function in the coefficients of the defining equations for the curve, evaluated at the critical points.

We will first compute the domain wall tensions by integrating the periods $\pi_{1,2}$ of the surface $\mathcal{D}_1$. Note that the K3 periods π_a depend on $\xi=\sqrt{z_3}$ via their dependence on u_1 and the sign of the square root correlates with the sign of α. To obtain the off-shell tension, we integrate $\pi_a(\xi)$ as

$$\mathcal{W}_a^{(\pm)}(z_1,z_2,z_3)=\frac{1}{2\pi i}\int_{\xi_0}^{\pm\sqrt{z_3}}\pi_a(\xi)\,\frac{d\xi}{\xi}\,,\tag{4.32}$$

[7] An explicit illustration of this fact is given in the case of the second family of divisors below.

[8] One parameter controlling the difference of these volumes is the direction of the off-shell modulus.

where ξ_0 denotes a fixed reference point. For example, the period π_1 integrates to

$$\frac{4\pi i\,\mathcal{W}_1^{(\pm)}}{c_1} = \int_{\xi_0}^{\pm\sqrt{z_3}} \sum_{n_1,n_2\geq 0} \frac{\Gamma\left(4+6n_1\right)\left(-\frac{z_1}{\xi^2}(1-\xi^2)^2\right)^{n_1+\frac{1}{2}} z_2^{n_2}}{\Gamma\left(2+2n_1\right)^2\Gamma\left(1+n_2\right)\Gamma\left(\frac{1}{2}-n_1+n_2\right)\Gamma\left(\frac{5}{2}+3n_1-2n_2\right)}\frac{d\xi}{\xi}$$
$$= \sum_{n_1,n_2\geq 0} \frac{\Gamma\left(4+6n_1\right)(-z_1)^{n_1+\frac{1}{2}} z_2^{n_2}\left(\xi^2-1\right)^{2n_1+2}{}_2F_1\left(1,\frac{3}{2}+n_1,\frac{1}{2}-n_1,\xi^2\right)}{(1+2n_1)\,\Gamma\left(2+2n_1\right)^2\Gamma\left(1+n_2\right)\Gamma\left(\frac{1}{2}-n_1+n_2\right)\Gamma\left(\frac{5}{2}+3n_1-2n_2\right)\xi^{2n_1+1}}\Bigg|_{\xi=\xi_0}^{\xi=\pm\sqrt{z_3}}$$

where the contribution from the reference point ξ_0 can be set to zero by choosing $\xi_0 = i$ as the lower bound. This will be used to split the result of the integral for the domain wall tension into two contributions of the superpotentials from the endpoints as in eq. (3.27). This split is not obvious in general, and ambiguous with respect to adding rational multiples of bulk periods. In the example we can use the $\mathbb{Z}_2$ symmetry acting on the curves to require that the superpotentials obey $\mathcal{W}_1^{(+)} = -\mathcal{W}_1^{(-)}$. With this convention and the particular choice of ξ_0 above, we obtain $\frac{1}{2\pi i}\int_{\xi_0}^{\pm\sqrt{z_3}}\pi_a(\xi)\,\frac{d\xi}{\xi} = \mathcal{W}_a^{(\pm)}$ or $\frac{1}{2\pi i}\int_{-\sqrt{z_3}}^{+\sqrt{z_3}}\pi_a(\xi)\,\frac{d\xi}{\xi} = \mathcal{W}_a^{(+)} - \mathcal{W}_a^{(-)} = 2\mathcal{W}_a^{(+)}$.

According to the discussion in chapter 3, the superpotentials $\mathcal{W}_a^{(\pm)}(z_1,z_2,z_3)$ restrict to the on-shell superpotentials $W_a^{(\pm)}(z_1,z_2)$ with vanishing derivative in the open-string direction z_3 at the critical point:

$$W_a^{(\pm)}(z_1,z_2) = \mathcal{W}_a^{(\pm)}\Big|_{z_3=1}, \qquad \xi\partial_\xi \mathcal{W}_a^{(\pm)}(z_1,z_2,\xi^2)\Big|_{z_3=1} = \pm\frac{1}{2\pi i}\;\pi_a\big|_{u_1=0} = 0\,. \tag{4.33}$$

For the above integrals one obtains

$$W_1^{(\pm)} = \mp\frac{c_1}{8\pi}\sum_{n_1,n_2\geq 0}\frac{(-1)^{n_1+1}\,\Gamma\left(-n_1-\frac{1}{2}\right)\Gamma\left(6n_1+4\right)\,z_1^{n_1+\frac{1}{2}}z_2^{n_2}}{\Gamma\left(n_1+\frac{3}{2}\right)\Gamma\left(2n_1+2\right)\Gamma\left(3n_1-2n_2+\frac{5}{2}\right)\Gamma\left(n_2+1\right)\Gamma\left(-n_1+n_2+\frac{1}{2}\right)}\,, \tag{4.34}$$
$$W_2^{(\pm)} = \mp\frac{c_2}{8\pi}\sum_{n_1,n_2\geq 0}\frac{(-1)^{n_1+1}\,\Gamma\left(-n_1-\frac{1}{2}\right)\Gamma\left(6n_1+4\right)\,z_1^{n_1+\frac{1}{2}}z_2^{n_2+\frac{1}{2}}}{\Gamma\left(n_1+\frac{3}{2}\right)\Gamma\left(2n_1+2\right)\Gamma\left(3n_1-2n_2+\frac{3}{2}\right)\Gamma\left(n_2+\frac{3}{2}\right)\Gamma\left(-n_1+n_2+1\right)}\,.$$

These functions can be expressed in terms of the bulk generating function as

$$W_1^{(\pm)} = \mp\frac{c_1}{8}\,B_{\{l\}}\left(z_1,z_2;\tfrac{1}{2},0\right)\,, \qquad W_2^{(\pm)} = \mp\frac{c_2}{8}\,B_{\{l\}}\left(z_1,z_2;\tfrac{1}{2},\tfrac{1}{2}\right)\,. \tag{4.35}$$

Complementary, the tensions $\mathcal{T}_a^{(\pm)}(z_1,z_2,z_3)$ and their on-shell restrictions $T_a^{(\pm)}(z_1,z_2)$ can be described as solutions to the large GKZ system for the relative cohomology problem specified by the vertices in Table 4.5. For the family (4.24) the additional charge vector is

	0	1	2	3	4	5	6	7	8
l^3	0	0	1	-1	0	0	0	-1	1

.

Together with the charge vectors l^1 and l^2 for the Calabi-Yau hypersurface this defines the extended hypergeometric system of the form (3.51). From the extended charge vectors one

$\Delta(Z)$	$\tilde{\nu}_0$	=	(	0	0	0	0	0	)
	$\tilde{\nu}_1$	=	(	2	2	3	4	0	)
	$\tilde{\nu}_2$	=	(	−1	0	0	0	0	)
	$\tilde{\nu}_3$	=	(	0	−1	0	0	0	)
	$\tilde{\nu}_4$	=	(	0	0	−1	0	0	)
	$\tilde{\nu}_5$	=	(	0	0	0	−1	0	)
	$\tilde{\nu}_6$	=	(	1	1	1	2	0	)
$\Delta_\flat(Z, L_1) = \Delta\cup$	$\tilde{\nu}_7$	=	(	−1	0	0	0	1	)
	$\tilde{\nu}_8$	=	(	0	−1	0	0	1	)

Table 4.5.: Vertices of the enhanced polyhedron $\Delta_\flat(Z, L_1)$ for the first divisor family $\mathcal{D}_1$ on the Calabi-Yau $\mathbf{X}_{12}^{(1,2,2,3,4)}$.

obtains after an appropriate factorization the system of differential operators[9]

$$\begin{aligned}
\mathcal{L}_1 &= (\theta_1+\theta_3)(\theta_1-\theta_3)(3\theta_1-2\theta_2) - 36z_1(6\theta_1+5)(6\theta_1+1)(\theta_2-\theta_1+2z_2(1+6\theta_1-2\theta_2))\,,\\
\mathcal{L}_2 &= \theta_2(\theta_2-\theta_1) - z_2(3\theta_1-2\theta_2-1)(3\theta_1-2\theta_2)\,,\\
\mathcal{L}_3 &= \theta_3(\theta_1+\theta_3) + z_3\theta_3(\theta_1-\theta_3)\,.
\end{aligned} \tag{4.36}$$

After a simple variable transformation $y = \ln(z_3)$, with the variable y centered at the critical point, the solutions to this system describe the expansion of the periods on the relative homology $H^3(Z^*, \mathcal{D}_1)$ around the critical point. These include the off-shell tensions $\mathcal{T}_a^{(\pm)}(z_1, z_2, z_3)$ (4.32), which restrict to the functions (4.35), and in addition the closed-string periods $\Pi(z_1, z_2)$. The integration from the geometric surface periods of the subsystem fixes the z_3-dependent piece. The GKZ system restricts the afore mentioned integration constant to a linear combination of the closed-string periods $\Pi(z_1, z_2)$. The rational coefficients appearing in this combination can be determined by a monodromy argument, as in [53] and as discussed for a non-compact limit of the Calabi-Yau three-fold in the Appendix G.

Finally one may also characterize the critical tensions $T_a^{(\pm)}$, or, for the above reasons also the critical superpotentials $W_a^{(\pm)}$, as the solution to the inhomogeneous Picard-Fuchs equation (3.54), which makes contact to the normal function approach of [54]. Due to

$$\mathcal{L}_1 = \mathcal{L}_1^{\text{bulk}}(\theta_1, \theta_2) - (3\theta_1 - 2\theta_2)\theta_3^2, \qquad \mathcal{L}_2 = \mathcal{L}_2^{\text{bulk}}(\theta_1, \theta_2)\,, \tag{4.37}$$

we observe that only the first operator may acquire a non-zero inhomogeneous term at the critical point. This term is determined by the leading behavior of the surface periods π_a in the limit $u_1 \to 0$. Acting with $\mathcal{L}_1^{\text{bdry}} = (3\theta_1 - 2\theta_2)\theta_3$ on the terms on the right hand side of

[9]The first operator is obtained after a factorization similar to the one described in [108] for the underlying three-fold.

eqs. (4.29) one obtains the inhomogeneous Picard-Fuchs equations

$$\begin{aligned}\mathcal{L}_1^{\text{bulk}} W_1^{(\pm)} &= \mp\frac{3c_1}{2\pi^2}\sqrt{z_1}\ {}_2F_1(\tfrac{1}{4},-\tfrac{1}{4},\tfrac{1}{2},4z_2) = f_1(\alpha_{1,\pm})\ ,\\ \mathcal{L}_1^{\text{bulk}} W_2^{(\pm)} &= \mp\frac{3c_2}{2\pi^2}\sqrt{z_1 z_2}\ {}_2F_1(\tfrac{3}{4},\tfrac{1}{4},\tfrac{3}{2},4z_2) = f_1(\alpha_{2,\pm})\ ,\end{aligned} \tag{4.38}$$

while $\mathcal{L}_2^{\text{bulk}}\, W_a^{(\pm)} = 0$. The roots (4.31) of the quartic equation are identified with the label $(a,\pm)$ of the curves in the right hand side of eq. (4.38). Indeed, as a consequence of eq. (4.30), the inhomogeneous terms can again be written in closed form as

$$\mathcal{L}_a^{\text{bulk}} W(\alpha) = f_a(z,\alpha)\,,$$

with $W(\alpha_{a,\pm}) = W_a^{(\pm)}$ and the $f_a(z,\alpha)$ rational functions in the coefficients of the defining equation:

$$f_1(z,\alpha) = \frac{3c}{2\pi^2}\cdot\frac{i\,\phi\,\alpha(\alpha^2+\phi)}{\psi^3}\,, \qquad f_2(z,\alpha) = 0\,, \tag{4.39}$$

for $c = c_1 = c_2$. As is apparent from (4.38), this function satisfies a hypergeometric equation $\mathcal{L}^{\text{inh}} f_1 = 0$. The hypergeometric operator is related to the surface operators by eq. (3.57). In the present case, the relevant operator arises from $\mathcal{L}_2^{\mathcal{D}}$, that is $\mathcal{L}^{\text{inh}} = (\mathcal{L}_2^{\mathcal{D}} + [\mathcal{L}_1^{\text{bdry}}, \mathcal{L}_2^{\mathcal{D}}]{\mathcal{L}_1^{\text{bdry}}}^{-1})|_{\hat{z}_{\text{crit}}}$, while $\mathcal{L}_1^{\mathcal{D}}$ becomes irrelevant. With

$$\mathcal{L}_2^{\mathcal{D}}|_{\hat{z}_{\text{crit}}} = \theta_2(\theta_2-\tfrac{1}{2}) - 4z_2(\theta_2-\tfrac{1}{4})(\theta_2-\tfrac{3}{4})\,, \qquad \mathcal{L}_1^{\text{bdry}}|_{\hat{z}_{\text{crit}}} = i(\theta_2-\tfrac{3}{4})\,,$$

one obtains

$$\mathcal{L}^{\text{inh}} = \theta_2(\theta_2-\tfrac{1}{2}) - 4z_2(\theta_2-\tfrac{1}{4})(\theta_2+\tfrac{1}{4})\,. \tag{4.40}$$

In the above we have used that the relevant surface period is the solution to the Picard-Fuchs system $\{\mathcal{L}_b^{\mathcal{D}}\}$ with index $\frac{1}{2}$ in the variable u_1 to set $\tilde{\theta}_1 = \frac{1}{2}$.

A-model expansion

By mirror symmetry, the above functions should have an integral instanton expansion when expressed in terms of the appropriate coordinates and taking appropriately into account the contributions from multi-covers [52]. For the critical branes at fixed $\hat{z}$, we use the modified multi-cover formulae of the type proposed in [53, 55, 73]:

$$\frac{W_1^{(\pm)}(z(q))}{\omega_0(z(q))} = \frac{1}{(2\pi i)^2}\sum_{k\ odd}\ \sum_{\substack{d_1\ odd\\ d_2\geq 0}} n_{d_1,d_2}^{(1,\pm)}\,\frac{q_1^{kd_1/2}\,q_2^{kd_2}}{k^2}\,, \tag{4.41}$$

$$\frac{W_2^{(\pm)}(z(q))}{\omega_0(z(q))} = \frac{1}{(2\pi i)^2}\sum_{k\ odd}\ \sum_{\substack{d_1\ odd\\ d_2\ odd}} n_{d_1,d_2}^{(2,\pm)}\,\frac{q_1^{kd_1/2}\,q_2^{kd_2/2}}{k^2}\,. \tag{4.42}$$

In this way one obtains the integer invariants in Table 4.6 for $c_a = 1$. As can be guessed from these numbers, the superpotentials for $a = 1,2$ are in fact not independent, but related by

$n^{(1,+)}_{d_1,d_2}$

$d_1 \backslash d_2$	0	1	2	3	4
1	16	48	0	0	0
3	−432	−480	38688	10800	0
5	45440	−78192	5472	92812032	146742768
7	−7212912	25141920	−165384288	61652832	327357559584
9	1393829856	−6895024080	49628432160	−426927933792	261880092960
11	−302514737008	1905539945472	−14487202588320	131586789107520	−1448971951799232
13	70891369116256	−538859226100800	4335978084777792	−39691782337561536	440278250387930640
15	−17542233743427360	155713098595732704	−1328641212531217728	12308540119113753936	−132576278776141577664

$n^{(2,+)}_{d_1,d_2}$

$d_1 \backslash d_2$	1	3	5	7	9	11	13
1	48	16	0	0	0	0	0
3	0	10800	38688	−480	−432	0	0
5	0	82080	26162880	146742768	92812032	5472	−78192
7	0	−10780160	241323840	88380335472	702830702688	1094178697056	327357559584
9	0	1843890480	−36172116480	932346639840	364829042312640	3751178206812144	*
11	0	−369032481792	6979488962400	−143329914498240	4246347124847520	*	*

Table 4.6.: Disc invariants for the on-shell superpotentials $W_a^{(+)}$ of the three-fold $\mathbf{X}_{12}^{(1,2,2,3,4)}$.

a $\mathbb{Z}_2$ symmetry. The family of Calabi-Yau hypersurfaces (4.21) develops a singularity at the discriminant locus $\Delta = 1 - 4z_2 = 0$, which is mirror to a curve of A_1 singularities [184, 185]. On the B-model side the $\mathbb{Z}_2$ monodromy around the singular locus $\Delta = 0$ exchanges the two sets of roots $\alpha_{1,\pm}$ and $\alpha_{2,\pm}$ in eq. (4.31). Accordingly, the superpotentials $W_1^{(\pm)}$ and $W_2^{(\pm)}$ are also exchanged as can be seen from the structure of the inhomogeneous terms. On the level of periods this monodromy action yields

$$t_1 \to t_1 + 3t_2, \qquad t_2 \to -t_2 \ . \tag{4.43}$$

As a result the invariants of W_2 are related to that of W_1 by the $\mathbb{Z}_2$ quantum symmetry $q_1 \to q_1 q_2^3$, $q_2 \to q_2^{-1}$ generated by (4.43).[10]

Extremal transition and a non-compact limit

The above results and the normalization obtained by integration from the subsystem can be verified by taking two different one-parameter limits. At the singular locus $\Delta = 0$, there is an extremal transition to the one parameter family mirror to a degree (6,4) complete intersection hypersurface in $\mathbb{P}^5_{(1,1,1,2,2,3)}$. From eq. (4.43) it follows that the transition takes place at $q_2 = 1$, predicting the relation

$$\sum_{\ell=0}^{3k} n_{k,\ell}^{(a,+)}(\mathbf{X}_{12}^{(1,2,2,3,4)}) = n_k(\mathbf{X}_{6,4}^{(1,1,1,2,2,3)}) \ , \quad a = 1, 2 \ , \tag{4.44}$$

[10]The $\mathbb{Z}_2$ symmetry is also realized on the closed-string invariants, see the results of [108].

where (k,ℓ) denote the degree in q_1 and q_2, respectively. The finiteness of the sum over ℓ follows from the symmetry (4.43). From the left hand side of the above equation one gets

$$n_k = 64,\ 48\,576,\ 265\,772\,480,\ 2\,212\,892\,036\,032,\ 22\,597\,412\,764\,939\,776,\ \ldots \tag{4.45}$$

for the first invariants of $\mathbf{X}_{6,4}^{(1,1,1,2,2,3)}$. This can be checked by a computation for the complete intersection manifold with the inhomogeneous Picard-Fuchs equation

$$\mathcal{L}\ W(z) = \frac{4\sqrt{z}}{(2\pi i)^2}\ , \qquad \mathcal{L} = \theta^4 - 48z(6\theta+5)(6\theta+1)(4\theta+3)(4\theta+1)\ . \tag{4.46}$$

Another interesting one modulus limit is obtained for $z_2 \to 0$, where X degenerates to the non-compact hypersurface

$$X^\flat: \qquad y_1^2 + y_2^3 + y_3^6 + y_4^6 + y_5^{-6} + \hat{\psi}\, y_1y_2y_3y_4y_5 = 0, \qquad \hat{\psi} = \frac{\psi}{\sqrt{\phi}} = z_1^{-1/6} \tag{4.47}$$

in weighted projective space $\mathbb{P}^4_{3,2,1,1,-1}$, with the new variables y_i related to the x_i by

$$y_1 = \phi^{1/2}x_4x_1^3,\ y_2 = x_5,\ y_3 = x_2,\ y_4 = x_3,\ y_5 = x_1^{-2}\ .$$

The non-compact three-fold $X^\flat$ is a local model for a certain type of singularity associated with the appearance of non-critical strings and has been studied in detail in [186].

In this limit the curves $C_{\alpha_{2,\pm},\kappa}$ of eq. (4.23) are pushed to the boundary of the local three-fold geometry $X^\flat$ and the domain wall tension between $C_{\alpha_{2,+},\kappa}$ and $C_{\alpha_{2,-},\kappa}$ becomes independent of the modulus z_1, which is reflected by the fact that all the disc invariants of W_2 vanish in the limit $z_2 \to 0$. The curves $C_{\alpha_{1,\pm},\kappa}$ become

$$C^\flat_{\varepsilon,\kappa} = \left\{ y_3 = \eta\, y_4\ ,\ y_1y_5^3 = \varepsilon\ ,\ y_2y_5 = \kappa\, y_4\sqrt{\varepsilon\eta\hat{\psi}} \right\}\ , \quad \varepsilon = \pm i\ ,\ \kappa = \pm i\ , \tag{4.48}$$

where $\varepsilon = \pm i$ distinguishes between the two roots $\alpha_{1,+}$ and $\alpha_{1,-}$. In Appendix G we show, that the 3-chain integral representing the domain wall tension in $X^\flat$ descends to an Abel-Jacobi map on a Riemann surface, which can be computed explicitly as an geometric integral. The invariants $n^{[6]}$ obtained for the superpotential in the non-compact geometry $X^\flat$ are reported in Appendix G and they agree with the q_2^0 term of T_1, $n_{k,0} = n_k^{[6]}$.

4.2.3. A second family of divisors and symmetric K3s

The same critical points can be embedded into a different family of divisors

$$Q(\mathcal{D}_2) = x_4^4 + z_3z_2^{-1/2}x_1^6x_4^2\,. \tag{4.49}$$

Our motivation to consider this second family in detail is two-fold. Firstly, the Hodge problem on the surface is equivalent to that of a two parameter family of K3 surfaces at a special point in the moduli, which can be studied explicitly without too many technicalities. We will

$\Delta(Z)$	$\tilde{\nu}_0$	=	(	0	0	0	0	0	)
	$\tilde{\nu}_1$	=	(	2	2	3	4	0	)
	$\tilde{\nu}_2$	=	(	−1	0	0	0	0	)
	$\tilde{\nu}_3$	=	(	0	−1	0	0	0	)
	$\tilde{\nu}_4$	=	(	0	0	−1	0	0	)
	$\tilde{\nu}_5$	=	(	0	0	0	−1	0	)
	$\tilde{\nu}_6$	=	(	1	1	1	2	0	)
$\Delta_\flat(Z, L_2) = \Delta\cup$	$\tilde{\nu}_7$	=	(	0	0	−1	0	1	)
	$\tilde{\nu}_8$	=	(	1	1	1	2	1	)

Table 4.7.: Vertices of the enhanced polyhedron $\Delta_\flat(Z, L_2)$ for the second divisor family $\mathcal{D}_2$ on the Calabi-Yau $\mathbf{X}_{12}^{(1,2,2,3,4)}$.

explicitly show that the relevant zero of the period vector arises at an orbifold point of the K3, which has been interpreted as a point with a half-integral B-field for the closed-string compactification on the local geometry [127]. Secondly, this family tests a different direction of the off-shell deformation space of the brane, leading to a different off-shell superpotential $\mathcal{W}$ for the deformation (4.49). However, since the family contains the curves $C_{\alpha,\kappa}$ for $z_3 = -\alpha^2 z_2^{1/2}$, the critical superpotential has to be the same as the one obtained for the family $\mathcal{D}_1$ in eq. (4.35). The agreement with the previous result and normalization gives an explicit illustration of the fact that different parametrizations of the off-shell directions, corresponding to a different choice of light fields represented by different relative cohomologies, fit together consistently near the critical locus.

As the critical point is determined by the vanishing condition (3.29), we again study the subsystem $P(Z^*) = Q(\mathcal{D}_2) = 0$. Solving for x_4 and changing coordinates to $\tilde{x}_1 = x_1^4$, the surface can be described as a cover[11] of a mirror family of K3 hypersurfaces

$$\tilde{x}_1^3 + x_2^6 + x_3^6 + x_5^3 + \tilde{\psi}\tilde{x}_1 x_2 x_3 x_5 + \tilde{\phi}(x_2 x_3)^3 = 0 \ .$$

Here $\tilde{\psi}^{-6} := u = -\frac{z_1 z_2}{z_3^3}(z_2 - z_3 + z_3^2)^2$ and the parameter $\tilde{\phi}$ is zero for the embedded surface. At $\tilde{\phi} = 0$, the GLSM for this family is defined by the charges

	0	1	2	3	5
$\tilde{l}$	-6	2	1	1	2

.

The GKZ system for this one modulus GLSM has an exceptional solution

$$\pi(u) = \frac{c}{2}\, B_{\{\tilde{l}\}}(u; \tfrac{1}{2}) = \frac{c}{2} \sum_{n=0}^{\infty} \frac{\Gamma(4+6n)}{\Gamma(2+2n)^2 \Gamma(\frac{3}{2}+n)^2} u^{n+\frac{1}{2}} \ , \tag{4.50}$$

[11] The change from x_1 to $\tilde{x}_1$ gives a fourfold cover acted on by a remaining $\mathbb{Z}_2$ action generated by g_1 in (4.22).

that vanishes at the critical point $u = 0$. To get a better understanding of this solution and of the integral periods on the surface, one may describe π as a regular solution of the two parameter family of K3 surfaces parametrized by $\tilde{\psi}$ and $\tilde{\phi}$, restricted to the symmetric point $\tilde{\phi} = 0$. The charges of the GLSM for the two parameter family of K3 manifolds are

	0	1	2	3	4	5
$\tilde{l}^1$	-3	1	0	0	1	1
$\tilde{l}^2$	0	0	1	1	0	-2

.

The two algebraic moduli of this family are $v_1 = -\tilde{\phi}\tilde{\psi}^{-3}$ and $v_2 = \tilde{\phi}^{-2}$ and these are related to the single modulus of the embedded surface by $u = \tilde{\psi}^{-6} = v_1^2 v_2$. The principal discriminant locus for this family has the two components

$$\Delta = \Delta_0 \cdot \Delta_1 = (1 + 54v_1 + 729v_1^2 - 2916v_1^2 v_2) \cdot (1 - 4v_2) \ .$$

The periods near $\tilde{\phi} = 0$ can be computed in the phase of the two parameter GLSM with coordinates $u_1 = v_1 v_2^{1/2}$ and $u_2 = v_2^{-1/2}$. The hypergeometric series

$$\tilde{\pi}(u_1, u_2) = \frac{c}{2\pi^2} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{\Gamma(1+3n)\Gamma(\frac{1}{2} - n + p)^2}{\Gamma(1+n)^2 \Gamma(2 - n + 2p)} u_1^n u_2^{1+2p-n} \tag{4.51}$$

is a solution of the Picard-Fuchs equation that restricts to $\pi(\sqrt{u})$ in the limit $u_2 = 0$. This series can be expressed with the help of a Barnes type integral as

$$\tilde{\pi}(u_1, u_2) = -\frac{c}{2\pi^2} \int_{\mathcal{C}_+} \sum_{n=0}^{\infty} \frac{\Gamma(1+3n)\Gamma(\frac{1}{2}+s)^2 \Gamma(1+s)\Gamma(-s)(-1)^s}{\Gamma(1+n)^2 \Gamma(2+n+2s)} (u_1 u_2)^n u_2^{1+2s} \tag{4.52}$$

$$+ \frac{c}{2\pi^2} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{\Gamma(1+3n)\Gamma(\frac{1}{2} - p)^2}{\Gamma(1+n)^2 \Gamma(2+n-2p)} (u_1 u_2)^n u_2^{1-2p} \ , \tag{4.53}$$

where the contour $\mathcal{C}_+$ encloses the poles of the Gamma functions on the positive real line including zero. To relate the special solution $\tilde{\pi}(u_1, u_2)$ to the integral periods on the K3, one may analytically continue it to large complex structure by closing the contour to the left and obtains

$$\begin{aligned} \tilde{\pi}(v_1, v_2) &= \frac{c}{2\pi i} \sum_{n,p=0}^{\infty} \frac{\Gamma(1+3n) v_1^n v_2^p \left(-i\pi + \ln(v_2) + 2(\Psi(1+n-2p) - \Psi(1+p))\right)}{\Gamma(1+n)^2 \Gamma(1+n-2p)\Gamma(1+p)^2} \\ &= c\,\omega_0 \left(t_2^{K3} - \frac{1}{2} \right) \ . \end{aligned} \tag{4.54}$$

Here $\omega_0 = B_{\tilde{l}}(v_a; 0, 0)$ is the fundamental integral period at large volume and the quantity $t_2^{K3} = (2\pi i \omega_0)^{-1} \partial_{\rho_2} B_{\tilde{l}}|_{\rho_a = 0}$ is the integral period associated with the volume of another 2-cycle C, which is mirror to the base of the elliptic fibration defined by the GLSM of the A-model side.

From the last expression it follows that the zero of the K3 period vector associated with the D-brane vacuum arises at the locus

$$J^{K3} = \operatorname{Im} t_2^{K3} = 0 \ , \qquad B^{K3} = \operatorname{Re} t_2^{K3} = \tfrac{1}{2} \ , \tag{4.55}$$

which, in the closed string compactification on this local K3 geometry, is interpreted as a 2-cycle of zero volume with a half-integral B-field. Indeed, in the limit $u = 0 = u_1$, eq. (4.51) becomes

$$\tilde{\pi}(u_1, u_2)|_{u_1=0} \sim \ln\left(\frac{1 - 2v_2 - \sqrt{1 - 4v_2}}{2v_2}\right) - i\pi \ ,$$

expanded around $v_2 = \infty$. The first term on the right hand side is the period for the compact cycle of the $\mathbb{C}^2/\mathbb{Z}_2$-quotient singularity studied in [127], which is zero on the discriminant locus $\Delta_1 = 0$, but a constant at $v_2 = \infty$. The zero associated with the critical point hence does not appear on the principal discriminant, but at an orbifold point with non-vanishing complex quantum volume. It has been argued in [53, 54], that the A-model data associated with the critical points of the present type include $\mathbb{Z}_2$-valued open-string degrees of freedom from the choice of a discrete gauge field on the A-brane. Here we see that to this discrete choice in the A-model there corresponds, at least formally, a half-integral valued B-field for the tension in the B-model geometry. It would be interesting to study this phenomenon and its $\mathbb{C}^2/\mathbb{Z}_n$ generalizations in more detail.

As in the previous parametrization, the tensions can be computed from the integrals

$$T_a = \frac{1}{2\pi i} \int_*^{\beta_a} \pi(u(\xi)) \frac{d\xi}{\xi} \ ,$$

where $\beta_{1/2} = \pm i z_2^{1/4} \alpha_{1/2}$, with $\alpha_{1/2}$ defined in eq. (4.31). We again choose the reference point such that $W^{(+,\alpha)} = -W^{(-,\alpha)}$ and find

$$W^{(\pm,\alpha_1)} = \mp \frac{c}{8} \cdot B_{\{l^1,l^2\}}\left(z_1, z_2; \tfrac{1}{2}, 0\right) \ , \quad W^{(\pm,\alpha_2)} = \mp \frac{c}{8} \cdot B_{\{l^1,l^2\}}\left(z_1, z_2; \tfrac{1}{2}, \tfrac{1}{2}\right) \ , \tag{4.56}$$

which is in agreement with (4.35) for $c = 1$.

4.3. Branes on $\mathbf{X}_{18}^{(1,1,1,6,9)}$

As a third example we study branes on the two moduli Calabi-Yau $Z = \mathbf{X}_{18}^{(1,1,1,6,9)}$. For this geometry closed string mirror symmetry has been studied in [117]. The manifold Z is an elliptic fibration over $\mathbb{P}^2$ with the elliptic fiber and the base parametrized by the coordinates x_4, x_5, x_6 and x_1, x_2, x_3 in (4.57), respectively. In the decompactification limit of large fiber, the compact Calabi-Yau approximates the non-compact Calabi-Yau $\mathcal{O}(-3)_{\mathbb{P}^2}$ with coordinates x_1, x_2, x_3, x_6. This limit is interesting, as it makes contact to the previous studies of branes on $\mathcal{O}(-3)_{\mathbb{P}^2}$ in [162, 49].

$\Delta(Z)$									
	$\tilde\nu_0$	=	(	0	0	0	0	)
	$\tilde\nu_1$	=	(	1	1	6	9	)
	$\tilde\nu_2$	=	(	−1	0	0	0	)
	$\tilde\nu_3$	=	(	0	−1	0	0	)
	$\tilde\nu_4$	=	(	0	0	−1	0	)
	$\tilde\nu_5$	=	(	0	0	0	−1	)
	$\tilde\nu_6$	=	(	0	0	2	3	)

Table 4.8.: Vertices of the bulk geometry $\Delta(Z)$ for $\mathbf{X}_{18}^{(1,1,1,6,9)}$.

4.3.1. Bulk geometry

The toric ambient geometry of $\mathbf{X}_{18}^{(1,1,1,6,9)}$ is given by the polyhedron $\Delta(Z)$ in Table 4.8. A maximal triangulation of $\Delta(Z)$ leads to the following relations

$$\begin{array}{c|c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \tilde l^1 & -6 & 0 & 0 & 0 & 2 & 3 & 1 \\ \tilde l^2 & 0 & 1 & 1 & 1 & 0 & 0 & -3 \end{array} . \tag{4.57}$$

In homogeneous coordinates of $\mathbb{P}^4_{(1,1,1,6,9)}$ the hypersurface constraint for the mirror manifold in $\mathbb{P}^4_{(1,1,1,6,9)}/(\mathbb{Z}_{18}\times\mathbb{Z}_6)$ becomes

$$P(Z^*) = x_1^{18} + x_2^{18} + x_3^{18} + x_4^3 + x_5^2 + \psi\,(x_1x_2x_3x_4x_5) + \phi\,(x_1x_2x_3)^6 \,, \tag{4.58}$$

where $\psi = z_1^{-1/6} z_2^{-1/18}$ and $\phi = z_2^{-1/3}$. The coordinates $z_1 = \frac{a_4^2 a_5^3 a_6}{a_0^6}$ and $z_2 = \frac{a_1a_2a_3}{a_6^3}$ are local coordinates in the neighborhood of the point of maximal unipotent monodromy. The Greene-Plesser orbifold group acts as $x_i \to \lambda_k^{g_{k,i}}\, x_i$ with $\lambda_1^{18}=1$, $\lambda_2^6=1$ and the weights

$$\mathbb{Z}_{18}:\, g_1 = (1,-1,0,0,0), \quad \mathbb{Z}_6:\, g_2 = (0,1,3,2,0)\,. \tag{4.59}$$

4.3.2. Brane geometry

We consider a family of A-branes parametrized by the relations

$$|\tilde x_3|^2 - |\tilde x_6|^2 = \hat c, \qquad \begin{array}{c|c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \hat l & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{array} . \tag{4.60}$$

This defines a family of D7-branes in the mirror parametrized by one complex modulus. To make contact with the non-compact branes we may add a second constraint $|x_2|^2 - |x_6|^2 = 0$ that selects a particular solution of the Picard-Fuchs system.[12] The brane geometry on the

[12] Since the constant in this equation must be zero to get a non-zero superpotential [142], there is no new modulus.

$\Delta(Z)$	$\tilde{\nu}_0$	$=(\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ )$
	$\tilde{\nu}_1$	$=(\ 1\ \ 1\ \ 6\ \ 9\ \ 0\ )$
	$\tilde{\nu}_2$	$=(\ -1\ \ 0\ \ 0\ \ 0\ \ 0\ )$
	$\tilde{\nu}_3$	$=(\ 0\ \ -1\ \ 0\ \ 0\ \ 0\ )$
	$\tilde{\nu}_4$	$=(\ 0\ \ 0\ \ -1\ \ 0\ \ 0\ )$
	$\tilde{\nu}_5$	$=(\ 0\ \ 0\ \ 0\ \ -1\ \ 0\ )$
	$\tilde{\nu}_6$	$=(\ 0\ \ 0\ \ 2\ \ 3\ \ 0\ )$
$\Delta_\flat(Z,L_1)=\Delta\cup$	$\tilde{\nu}_7$	$=(\ 0\ \ -1\ \ 0\ \ 0\ \ -1\ )$
	$\tilde{\nu}_8$	$=(\ 0\ \ 0\ \ 2\ \ 3\ \ -1\ )$

Table 4.9.: Vertices of the enhanced polyhedron $\Delta_\flat(Z,L_1)$ for the family $\mathcal{D}_1$ on $\mathbf{X}_{18}^{(1,1,1,6,9)}$.

B-model side is defined by the two equations

$$\begin{aligned} &P(Z^*)=\sum a_i y_i = a_0x_1x_2x_3x_4x_5+a_1x_1^{18}+a_2x_2^{18}+a_3x_3^{18}+a_4x_4^3+a_5x_5^2+a_6(x_1x_2x_3)^6,\\ &Q(\mathcal{D}_1):\qquad y_3=y_6 \qquad \text{or} \qquad (x_1x_2x_3)^6=x_3^{18}\ . \end{aligned} \tag{4.61}$$

As in the previous cases one observes that the complex deformations of the brane geometry are related to the periods of a K3 surface defined by

$$P_{\mathcal{D}_1}=a_0x_1'x_2'x_3'x_4'+(a_3+a_6)(x_1'x_2')^6+a_1{x_1'}^{12}+a_2{x_2'}^{12}+a_4{x_4'}^3+a_5{x_5'}^2\ .$$

The GLSM for the above brane geometry corresponds to the enhanced polyhedron $\Delta_\flat(Z,L_1)$ in Table 4.9. Choosing a triangulation of $\Delta_\flat(Z,L_1)$ that represents a large complex structure phase yields the following basis of the linear relations:

$$\begin{array}{c|c|cccccc|cc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\\ \hline l^1 & -6 & 0 & 0 & 0 & 2 & 3 & 1 & 0 & 0\\ l^2 & 0 & 1 & 1 & 0 & 0 & 0 & -2 & 1 & -1\\ l^3 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 \end{array} \tag{4.62}$$

The last two charge vectors define a GLSM for the "inner phase" of the brane in the non-compact Calabi-Yau described in [49]. The differential operators (3.51) for the relative periods are given by

$$\begin{aligned} \mathcal{L}_1&=\theta_1(\theta_1-2\theta_2-\theta_3)-12z_1(6\theta_1+5)(6\theta_1+1),\\ \mathcal{L}_2&=\theta_2^2(\theta_2-\theta_3)+z_2(\theta_1-2\theta_2-\theta_3)(\theta_1-2\theta_2-1-\theta_3)(\theta_2-\theta_3),\\ \mathcal{L}_3&=-\theta_3(\theta_2-\theta_3)-z_3(\theta_1-2\theta_2-\theta_3)(\theta_2-\theta_3)\ . \end{aligned} \tag{4.63}$$

4.3.3. Large volume brane

The elliptic fiber compactifies the non-compact fiber direction x_6 of the non-compact Calabi-Yau $\mathcal{O}(-3)_{\mathbb{P}^2}$. In the limit of large elliptic fiber we therefore expect to find a deformation of

$l \backslash m$	0	1	2	3	4	5	6
0	$*$	1	0	0	0	0	0
1	1	$*$	−1	−1	−1	−1	−1
2	−1	−2	$*$	5	7	9	12
3	1	4	12	$*$	−40	−61	−93
4	−2	−10	−32	−104	$*$	399	648
5	5	28	102	326	1085	$*$	−4524
6	−13	−84	−344	−1160	−3708	−12660	$*$
7	35	264	1200	4360	14274	45722	159208
8	−100	−858	−4304	−16854	−57760	−185988	−598088
9	300	2860	15730	66222	239404	793502	2530946
10	−925	−9724	−58208	−262834	−1004386	−3460940	−11231776

Table 4.10.: Invariants $N_{0,l,m}$ for the geometry (4.62).

the brane studied in [162, 49]. Large volume corresponds to $z_a = 0$ in the coordinates defined by eqs. (4.62),(2.82).

The mirror maps and the superpotential can be computed from (4.63). Expressing the superpotential in the flat coordinates t_a gives the Ooguri-Vafa invariants. The homology class β can be labelled by three integers (k, l, m) that determine the Kähler volume $kt_1+lt_2+mt_3$ of a curve in this class. Here t_1 is the volume of the elliptic fiber and t_2, t_3 are the (D4-)volumes of two homologically distinct discs in the brane geometry. The Kähler class of the section, which measures the volume of the fundamental sphere in $\mathbb{P}^2$, is $t_2 + t_3$.

For the discs that do not wrap the elliptic fiber $\beta = (0, l, m)$ we obtain the invariants in Table 4.10. This result agrees with the results of [162, 49] for the disc invariants in the "inner phase" of the non-compact Calabi-Yau $\mathcal{O}(-3)_{\mathbb{P}^2}$ and can be explained heuristically as follows. The holomorphic discs ending on the non-compact A-brane in $\mathcal{O}(-3)_{\mathbb{P}^2}$ lie within the zero section of $\mathcal{O}(-3)_{\mathbb{P}^2}$. Similarly discs with $k = 0$ in $\mathbf{X}_{18}^{(1,1,1,6,9)}$ are holomorphic curves that must map to the section $x_6 = 0$ of the elliptic fibration. The moduli space of maps into the sections of the non-compact and compact manifolds, respectively, does not see the compactification in the fiber, explaining the agreement. The agreement of the two computations can be viewed as a statement of local mirror symmetry in the open string setup. For worldsheets that wrap the fiber on obtains the numbers in Table 4.11. It would be interesting to confirm some of these numbers by an independent computation.

4.3.4. Deformation of the non-compact involution brane

In [187] an involution brane in the local model $\mathcal{O}(-3)_{\mathbb{P}^2}$ has been studied. Similarly as in the previous case one expects to find a deformation of this brane by embedding it in the compact manifold and taking the limit of large elliptic fiber, $z_1 = 0$. In order to recover the involution

$l \backslash m$	0	1	2	3	4	5
			$k=1$			
0	∗	252	0	0	0	0
1	−240	∗	300	300	300	300
2	240	780	∗	−2280	−3180	−4380
3	−480	−2040	−6600	∗	24900	39120
4	1200	6300	22080	74400	∗	−315480
5	−3360	−21000	−82200	−276360	−957600	∗
6	10080	73080	319200	1134000	3765000	13300560
7	−31680	−261360	−1265040	−4818240	−16380840	−54173880

$l \backslash m$	0	1	2	3	4
			$k=2$		
0	∗	5130	−18504	0	0
1	−141444	∗	−73170	−62910	−62910
2	−28200	−108180	∗	544140	778560
3	85320	403560	1557000	∗	−7639920
4	−285360	−1647540	−6485460	−24088680	∗
5	1000440	6815160	29214540	106001100	392435460
6	−3606000	−28271880	−133294440	−505417320	−1773714840

Table 4.11.: Invariants $N_{k,l,m}$ for the geometry (4.62).

brane of the local geometry we study the critical points near $z_3 = 1$ in the local coordinates

$$\tilde{z}_1 = z_1(-z_2)^{1/2}, \quad u = (-z_2)^{-1/2}(1-z_3)\,, \quad v = (-z_2)^{1/2}\,.$$

After transforming the Picard-Fuchs system to these variables, the solution corresponding to the superpotential has the following expansion

$$c\,\mathcal{W}(u,v,\tilde{z}_1) = -v - \frac{35v^3}{9} + \frac{1}{2}uv^2 + \frac{200}{3}\tilde{z}_1 v^2 - \frac{u^2 v}{8} - 12320\tilde{z}_1^2 v - 60u\tilde{z}_1 v + \dots\,, \tag{4.64}$$

where c is a constant that will be fixed again by comparing the critical value with the results of [187]. In the decompactification limit $\tilde{z}_1 = 0$, the critical point of the superpotential is at $u = 0$, where we obtain the following expansion

$$c\,W_{\mathrm{crit}}(z_2) = -\sqrt{z_2} - \frac{35}{9}z_2^{3/2} - \frac{1001}{25}z_2^{5/2} + \dots\,, \tag{4.65}$$

The restricted superpotential satisfies the differential equation

$$\mathcal{L}_{\mathrm{bulk}} W_{\mathrm{crit}}(z_2) = -\frac{\sqrt{z_2}}{8c}\,,$$

with $\mathcal{L}_{\mathrm{bulk}}$ the Picard-Fuchs operator of the local geometry $\mathcal{O}(-3)_{\mathbb{P}^2}$. The above expressions at the critical point agree with the ones given in [187] for $c = 1$. Using the closed string mirror map the real invariants for the involution brane on $\mathcal{O}(-3)_{\mathbb{P}^2}$ are given by [187]

$$W_{\rm crit}(z(q)) = \sum_{k\,\rm odd}\left(\frac{(-1)}{k^2}q^{k/2}+\frac{1}{k^2}q^{3k/2}+\frac{(-5)}{k^2}q^{5k/2}+\frac{42}{k^2}q^{7k/2}+\frac{(-429)}{k^2}q^{9k/2}+\ldots\right) \quad (4.66)$$

As might have been expected, the full superpotential (4.64) shows that the involution brane of the local model is non-trivially deformed in the compact Calabi-Yau manifold for $z_1 \neq 0$. It is not obvious that the modified multi-cover description of [53], which is adapted to real curves and differs from the original proposal of [52], can be generalized to obtain integral invariants for the deformations of the critical point in the z_1 direction. One suspects that an integral expansion in the sense of [53] exists only at critical points with an extra symmetry and for deformations that respect this symmetry.

4.3.5. Further domain wall tensions

In the bulk geometry of $\mathbf{X}_{18}^{(1,1,1,6,9)}$ we consider domain walls which connect the curves

$$\begin{aligned} C_{\alpha,\pm} &= \{x_2=\eta_1 x_1,\ x_5=\eta_2 x_3^9-\frac{\psi}{2}x_1x_2x_3x_4,\ x_4=\psi^2\alpha(x_1x_2x_3)^2\}\,, \\ &\qquad \eta_1^{18}=\eta_2^2=-1, \quad \alpha^3-\frac{1}{4}\alpha^2+\frac{\phi}{\psi^6}=0\,, \end{aligned} \quad (4.67)$$

where different choices for $\eta_{1,2}$ are identified under the Greene-Plesser group as $(\eta_1,\eta_2,\alpha)\sim(\eta_1\lambda_1^2\lambda_2^{-1},\eta_2\lambda_2^3,\alpha)$, and we distinguish the curves $C_{\alpha,+}$ and $C_{\alpha,-}$ by the orbits of the labels (η_1,η_2,α) under this orbifold action. Specifically the orbits $C_{\alpha,\pm}$ contain the components $\eta_1^9=\pm i$ for fixed $\eta_2=i$ and fixed α, respectively.

Divisor geometry and tensions

To capture the deformation of the above curves we study the family of divisors

$$Q(\mathcal{D}_2)=x_2^{18}+z_3x_1^{18}\,. \quad (4.68)$$

The periods on this family are captured by the GLSM with charges

	0	1	2	3	4	5
$\tilde{l}^1$	-6	0	0	2	3	1
$\tilde{l}^2$	0	1	2	0	0	-3

,

where the two algebraic moduli are $u_1=z_1$ and $u_2=-\frac{z_2}{z_3}(1-z_3)^2$. The exceptional solutions

$$\begin{aligned} \pi_1(u) &= \frac{c_1}{2}\,B_{\{\tilde{l}^1,\tilde{l}^2\}}(u_1,u_2;0,\tfrac{1}{2}) = -\frac{c_1}{2\pi}\sqrt{u_2}\ {}_2F_1\left(\tfrac{1}{6},\tfrac{5}{6},-\tfrac{1}{2},432u_1\right)+\mathcal{O}(u_2^{3/2})\,, \\ \pi_2(u) &= \frac{c_2}{2}\,B_{\{\tilde{l}^1,\tilde{l}^2\}}(u_1,u_2;\tfrac{1}{2},\tfrac{1}{2}) = \frac{2048c_2}{\pi}u_1^{3/2}\sqrt{u_2}\ {}_2F_1\left(\tfrac{5}{3},\tfrac{7}{3},\tfrac{5}{2},432u_1\right)+\mathcal{O}(u_2^{3/2})\,, \end{aligned} \quad (4.69)$$

vanish at the critical point $u_2=0$. Similarly to eq. (4.32) we define off-shell superpotentials by

$$\mathcal{W}_a^{(\pm)}(z_1,z_2,z_3)=\frac{1}{2\pi i}\int_{\xi_0}^{\pm\sqrt{z_3}}\pi_a(u(z_1,z_2,\xi^2))\,\frac{d\xi}{\xi}\ ,$$

$\Delta(Z)$	$\tilde{\nu}_0$	=	(	0	0	0	0	0	)
	$\tilde{\nu}_1$	=	(	1	1	6	9	0	)
	$\tilde{\nu}_2$	=	(	−1	0	0	0	0	)
	$\tilde{\nu}_3$	=	(	0	−1	0	0	0	)
	$\tilde{\nu}_4$	=	(	0	0	−1	0	0	)
	$\tilde{\nu}_5$	=	(	0	0	0	−1	0	)
	$\tilde{\nu}_6$	=	(	0	0	2	3	0	)
$\Delta_\flat(Z,L_2)=\Delta\cup$	$\tilde{\nu}_7$	=	(	1	1	6	9	−1	)
	$\tilde{\nu}_8$	=	(	−1	0	0	0	0	)

Table 4.12.: Vertices of the enhanced polyhedron $\Delta_\flat(Z,L_2)$ for the family $\mathcal{D}_2$ on $\mathbf{X}_{18}^{(1,1,1,6,9)}$.

with the fixed reference point ξ_0. For $\xi_0 = i$ the contribution of the reference point vanishes, and at the critical value $z_3 = 1$ we arrive at the on-shell superpotentials $W_a^{(\pm)}$, where the $\pm$-label is now correlated with the orbits of the curves (4.67)

$$\begin{aligned} W_1^{(\pm)} &= \pm\frac{c_1}{8}\sum_{n_i\geq 0}\frac{\Gamma(6n_1+1)\; z_1^{n_1} z_2^{n_2+\frac{1}{2}}}{\Gamma(2n_1+1)\,\Gamma(3n_1+1)\,\Gamma\left(n_1-3n_2-\frac{1}{2}\right)\Gamma\left(n_2+\frac{3}{2}\right)^3}\,, \\ W_2^{(\pm)} &= \pm\frac{c_2}{8}\sum_{n_i\geq 0}\frac{\Gamma(6n_1+4)\; z_1^{n_1+\frac{1}{2}} z_2^{n_2+\frac{1}{2}}}{\Gamma(2n_1+2)\,\Gamma\left(3n_1+\frac{5}{2}\right)\Gamma(n_1-3n_2)\,\Gamma\left(n_2+\frac{3}{2}\right)^3}\,. \end{aligned} \tag{4.70}$$

They can be expressed in terms of the bulk generating function as

$$W_1^{(\pm)} = \pm\frac{c_1}{8}B_{\{l^1,l^2\}}\left(z_1,z_2;0,\tfrac{1}{2}\right)\,,\quad W_2^{(\pm)} = \pm\frac{c_2}{8}B_{\{l^1,l^2\}}\left(z_1,z_2;\tfrac{1}{2},\tfrac{1}{2}\right)\,. \tag{4.71}$$

Again these functions can be also obtained as solutions to the large GKZ system (3.51) of the relative cohomology problem. The vertices of the enhanced polyhedron $\Delta_\flat(Z,L_2)$ are given in Table 4.12. For the family (4.68) triangulation of $\Delta_\flat(Z,L_2)$ gives the additional charge vector

	0	1	2	3	4	5	6	7	8
l^3	0	-1	1	0	0	0	0	1	-1

.

This leads to the generalized hypergeometric system

$$\begin{aligned} \tilde{\mathcal{L}}_1 &= \theta_1(\theta_1-3\theta_2) - 12z_1(6\theta_1+1)(6\theta_1+5)\,, \\ \tilde{\mathcal{L}}_2 &= \theta_2(\theta_2-\theta_3)(\theta_2+\theta_3) - z_2(\theta_1-3\theta_2)(\theta_1-3\theta_2-1)(\theta_1-3\theta_2-2)\,, \\ \tilde{\mathcal{L}}_3 &= \theta_3(\theta_2+\theta_3) + z_3\theta_3(\theta_2-\theta_3)\,, \end{aligned} \tag{4.72}$$

annihilating the relative period integrals. There are two solutions with a minimum at the critical locus $\ln(z_3) = 0$ that restrict to the on-shell superpotentials $W_1^{(\pm)}$ and $W_2^{(\pm)}$, respectively.

To characterize the on-shell superpotentials $W_a^{(\pm)}$ as solutions to an inhomogeneous Picard-Fuchs equation we note that

$$\tilde{\mathcal{L}}_1 = \mathcal{L}_1^{\text{bulk}}, \qquad \tilde{\mathcal{L}}_2 = \mathcal{L}_2^{\text{bulk}} - \theta_2\theta_3^2 .$$

So only the second operator acquires an inhomogeneous term, which is determined by the leading part of the surface periods $\pi_a(u)$. Acting with $\theta_2\theta_3$ on the terms in (4.69) one obtains the inhomogeneous Picard-Fuchs equations

$$\begin{aligned} A_1^{(\pm)} &:= \mathcal{L}_2^{\text{bulk}} W_1^{(\pm)} &&= \pm\frac{-c_1}{16\pi^2}\sqrt{z_2}\,{}_2F_1\left(\tfrac{1}{6},\tfrac{5}{6};-\tfrac{1}{2};432z_1\right) , \\ A_2^{(\pm)} &:= \mathcal{L}_2^{\text{bulk}} W_2^{(\pm)} &&= \pm\frac{4096c_2}{16\pi^2} z_1^{3/2}\sqrt{z_2}\,{}_2F_1\left(\tfrac{5}{3},\tfrac{7}{3};\tfrac{5}{2};432z_1\right) . \end{aligned} \tag{4.73}$$

To find the geometric domain wall tensions, we note that the three roots α_ℓ of the cubic equation (4.67) can be written as

$$\alpha_\ell = \frac{1}{12}\left(1 + e^{\frac{2\pi i}{3}(\ell-1)}\,\Delta + e^{-\frac{2\pi i}{3}(\ell-1)}\,\frac{1}{\Delta}\right) , \quad \ell = 1,2,3 ,$$

with

$$\Delta = \sqrt[3]{1 - 864z_1 + 2\sqrt{432z_1\,(432z_1-1)}} .$$

Under a monodromy $z_1^{-1} \to e^{2\pi i}\,z_1^{-1}$ around $z_1 = \infty$, Δ transforms as $\Delta \to e^{\frac{2\pi i}{3}}\Delta$ and the three roots are permuted according to $\alpha_\ell \to \alpha_{\ell+1}$. On the curves $C_{\alpha,\pm}$ the monodromy acts as the $\mathbb{Z}_6$ symmetry

$$M(z_1 = \infty) : \quad \begin{pmatrix} \alpha_{1,\pm} \\ \alpha_{2,\pm} \\ \alpha_{3,\pm} \end{pmatrix} \mapsto \begin{pmatrix} \alpha_{2,\mp} \\ \alpha_{3,\pm} \\ \alpha_{1,\pm} \end{pmatrix} .$$

It follows that the domain walls between the curves $C_{\alpha,\pm}$ for fixed α must be permuted under the monodromy as well. To this end note that the hypergeometric functions in eq. (4.73) are solutions of the same hypergeometric differential equation and in fact are related by monodromy. Indeed, for $c \equiv c_1 \equiv c_2$, the inhomogeneous pieces can be expressed in terms of the single rational expression

$$f_1(\alpha) = 0 , \qquad f_2(\alpha) = \frac{c}{4\pi^2}\frac{1-12\alpha}{\psi^9\,\alpha^3\,(1-6\alpha)^3} , \tag{4.74}$$

as

$$f_2(\alpha_1) = -2A_2^{(+)} , \qquad f_2(\alpha_2) = -A_1^{(+)} + A_2^{(+)} , \qquad f_2(\alpha_3) = A_1^{(+)} + A_2^{(+)} . \tag{4.75}$$

From the above we obtain the following linear combinations for the geometric superpotentials

$$W_{\alpha_1}^{(\pm)} = 2\,W_2^{(\pm)} , \qquad W_{\alpha_2}^{(\pm)} = W_1^{(\pm)} - W_2^{(\pm)} , \qquad W_{\alpha_3}^{(\pm)} = W_1^{(\pm)} + W_2^{(\pm)} , \tag{4.76}$$

which satisfy $\mathcal{L}_2^{\text{bulk}}\,W_\alpha^{(\pm)} = \pm f(\alpha)$.

The inhomogeneous term (4.75) becomes singular at the zeros of the open-string discriminant

$$\Delta_\alpha = z_1(1-432z_1)\,,$$

of the cubic equation, where two roots coincide. This leads to the appearance of tensionless domain walls

$$\begin{array}{lllll} z_1 = 0 & \Rightarrow & (\alpha_\ell) = (\frac{1}{4},\,0,\,0) & \Rightarrow & \begin{array}{l} T^{(+,-)}_{\alpha_1,\alpha_1} = W^{(+)}_{\alpha_1} - W^{(-)}_{\alpha_1} = 0 \\ T^{(\pm,\pm)}_{\alpha_2,\alpha_3} = W^{(\pm)}_{\alpha_2} - W^{(\pm)}_{\alpha_3} = 0 \end{array}\,, \\ z_1 = \frac{1}{432} & \Rightarrow & (\alpha_\ell) = (\frac{1}{6},-\frac{1}{12},\frac{1}{6}) & \Rightarrow & \begin{array}{l} T^{(+,-)}_{\alpha_2,\alpha_2} = W^{(+)}_{\alpha_2} - W^{(-)}_{\alpha_2} = 0 \\ T^{(\pm,\pm)}_{\alpha_1,\alpha_3} = W^{(\pm)}_{\alpha_1} - W^{(\pm)}_{\alpha_3} = 0 \end{array}\,. \end{array} \tag{4.77}$$

A-model expansion

As before by mirror symmetry, the functions (4.70) should have an integral instanton expansion when expressed in terms of the appropriate coordinates and taking into account appropriately the contributions from multi-covers [52]. For the critical branes at fixed $\hat{z}$, we use again the modified multi-cover formulae of the type proposed in [53, 55, 73]:

$$\frac{W_1^{(\pm)}(z(q))}{\omega_0(z(q))} = \frac{1}{(2\pi i)^2}\sum_{k\,odd}\sum_{\substack{d_2\,odd\\ d_1\geq 0}} n^{(1,\pm)}_{d_1,d_2}\,\frac{q_1^{kd_1}\,q_2^{kd_2/2}}{k^2}\,, \tag{4.78}$$

$$\frac{W_2^{(\pm)}(z(q))}{\omega_0(z(q))} = \frac{1}{(2\pi i)^2}\sum_{k\,odd}\sum_{\substack{d_1\,odd\\ d_2\,odd}} n^{(2,\pm)}_{d_1,d_2}\,\frac{q_1^{kd_1/2}\,q_2^{kd_2/2}}{k^2}\,. \tag{4.79}$$

In the Table 4.13 we list the integer invariants of the superpotentials $W_a^{(+)}$ obtained with the modified multicover formulas (4.78) and (4.79) for the normalization $c = c_1 = c_2 = 1$.

In the limit $q_1 \to 0$ the superpotential $W_1^{(+)}$ reproduces the numbers $n_k^{[3]}$ of the local Calabi-Yau geometry $\mathcal{O}(-3)_{\mathbb{P}^2}$ given in Table G.1 in Appendix G and (4.66). Therefore in this local limit the domain wall between the curves $C_{\alpha_2,+}$ and $C_{\alpha_3,-}$, which yields the on-shell tension $T^{(+-)}_{\alpha_2,\alpha_3} \equiv W^{(+)}_{\alpha_2} - W^{(-)}_{\alpha_3} = 2\,W_1^{(+)}$, becomes equivalent to the local domain wall of the local three-fold $\mathcal{O}(-3)_{\mathbb{P}^2}$ for the numbers $n_k^{[3]}$. The on-shell superpotentials $W_2^{(\pm)}$ vanish in this limit and give rise to tensionless domain walls (4.77).

Non-compact limit

We exhibit the non-compact limit by redefining the projective coordinate of $\mathbb{P}_{1,1,1,6,9}/(\mathbb{Z}_{18}\times\mathbb{Z}_6)$ according to

$$y_1 = x_1^6\,,\quad y_2 = x_2^6\,,\quad y_3 = x_3^6\,,\quad x = x_5\,,\quad z = x_5 + \psi\,x_1x_2x_3x_4\,,$$

where $y_\ell \in \mathbb{C}^*,\ x,z \in \mathbb{C}$. In these local coordinates the Greene-Plesser orbifold group reduces to $\mathbb{Z}_3$. It acts on the coordinates y_ℓ as $y_\ell \to \lambda^\ell y_\ell$, with $\lambda^3 = 1$, while the coordinates x, z

$\frac{1}{2}\cdot n^{(1,+)}_{d_1,d_2}$

$d_1 \backslash d_2$	1	3	5	7	9
0	1	−1	5	−42	429
1	−270	270	−2430	27270	−351000
2	−35235	0	467775	−7767495	131193270
3	−1129110	−3171960	−56432160	1346568000	−30388239450
4	−19625112	−9840669480	18001000575	−268964593065	6132575901195
5	−237548052	−4228413761754	2588348258640	38534260978296	−1115308309663386

$\frac{1}{8192}\cdot n^{(2,+)}_{d_1,d_2}$

$d_1 \backslash d_2$	1	3	5	7	9
1	0	0	0	0	0
3	−1	0	0	0	0
5	−54	108	−270	1728	−15444
7	−1215	−24300	99630	−918540	10783125
9	−17290	−60310547	−15819570	220135880	−3485260710

Table 4.13.: Disc invariants for the on-shell superpotentials $W_a^{(+)}$ of the three-fold $\mathbf{X}_{18}^{(1,1,1,6,9)}$.

remain invariant. In the limit $z_1 \to 0$, which is mirror symmetric to the limit $q_1 \to 0$, we arrive at the local Calabi-Yau geometry

$$0 \;=\; y_1^3 + y_2^3 + y_3^3 + z_2^{1/3}\, y_1 y_2 y_3 + x\, z + \mathcal{O}(\sqrt{z_1}) \ ,$$

together with the associated local holomorphic three-form Ω. The local geometry is related to the (mirror) cubic elliptic curve with the points $y_\ell = 0$ removed, and it captures the local mirror of the non-compact three-folds $\mathcal{O}(-3)_{\mathbb{P}^2}$ (see Appendix G for details). This explains the appearance of the disc invariants $n_k^{[3]} = n_{0,d_2}^{(1,+)}$ in Table 4.13.

4.4. Branes on $\mathbf{X}_9^{(1,1,1,3,3)}$

As a further example we study branes on the two moduli Calabi-Yau manifold $Z = \mathbf{X}_9^{(1,1,1,3,3)}$. This manifold is also an elliptic fibration over $\mathbb{P}^2$ and one can consider a similar compactification of the non-compact brane in $\mathcal{O}(-3)_{\mathbb{P}^2}$ as in the preceding example.

4.4.1. Bulk geometry

The toric ambient geometry of $\mathbf{X}_9^{(1,1,1,3,3)}$ is given by the polyhedron $\Delta(Z)$ in Table 4.14.

A maximal triangulation of $\Delta(Z)$ leads to the following relations

	0	1	2	3	4	5	6
$\tilde{l}^1$	−3	0	0	0	1	1	1
$\tilde{l}^2$	0	1	1	1	0	0	−3

(4.80)

$\Delta(Z)$	$\tilde{\nu}_0$	=	(	0	0	0	0	)
	$\tilde{\nu}_1$	=	(	−1	0	1	1	)
	$\tilde{\nu}_2$	=	(	0	−1	1	1	)
	$\tilde{\nu}_3$	=	(	1	1	1	1	)
	$\tilde{\nu}_4$	=	(	0	0	−1	0	)
	$\tilde{\nu}_5$	=	(	0	0	0	−1	)
	$\tilde{\nu}_6$	=	(	0	0	1	1	)

Table 4.14.: Vertices of the bulk geometry $\Delta(Z)$ for $\mathbf{X}_9^{(1,1,1,3,3)}$.

The hypersurface constraint for the mirror manifold in $\mathbb{P}_{1,1,1,3,3}/(\mathbb{Z}_9^2 \times \mathbb{Z}_3)$, written in homogeneous coordinates of $\mathbb{P}_{1,1,1,3,3}$ is

$$P(Z^*) = x_1^9 + x_2^9 + x_3^9 + x_4^3 + x_5^3 - \psi\,(x_1x_2x_3x_4x_5) + \phi\,(x_1x_2x_3)^3\,, \tag{4.81}$$

where $\psi = z_1^{-1/3} z_2^{-1/9}$ and $\phi = z_2^{-1/3}$. The coordinates $z_1 = -\frac{a_4a_5a_6}{a_0^3}$ and $z_2 = \frac{a_1a_2a_3}{a_6^3}$ are again local coordinates of the point of maximal unipotent monodromy. The Greene-Plesser orbifold group acts as $x_i \to \lambda_k^{g_{k,i}}\, x_i$ with $\lambda_1^9 = \lambda_2^9 = 1$, $\lambda_3^3 = 1$ and weights

$$\mathbb{Z}_9 : g_1 = (1,-1,0,0,0), \quad \mathbb{Z}_9 : g_2 = (1,0,-1,0,0), \quad \mathbb{Z}_3 : g_2 = (0,0,0,1,-1)\,. \tag{4.82}$$

4.4.2. Brane geometry

We consider a family of D7-branes which we expect to include a brane that exists at the Landau Ginzburg point of the two moduli Calabi–Yau. The mirror A-brane is defined by

$$-\,|\tilde{x}_0|^2 + |\tilde{x}_4|^2 = \hat{c}, \qquad \begin{array}{c|c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \hat{l} & -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\,. \tag{4.83}$$

The enhanced polyhedron $\Delta_\flat(Z, L_1)$ for this brane geometry is given in Table 4.15. A suitable basis of relations for the charge vectors is

$$\begin{array}{c|c|cccccc|cc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline l^1 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 \\ l^2 & 0 & 1 & 1 & 1 & 0 & 0 & -3 & 0 & 0 \\ l^3 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \end{array} \tag{4.84}$$

leading to the differential operators

$$\begin{aligned} \mathcal{L}_1 &= \theta_1(\theta_1 - 3\theta_2)(\theta_1 - \theta_3) + z_1(\theta_1 - \theta_3)(2\theta_1 + 1 + \theta_3)(2\theta_1 + 2 + \theta_3), \\ \mathcal{L}_2 &= \theta_2^3 - z_2(\theta_1 - 3\theta_2)(\theta_1 - 3\theta_2 - 1)(\theta_1 - 3\theta_2 - 2), \\ \mathcal{L}_3 &= -\theta_3(\theta_1 - \theta_3) - z_3(\theta_1 - \theta_3)(2\theta_1 + 1 + \theta_3)\;. \end{aligned} \tag{4.85}$$

$\Delta(Z)$	$\tilde{\nu}_0$	=	(	0	0	0	0	0	)
	$\tilde{\nu}_1$	=	(	−1	0	1	1	0	)
	$\tilde{\nu}_2$	=	(	0	−1	1	1	0	)
	$\tilde{\nu}_3$	=	(	1	1	1	1	0	)
	$\tilde{\nu}_4$	=	(	0	0	−1	0	0	)
	$\tilde{\nu}_5$	=	(	0	0	0	−1	0	)
	$\tilde{\nu}_6$	=	(	0	0	1	1	0	)
$\Delta_\flat(Z, L_1) = \Delta\cup$	$\tilde{\nu}_7$	=	(	0	0	0	0	−1	)
	$\tilde{\nu}_8$	=	(	0	0	−1	0	−1	)

Table 4.15.: Vertices of the enhanced polyhedron $\Delta_\flat(Z, L_1)$ for on $\mathbf{X}_9^{(1,1,1,3,3)}$.

The brane geometry on the B-model side is defined by the two equations

$$\begin{aligned} P(Z^*) &= \sum a_i y_i = a_0 x_1 x_2 x_3 x_4 x_5 + a_1 x_1^9 + a_2 x_2^9 + a_3 x_3^9 + a_4 x_4^3 + a_5 x_5^3 + a_6 (x_1 x_2 x_3)^3 , \\ Q(\mathcal{D}_1) &: \qquad y_0 = y_1 \qquad \text{or} \qquad x_1 x_2 x_3 x_4 x_5 = x_4^3 \, . \end{aligned} \tag{4.86}$$

As in the previous cases, the deformations of the hypersurface $Q(\mathcal{D}_1)$ are described by the periods on a K3 surface.

We are interested in a brane superpotential with critical point at $z_3 = -1$. Choosing the following local coordinates centered around $z_3 = -1$

$$u = (-z_1)^{-1/2} z_2^{-1/6} (z_3 + 1) \, , \quad v = (-z_1)^{1/2} z_2^{1/6} \quad x_2 = z_2^{-1/3} \, ,$$

we obtain the superpotential

$$c\,\mathcal{W}(u, v, x_2) = -\frac{1}{2} u x_2 + \frac{1}{24} u^3 + 210 v^3 + \frac{3}{4} v x_2^2 - \frac{3}{8} u^2 v x_2 + \dots \, . \tag{4.87}$$

This superpotential has a critical point at $u = 0$ and $x_2 = 0$. At the critical locus we have $v = z^{1/6}$, where z denotes the closed string modulus

$$z = -\frac{a_4^3 a_5^3 a_1 a_2 a_3}{a_0^9} \, .$$

The expansion of the superpotential at this critical locus reads

$$c\,W_{\text{crit}}(z) = 210\sqrt{z} + \frac{53117350}{3} z^{3/2} + \frac{18297568296042}{5} z^{5/2} + \frac{7182631458065952702}{7} z^{7/2} + \dots \, .$$

4.4.3. Large volume brane

As above one can extract large volume Ooguri-Vafa invariants. Some of these large volume invariants for the brane geometry (4.84) are given in Table 4.16. There is another interesting large volume brane, which is the compactification of the local brane in $\mathcal{O}(-3)_{\mathbb{P}^2}$. It is described

$l = 0$

$k \backslash m$	0	1	2	3	4	5
0	$*$	54	0	0	0	0
1	−36	$*$	54	−18	0	0
2	18	−54	$*$	36	0	0
3	0	0	−54	$*$	54	0
4	0	0	0	−36	$*$	54
5	0	0	0	18	−54	$*$
6	0	0	0	0	0	−54
7	0	0	0	0	0	0

$l = 1$

$k \backslash m$	0	1	2	3	4
0	$*$	0	0	0	0
1	72	$*$	−108	36	0
2	−36	−1728	$*$	2772	−1026
3	−1224	17280	−80460	$*$	243756
4	5508	−64800	340092	−1075140	$*$

$l = 2$

$k \backslash m$	0	1	2	3
0	$*$	0	0	0
1	−180	$*$	270	−90
2	108	7020	$*$	−11160
3	−108	−5832	−97686	$*$
4	−10944	133488	−588276	2643372

Table 4.16.: Invariants $N_{k,l,m}$ for the geometry (4.84).

$\Delta(Z)$	$\tilde{\nu}_0$	=	(0	0	0	0	0)
	$\tilde{\nu}_1$	=	(−1	0	1	1	0)
	$\tilde{\nu}_2$	=	(0	−1	1	1	0)
	$\tilde{\nu}_3$	=	(1	1	1	1	0)
	$\tilde{\nu}_4$	=	(0	0	−1	0	0)
	$\tilde{\nu}_5$	=	(0	0	0	−1	0)
	$\tilde{\nu}_6$	=	(0	0	1	1	0)
$\Delta_\flat(Z, L_2) = \Delta \cup$	$\tilde{\nu}_7$	=	(1	1	1	1	−1)
	$\tilde{\nu}_8$	=	(0	0	1	1	−1)

Table 4.17.: Vertices of the enhanced polyhedron $\Delta_\flat(Z, L_2)$ for on $\mathbf{X}_9^{(1,1,1,3,3)}$.

by the enhanced polyhedron $\Delta_\flat(Z, L_2)$ given in Table 4.17. A maximal triangulation of $\Delta_\flat(Z, L_2)$ leads to the following charge vectors

	0	1	2	3	4	5	6	7	8
l^1	−3	0	0	0	1	1	1	0	0
l^2	0	1	1	0	0	0	−2	1	−1
l^3	0	0	0	1	0	0	−1	−1	1

(4.88)

Some invariants for this geometry are given in Table 4.18. The invariants for $k = 0$ are three times the invariants in Table 4.10, where the overall factor comes from the three global

$k=0$

$l \backslash m$	0	1	2	3	4	5
0	$*$	3	0	0	0	0
1	3	$*$	-3	-3	-3	-3
2	-3	-6	$*$	15	21	27
3	3	12	36	$*$	-120	-183
4	-6	-30	-96	-312	$*$	1197
5	15	84	306	978	3255	$*$
6	-39	-252	-1032	-3480	-11124	-37980
7	105	792	3600	13080	42822	137166

	$k=1$				$k=2$			
$l \backslash m$	0	1	2	3	0	1	2	3
0	$*$	27	0	0	$*$	81	-108	0
1	-72	$*$	90	90	-1269	$*$	-1539	-1377
2	72	234	$*$	-684	-684	-2808	$*$	13554
3	-144	-612	-1980	$*$	2268	11232	42336	$*$
4	360	1890	6624	22320	-7848	-46656	-182916	-671922
5	-1008	-6300	-24660	-82908	27972	194832	835758	3020382
6	3024	21924	95760	340200	-102024	-813456	-3844512	-14554242
7	-9504	-78408	-379512	-1445472	377784	3390336	17598600	70975872

Table 4.18.: Invariants $N_{k,l,m}$ for the geometry (4.88).

sections of the elliptic fibration $\mathbf{X}_9^{(1,1,1,3,3)}$. It appears that the invariants for $k=1,\ l\neq 0$ are generally 3/10 times the invariants in Table 4.11.

4.4.4. Further domain wall tensions

We are going to furthermore study the family of divisors

$$Q(\mathcal{D}_3) = x_2^9 + z_3 x_1^9\,, \tag{4.89}$$

near the point $z_3=-1$. The surface periods defined by the family (4.89) are captured by the GLSM with charges

	0	1	3	4	5	6
$\tilde{l}^1$	-3	0	0	1	1	1
$\tilde{l}^2$	0	2	1	0	0	-3

,

$\Delta(Z)$	$\tilde\nu_0$	=	(	0	0	0	0	0)
	$\tilde\nu_1$	=	(	−1	0	1	1	0)
	$\tilde\nu_2$	=	(	0	−1	1	1	0)
	$\tilde\nu_3$	=	(	1	1	1	1	0)
	$\tilde\nu_4$	=	(	0	0	−1	0	0)
	$\tilde\nu_5$	=	(	0	0	0	−1	0)
	$\tilde\nu_6$	=	(	0	0	1	1	0)
$\Delta_\flat(Z,L_3)=\Delta\cup$	$\tilde\nu_7$	=	(	−1	0	1	1	−1)
	$\tilde\nu_8$	=	(	0	−1	1	1	−1)

Table 4.19.: Vertices of the enhanced polyhedron $\Delta_\flat(Z,L_3)$ for the family $\mathcal{D}_3$ on $\mathbf{X}_9^{(1,1,1,3,3)}$.

depending on the two algebraic moduli $u_1=z_1$ and $u_2=-\frac{z_2}{z_3}(1-z_3)^2$. The exceptional solutions

$$\begin{aligned}\pi_1&=\frac{c_1}{2}\,B_{\{\tilde l^1,\tilde l^2\}}(u_1,u_2;0,\tfrac12)=\frac{-c_1}{2\pi}\sqrt{u_2}\;{}_2F_1(\tfrac13,\tfrac23,-\tfrac12,27z_1)+\mathcal{O}(u_2^{3/2})\,,\\ \pi_2&=\frac{c_2}{2}\,B_{\{\tilde l^1,\tilde l^2\}}(u_1,u_2;\tfrac12,\tfrac12)=\frac{105c_2}{2\pi}\sqrt{u_2}\;z_1^{3/2}\,{}_2F_1(\tfrac{11}{6},\tfrac{13}{6},\tfrac52,27z_1)+\mathcal{O}(u_2^{3/2})\,,\end{aligned}\tag{4.90}$$

vanish at the point $z_3-1=0=u_2$. The sign of the root $\sqrt{z_3}$ distinguishes the two sheets of the coordinate change $x_1^2=\tilde x_1$ similarly as in eq. (4.32). Integrating along similar contours as in that case, we obtain the superpotentials

$$\begin{aligned}W_1^{(\pm)}&=\pm\frac{c_1}{8}\sum_{n_i\geq0}\frac{\Gamma(3n_1+1)\;z_1^{n_1}z_2^{n_2+\frac12}}{\Gamma(n_1+1)^2\,\Gamma\left(n_1-3n_2-\frac12\right)\Gamma\left(n_2+\frac32\right)^3}\,,\\ W_2^{(\pm)}&=\pm\frac{c_2}{8}\sum_{n_i\geq0}\frac{\Gamma\left(3n_1+\frac52\right)\;z_1^{n_1+\frac12}z_2^{n_2+\frac12}}{\Gamma\left(n_1+\frac32\right)^2\Gamma(n_1-3n_2)\,\Gamma\left(n_2+\frac32\right)^3}\,,\end{aligned}\tag{4.91}$$

or equivalently, expressed in terms of the bulk generating function

$$W_1^{(\pm)}=\pm\frac{c_1}{8}\,B_{\{l^1,l^2\}}\left(z_1,z_2;0,\tfrac12\right)\,,\qquad W_2^{(\pm)}=\pm\frac{c_2}{8}\,B_{\{l^1,l^2\}}\left(z_1,z_2;\tfrac12,\tfrac12\right)\,.\tag{4.92}$$

These functions are solutions to the large GKZ system (3.51) of the relative cohomology problem. The enhanced polyhedron $\Delta_\flat(Z,L_3)$ for the family $\mathcal{D}_3$ in (4.89) is given in Table 4.19. For the family in (4.89) maximal triangulation of the enhanced polyhedron $\Delta_\flat(Z,L_3)$ leads to the additional extended charge vector

	0	1	2	3	4	5	6	7	8
l^3	0	-1	1	0	0	0	0	1	-1

,

$\frac{1}{2} \cdot n^{(1,+)}_{d_1,d_2}$

$q_1 \backslash q_2^{1/2}$	1	3	5	7	9	11
0	1	−1	5	−42	429	−4939
1	−27	27	−243	2727	−35100	487647
2	−243	0	4131	−71442	1230795	−21333942
3	−1347	−2295	−33804	979800	−24220836	544584789
4	−6021	−231876	532575	−10061955	319551804	−9298367514
5	−22356	−7276878	5101407	73610289	−3196953927	117194205483

$\frac{1}{2} \cdot n^{(2,+)}_{d_1,d_2}$

$q_1^{1/2} \backslash q_2^{1/2}$	1	3	5	7	9
1	0	0	0	0	0
3	−105	0	0	0	0
5	−567	1134	−2835	18144	−162162
7	−2916	−18954	81648	−826686	10133100
9	−11904	−1421850	−498555	13289664	−255008817

Table 4.20.: Disc invariants for the on-shell superpotentials $W_a^{(+)}$ of the brane geometry $\mathcal{D}_3$ of the three-fold $\mathbf{X}_9^{(1,1,1,3,3)}$.

which together with the charge vectors of the three-fold, gives rise to the differential operators according to eq. (3.51)

$$\begin{aligned}
\tilde{\mathcal{L}}_1 &= \theta_1(\theta_1 - 3\theta_2) - 3z_1(3\theta_1 + 1)(3\theta_1 + 2) \ , \\
\tilde{\mathcal{L}}_2 &= (\theta_2 - \theta_3)(\theta_2 + \theta_3)\theta_2 - z_2(\theta_1 - 3\theta_2)(\theta_1 - 3\theta_2 - 1)(\theta_1 - 3\theta_2 - 2) \ , \\
\tilde{\mathcal{L}}_3 &= \theta_3(\theta_2 + \theta_3) + z_3\,\theta_3(\theta_2 - \theta_3) \ .
\end{aligned} \tag{4.93}$$

The solutions to these operators are the relative period integrals. In particular there are two solutions with a minimum at the critical locus $\ln(z_3) = 0$, which restrict to the on-shell superpotentials $W_1^{(\pm)}$ and $W_2^{(\pm)}$, respectively.

To characterize the critical superpotentials $W_a^{(\pm)}$ as solutions to an inhomogeneous Picard-Fuchs equation, we observe

$$\tilde{\mathcal{L}}_1 = \mathcal{L}_1^{\text{bulk}}, \qquad \tilde{\mathcal{L}}_2 = \mathcal{L}_2^{\text{bulk}} - \theta_2\theta_3^2 \ .$$

So only the second operator acquires an inhomogeneous term, which is determined by the leading part of the surface periods π_a. Acting with $\theta_2\theta_3$ on the leading coefficients of (4.90) one obtains the inhomogeneous Picard-Fuchs equations

$$\begin{aligned}
\mathcal{L}_2^{\text{bulk}} W_1^{(\pm)} &= \mp\frac{c_1}{16\pi^2}\sqrt{z_2}\;{}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; -\tfrac{1}{2}; 27z_1\right) \ , \\
\mathcal{L}_2^{\text{bulk}} W_2^{(\pm)} &= \pm\frac{105c_2}{16\pi^2} z_1^{3/2}\sqrt{z_2}\;{}_2F_1\left(\tfrac{11}{6}, \tfrac{13}{6}; \tfrac{5}{2}; 27z_1\right) \ .
\end{aligned} \tag{4.94}$$

The inhomogeneous terms are again solutions to the same hypergeometric equation and related by monodromy. The differential operator is obtained by specializing the Picard-Fuchs operator of the surface $\mathcal{L}_1^{\mathcal{P}} = \tilde{\theta}_1(\tilde{\theta}_1 - 3\tilde{\theta}_2) - 3u_1(3\tilde{\theta}_1 + 1)(3\tilde{\theta}_1 + 2)$ to the critical point $u_2 = 0$:

$$\mathcal{L}^{\text{inh}} f_2(z_1, z_2) = 0 \,, \qquad \mathcal{L}^{\text{inh}} = \mathcal{L}_1^{\mathcal{P}}|_{\hat{z}_{\text{crit}}} = (1-z)z\frac{d^2}{dz^2} + \left(-\frac{1}{2} - 2z\right)\frac{d}{dz} - \frac{2}{9} \,,$$

with $\tilde{\theta}_a = u_a \frac{d}{du_a}$, $z = 27z_1$. The specialization to the leading term in the limit $u_2 = 0$ is achieved by setting $\theta_2 = \frac{1}{2}$. Similarly as in the other examples one can verify that the hypergeometric functions (4.94) can be written in closed form.

In Table 4.20 we list the integer invariants obtained with the modified multicover formula (4.78), (4.79) for the normalization $c_1 = c_2 = 1$. Similarly as in the previous examples, the hypersurface degenerates to the non-compact mirror three-fold of $\mathcal{O}(-3)_{\mathbb{P}^2}$ in the limit $z_1 \to 0$. This explains the appearance of the invariants $n^{[3]}$ (c.f. Table G.1) in the superpotential $W_1^{(+)}$, which are listed in the first row of the first table in Table 4.20.

4.5. Domain wall tensions on $\mathbf{X}_8^{(1,1,2,2,2)}$

As a last example we consider certain domain wall tensions on the Calabi-Yau $\mathbf{X}_8^{(1,1,2,2,2)}$. Closed string mirror symmetry for this geometry was studied in [116]. This example is qualitatively very similar to the two-parameter on-shell brane geometry considered in [73].

4.5.1. Bulk geometry

The toric ambient geometry of the hypersurface $\mathbf{X}_8^{(1,1,2,2,2)}$ is given by the polyhedron $\Delta(Z)$ in Table 4.21. A maximal triangulation of $\Delta(Z)$ leads to the following relations

$$\begin{array}{c|c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline l^1 & -4 & 0 & 0 & 1 & 1 & 1 & 1 \\ l^2 & 0 & 1 & 1 & 0 & 0 & 0 & -2 \end{array} \,. \tag{4.95}$$

The hypersurface constraint for the mirror manifold written in homogeneous coordinates of $\mathbb{P}^4_{1,1,2,2,2}/(\mathbb{Z}_8 \times \mathbb{Z}_4^2)$ is

$$P(Z^*) = x_1^8 + x_2^8 + x_3^4 + x_4^4 + x_5^4 + \psi\, x_1x_2x_3x_4x_5 + \phi\,(x_1x_2)^4 \,, \tag{4.96}$$

where $\psi = z_1^{-1/4} z_2^{-1/8}$ and $\phi = z_2^{-1/2}$. The Greene-Plesser orbifold group acts as $x_i \to \lambda_k^{g_{k,i}}\, x_i$ with generators $\lambda_1^8 = 1$, $\lambda_2^4 = \lambda_3^4 = 1$ and weights

$$\mathbb{Z}_8 : g_1 = (1,-1,0,0,0), \quad \mathbb{Z}_4 : g_2 = (1,0,-1,0,0), \quad \mathbb{Z}_4 : g_2 = (1,0,0,-1,0) \,. \tag{4.97}$$

In this geometry we consider the curves

$$\begin{aligned} C_\alpha &= \{x_3 = \eta_1 x_1^2 \,,\, x_4 = \eta_2 x_2^2 \,,\, \eta_1\eta_2 x_5 = \alpha x_1 x_2\} \,, \\ &\quad \eta_1^4 = \eta_2^4 = -1 \,, \alpha^4 + \psi\,\alpha + \phi = 0 \,, \end{aligned} \tag{4.98}$$

$\Delta(Z)$								
	$\tilde{\nu}_0$	= (	0	0	0	0	)
	$\tilde{\nu}_1$	= (	1	2	2	2	)
	$\tilde{\nu}_2$	= (	−1	0	0	0	)
	$\tilde{\nu}_3$	= (	0	−1	0	0	)
	$\tilde{\nu}_4$	= (	0	0	−1	0	)
	$\tilde{\nu}_5$	= (	0	0	0	−1	)
	$\tilde{\nu}_6$	= (	0	1	1	1	)

Table 4.21.: Vertices of the bulk geometry $\Delta(Z)$ for $\mathbf{X}_8^{(1,1,2,2,2)}$.

where $\eta_1^4 = \eta_2^4 = -1$. Under the orbifold action the curves are identified as $(\eta_1, \eta_2, \alpha) \sim (\eta_1\lambda_1^2\lambda_2^3\lambda_3^2, \eta_2\lambda_1^{-2}\lambda_3, \alpha)$. The curves are labeled by the four roots α, while under the $\mathbb{Z}_8 \times \mathbb{Z}_4^2$ orbifold action the 16 distinct choices for the phases η_1 and η_2 are identified. Thus we find four distinct orbits of curves C_α.

4.5.2. Brane geometry and tensions

To compute domain wall tensions for these curves we study the family of divisors

$$Q(\mathcal{D}) = x_5^4 - z_3\, z_1^{-1/4} z_2^{-1/8}\, x_1x_2x_3x_4x_5\ . \tag{4.99}$$

The curves C_α are included in $\mathcal{D}$ for the critical values $z_3 = z_1^{1/4} z_2^{1/8}\,\alpha^3 \equiv \tilde{\alpha}$, where the new label $\tilde{\alpha}$ obeys the fourth order equation

$$\tilde{\alpha}(1+\tilde{\alpha})^3 + y = 0\ , \qquad y \equiv \frac{z_1}{z_2}\ . \tag{4.100}$$

Note that the roots $\tilde{\alpha}$ of this fourth order equation are in one-to-one correspondence with the curves C_α. The chosen open-string coordinate z_3 is the natural coordinate on the non-compact fourfold defined by the enhanced polyhedron in Table 4.22. This adds the following further constraint

	0	1	2	3	4	5	6	7	8
l^3	-1	0	0	0	0	1	0	1	-1

to the GLSM of the A-model manifold. Periods on the intersection $Q(\mathcal{D}) = P = 0$ are captured by a GLSM with charges

	0	1	2	3	4	5
$\tilde{l}^1$	-3	0	0	1	1	1
$\tilde{l}^2$	0	1	1	0	0	-2

where the algebraic moduli are $u_1 = -\frac{z_1}{z_3(1+z_3)^3}$ and $u_2 = z_2$. In these coordinates the critical points $z_3 = \tilde{\alpha}$ arise at $u_1 = u_2$. This condition corresponds to the fourth order equation (4.100) for the label $\tilde{\alpha}$.

$\Delta(Z)$	$\tilde{\nu}_0$	=	(	0	0	0	0	0	)
	$\tilde{\nu}_1$	=	(	1	2	2	2	0	)
	$\tilde{\nu}_2$	=	(	−1	0	0	0	0	)
	$\tilde{\nu}_3$	=	(	0	−1	0	0	0	)
	$\tilde{\nu}_4$	=	(	0	0	−1	0	0	)
	$\tilde{\nu}_5$	=	(	0	0	0	−1	0	)
	$\tilde{\nu}_6$	=	(	0	1	1	1	0	)
$\Delta_\flat(Z,L)=\Delta\cup$	$\tilde{\nu}_7$	=	(	0	0	0	−1	1	)
	$\tilde{\nu}_8$	=	(	0	0	0	0	1	)

Table 4.22.: Vertices of the enhanced polyhedron $\Delta_\flat(Z,L)$ for the divisor family $\mathcal{D}$ on $\mathbf{X}_8^{(1,1,2,2,2)}$.

The solutions of this subsystem can be generated with the Frobenius method from the generating function

$$B_{\{\tilde{l}^1,\tilde{l}^2\}}(u_1,u_2;\rho_1,\rho_2)=\sum_{n_i\in\mathbb{Z}+\rho_i}\frac{\Gamma(1+3n_1)\,u_1^{n_1}u_2^{n_2}}{\Gamma(1+n_1)^2\,\Gamma(1+n_1-2n_2)\,\Gamma(1+n_2)^2}\,. \tag{4.101}$$

The linear combination

$$\tau=\frac{c}{2\pi i}(\partial_{\rho_1}B_{\{\tilde{l}^1,\tilde{l}^2\}}-\partial_{\rho_2}B_{\{\tilde{l}^1,\tilde{l}^2\}})\big|_{\rho_i=0}:=c(t_1-t_2)\,, \tag{4.102}$$

vanishes at the critical locus $u_1=u_2$. The moduli t_1 and t_2 measure the volumes of two generators of $H_2(K3,\mathbb{Z})$ and at criticality the zero of the period arises because their volumes coincide. The four critical points $\tilde{\alpha}_k, k=0,1,2,3$, which are given in terms of the fourth order equation (4.100), enjoy in terms of $y=\frac{z_1}{z_2}$ the expansion

$$\begin{aligned}\tilde{\alpha}_0(y)&=-y\left(1+3y+15y^2+91y^3+612y^4+4389y^5+32890y^6+\dots\right)\,,\\ \tilde{\alpha}_\ell(y)&=-1+\nu_\ell y^{1/3}+\frac{1}{3}\nu_\ell^2y^{2/3}+\frac{\nu_\ell^3y}{3}+\frac{35}{81}\nu_\ell^4y^{4/3}+\frac{154}{243}\nu_\ell^5y^{5/3}+\nu_\ell^6y^2+\dots\,,\end{aligned} \tag{4.103}$$

with $\nu_\ell=e^{\frac{2\pi i}{3}(\ell-1)}, \ell=1,2,3$. There appears an additional measure factor for the integration of subsystem period to the superpotential, namely

$$2\pi i\,\theta_{z_3}\mathcal{W}(z_1,z_2,z_3)=\frac{1}{1+z_3}\pi(u_1,u_2)\,.$$

Hence, integrating the discussed subsystem period (4.102) with the additional measure factor, we obtain for the critical point $\tilde{\alpha}_0$ the on-shell superpotential

$$W^{(\alpha_0)}(y,z_2)=-\frac{c}{4\pi^2}\left(\frac{1}{2}S_0\,\log(-y)^2+(S_1-S_2)\,\log(-y)+S_{\alpha_0}(y,z_2)\right)\,. \tag{4.104}$$

Here S_0, S_1 and S_2 are the power-series[13]

$$\begin{aligned} S_0 &= 1 + 24yz_2 + 2520y^2z_2^2 + 5040y^2z_2^3 + 369600y^3z_2^3 + 2217600y^3z_2^4 + \ldots\,, \\ S_1 &= -z_2 + 104yz_2 - \frac{3}{2}z_2^2 + 24yz_2^2 + 12276y^2z_2^2 - \frac{10}{3}z_2^3 + 12yz_2^3 - \frac{35}{4}z_2^4 + \ldots\,, \\ S_2 &= 2z_2 + 48yz_2 + 3z_2^2 - 48yz_2^2 + 7560y^2z_2^2 + \frac{20}{3}z_2^3 - 24yz_2^3 + \frac{35}{2}z_2^4 + \ldots\,. \end{aligned} \tag{4.105}$$

For the instanton part S_{α_0} we get

$$S_{\alpha_0}(y,z_2) = (4y+3z_2) + \left(7y^2 - 64yz_2 + \frac{45z_2^2}{4}\right) + \left(\frac{220y^3}{9} + 210y^2z_2 + 528yz_2^2 + \frac{191z_2^3}{6}\right) + \ldots\,. \tag{4.106}$$

Finally, we note that in terms of the GLSM charges suitable for the coordinates y, z_2

	0	1	2	3	4	5	6
h^1	-4	-1	-1	1	1	1	3
h^2	0	1	1	0	0	0	-2

we can express the superpotential $W^{(\alpha_0)}$ as

$$W^{(\alpha_0)}(y,z_2) \;=\; -\frac{c}{8\pi^2}\,\partial^2_{\rho_1}\,B_{\{h^1,h^2\}}(y,z_2,\rho_1,\rho_2)|_{\rho_i=0}\;. \tag{4.107}$$

Integrating the subsystem period with the additional measure factor to the other roots $\tilde{\alpha}_\ell(y)$ one finds similar expansions for the on-shell superpotentials $W^{(\alpha_\ell)}(y,z_2)$ associated to these roots.

To characterize the superpotential $W^{(\alpha_0)}$ by an inhomogeneous Picard-Fuchs equation we calculate the inhomogeneous pieces with the following bulk operators

$$\mathcal{L}_1^{\text{bulk}} = \theta_1^2(\theta_1 - 2\theta_2) - 4z_1(4\theta_1+1)(4\theta_1+2)(4\theta_1+3)\,, \tag{4.108}$$

$$\mathcal{L}_2^{\text{bulk}} = \theta_2^2 - z_2(\theta_1 - 2\theta_2)(\theta_1 - 2\theta_2 - 1)\,, \tag{4.109}$$

and we obtain

$$\begin{aligned} \mathcal{L}_1^{\text{bulk}}\,W^{(\alpha_0)} &= -\frac{3\,c}{4\pi^2}\;\theta_y\,{}_3F_2\left(\tfrac{1}{4},\tfrac{2}{4},\tfrac{3}{4};\tfrac{1}{3},\tfrac{2}{3};\tfrac{256\,y}{27}\right) = 3\,\theta_y f(\alpha_0)\;, \\ \mathcal{L}_2^{\text{bulk}}\,W^{(\alpha_0)} &= -\frac{c}{4\pi^2}\;{}_3F_2\left(\tfrac{1}{4},\tfrac{2}{4},\tfrac{3}{4};\tfrac{1}{3},\tfrac{2}{3};\tfrac{256\,y}{27}\right) = f(\alpha_0)\;, \end{aligned} \tag{4.110}$$

where the label α_0 refers to the root of the quartic equation in (4.98) associated to the corresponding root $\tilde{\alpha}_0$ in eq. (4.103). As in previous examples we can also express the inhomogeneous terms as functions in the coefficients of the defining equations, i.e.

$$f(\alpha) \;=\; \frac{c}{4\pi^2}\,\frac{z_2^{1/8}\alpha}{4y^{1/4}+3z_2^{1/8}\alpha} \;=\; -\frac{c}{4\pi^2}\cdot\frac{1}{4\tilde{\alpha}+1}\;. \tag{4.111}$$

[13] S_0 is the fundamental closed string period and S_a, $a=1,2$, the series part of the single logarithmic closed string periods $(2\pi i)t_a = \log(z_a) + S_a$, which determine the closed string mirror map. However there is no double logarithmic closed-string period that has the same classical terms as eq. (4.104).

$q_y \backslash q_{z_2}$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	4	188	188	4	0	0	0	0	0
2	6	−68	5194	19024	5194	−68	6	0	0
3	24	−292	−3232	259524	3569704	3569704	259524	−3232	−292
4	112	−1660	−10996	−4092	13712184	555071696	1455120104	555071696	13712184
5	620	−10768	−42752	383424	−256440	695568492	74900481736	418921719720	418921719720
6	3732	−75468	−140150	4170468	6794752	−464516720	32227348614	9235136625472	97930146122188
7	24164	−556600	5648	37548816	24834800	−2671560544	−62352854944	991475402468	1066545645786456
8	164320	−4256460	7444296	318651284	−286806192	−20467318044	−282718652536	−7115509903004	−64593220192464
9	1162260	−33442800	114057840	2622725460	−7347237536	−170307380384	−1384203066912	−28014543398208	−915396773309428

Table 4.23.: Symmetric disc invariants for the on-shell superpotentials $W^{(\alpha_0)}$ of the three-fold $\mathbf{X}_8^{(1,1,2,2,2)}$.

The open string discriminant is $\Delta_\alpha = y(1 - \frac{256y}{27})$, with the three roots $\tilde{\alpha}_\ell$, $\ell > 0$ colliding for $y = 0$ at $\tilde{\alpha}_\ell = -1$, see eq. (4.103), while at $y = \frac{27}{256}$ one has $\tilde{\alpha}_0 = -\frac{1}{4} = \tilde{\alpha}_1$. The inhomogeneous term (4.111) become singular at the second zero, indicating a tensionless domainwall between the curves associated with $\tilde{\alpha}_{0,1}$.

For the other on-shell superpotentials $W^{(\alpha_\ell)}(y, z_2)$, we find the same inhomogeneous terms

$$\mathcal{L}_1^{\text{bulk}}\, W^{(\alpha_\ell)} \;=\; 3\,\theta_y f(\alpha_\ell)\ , \quad \mathcal{L}_2^{\text{bulk}}\, W^{(\alpha_\ell)} \;=\; f(\alpha_\ell)\ , \quad \ell = 1, 2, 3\ , \tag{4.112}$$

where again the roots α_ℓ are associated to the corresponding roots $\tilde{\alpha}_\ell$.

A-model expansion

Using the standard multicover formula

$$\frac{W^{(\alpha_0)}(z(q))}{\omega_0(z(q))} = \frac{1}{(2\pi i)^2} \sum_k \sum_{d_{1,2} \geq 0} n_{d_1,d_2}^{(\alpha_1)} \frac{q_1^{kd_1}\, q_2^{kd_2}}{k^2}$$

we obtain for $c = 1$ the integer invariants in Tab. 4.23. Here $q_y = z_1 - z_2 + \dots$ and $q_2 = z_2 + \dots$. We have added a rational multiple of a closed-string period with leading behavior $\Pi = \frac{3}{2} t_1 t_2 + \dots$ to get invariants n_{d_1,d_2} symmetric under the $\mathbb{Z}_2$ symmetry $t_1 \to t_1 + t_2,\ t_2 \to -t_2;\ q_y \to q_y q_2,\ q_2 \to q_2^{-1}$. This is the Weyl symmetry of a non-perturbative $SU(2)$ gauge symmetry appearing in the type II compactification at the transition point [184, 185]. The domain wall is a singlet under this global symmetry as can be seen from the defining equation (4.98).

For the candidate superpotentials $W^{(\alpha_\ell)}$, $\ell = 1, 2, 3$, which have an expansion in fractional powers $q_y^{d_y/3}$ with $d_y \in \mathbb{Z}$, we did not find integral invariants with the multi-cover formula used in the other examples and in [73]. It appears that only the numbers $n_{d_y,d_2} \cdot Z^{d_y}$, with

Z a small power of 3, are integral. The solution to this problem might require a shift of the open string mirror map or a refinement of the multi-cover formula.

Extremal transition

At the singular locus $\phi^2 = 1$ ($z_2 = \frac{1}{4}$) there is an extremal transition to the mirror of the one-parameter model $\mathbf{X}_{2,4}^{(1,1,1,1,1,1)}$ [116]. The large complex structure parameters $z^{(1)}$ of the one-parameter model and the two-parameter model are related by

$$z^{(1)} = \frac{1}{2} z_1 \ . \tag{4.113}$$

To restrict the superpotential found in the two-parameter model to that of the one-parameter model we have to add as in [73] an additional linear combination of bulk periods

$$\tilde{W}^{(\alpha_0)}(y, z_2) = W^{(\alpha_0)}(y, z_2) + 3F_1(y, z_2) + \frac{3}{2} F_2(y, z_2) \ , \tag{4.114}$$

where $(2\pi i)^2 F_1 = \partial_{\rho_1}\partial_{\rho_2} B_{\{l^1,l^2\}}|_{\rho_i=0}$ and $(2\pi i)^2 F_2 = \partial_{\rho_1}^2 B_{\{l^1,l^2\}}|_{\rho_i=0}$. We then obtain

$$W^{(1,\alpha_0)}(z^{(1)}) = \tilde{W}^{\alpha_0}(8z^{(1)}, \tfrac{1}{4}) \ . \tag{4.115}$$

Using the Picard-Fuchs operator of the one-parameter model

$$\mathcal{L}^{(1)} = \theta^4 - 8z^{(1)}(4\theta + 1)(4\theta + 2)(4\theta + 3)(2\theta + 1) \tag{4.116}$$

with $\theta = z^{(1)}\partial_{z^{(1)}}$ one obtains the inhomogeneous term

$$\mathcal{L}^{(1)} W^{(1,\alpha_0)} = \frac{224 z^{(1)}}{(2\pi i)^2} \left(1 + 272 z^{(1)} + \frac{285120 (z^{(1)})^2}{7} + 4925440 (z^{(1)})^3 + \dots \right) \ . \tag{4.117}$$

For the integer invariants we expect the following relation

$$\sum_{l=0}^{3k} n_{k,l}^{(\alpha_0)}(\mathbf{X}_8^{(1,1,2,2,2)}) = n_k^{(\alpha_0)}(\mathbf{X}_{2,4}^{(1,1,1,1,1,1)}) \, . \tag{4.118}$$

However such a relation only emerges after the addition of an additional bulk period, again as in [73]

$$\tilde{W}^{(1,\alpha_0)}(z^{(1)}) = W^{(1,\alpha_0)}(z^{(1)}) - \frac{3}{2} F^{(1)}(z^{(1)}) \, , \tag{4.119}$$

where $(2\pi i)^2 F^{(1)} = \partial_\rho^2 B_{\{l^{(1)}\}}|_{\rho=0}$ with $l^{(1)} = (-4, -2 \,|\, 1, 1, 1, 1, 1, 1)$. The invariants of $\tilde{W}^{(1,\alpha_0)}$ are given by

$$n_k^{(\alpha_0)} = 384, 29288, 7651456, 2592654592, 989035688064, \dots \ . \tag{4.120}$$

It would be interesting to also get a better understanding of the restriction of the superpotentials $W^{(\alpha_\ell)}$, $\ell = 1, 2, 3$, to the one-parameter model.

Summary and outlook of Part II

In this part of the thesis we analyzed the deformation problem of certain families of D-branes in compact Calabi-Yau three-folds. This was achieved by studying the variation of mixed Hodge structure as described by the periods of the holomorphic three-form of the Calabi-Yau manifold while keeping track of the boundary contributions relative to a family $\mathcal{D}$ of four-cycles describing the B-brane geometry.

We found that, similarly to the well-studied deformation problem in the pure closed-string sector, the notions of flatness and integrability of the Gauss-Manin connection continue to make sense on the open-closed deformation space $\mathcal{M}$ of the family and lead to sensible results for open string enumerative invariants. Amongst others, the Gauss-Manin connection in flat coordinates displays an interesting ring structure on the infinitesimal deformations in F^2, which is compatible with CFT expectations and gives evidence for the existence of an A-model quantum product defined by the Ooguri-Vafa invariants. Other hints in this direction are the integrability condition and the meaningful definition of the mirror map via a flatness condition.

For geometries with a single open string modulus the integrability conditions imply that the relative period matrices and the Gauss-Manin connection matrices can all be expressed in terms of functional relations involving only the holomorphic prepotential $\mathcal{F}$ and one additional holomorphic function W. The analyzed open-closed deformation problem can also be related to CFT correlators. We explained how an a priori unobstructed deformation problem of B-branes wrapping a holomorphic family of four-cycles describes an obstructed deformation problem after turning on D5-brane charges. In particular this effect can be described in the CFT by the change of boundary conditions induced by non-trivial fluxes on the worldvolume of the B-brane. The afore mentioned holomorphic function W in the Gauss-Manin connection then turns into a superpotential encoding these obstructions.

By mirror symmetry our analysis carries over to the quantum integrable structure of the obstructed deformation space of the mirror A-branes in the mirror three-fold. For our explicit examples we obtain predictions for the Ooguri-Vafa invariants of the open-closed deformation space that satisfy the expected integrality constraints and further consistency conditions. We also analyzed the critical points of the type studied in [53, 54, 55, 188, 172, 189], where the A-model expansion emerges only after integrating out the open-string directions. The on-shell computations of [53, 54, 55] are conceptually well understood and provided the first examples of open-string mirror symmetry in compact Calabi–Yau manifolds. Our main motivation to study the type of critical points accessible also in the on-shell formalism was to gain a better understanding of the minimization in the open-string direction, which relates the on-shell computation to the described off-shell framework. On the B-model side, the relation is provided by the connection between integral relative period integrals and normal functions described in section 3.3.3. An important datum in this correspondence is the period vector on the surface, that is the brane 4-cycle. It classifies the D-brane vacua by the vanishing condition (3.29) and determines the inhomogeneous term in the Picard-Fuchs equation for the normal function.

In the relative cohomology approach the open-string deformations are off-shell yet one avoids working in string field theory by perturbing the unobstructed F-theory moduli space associated with the family of surfaces $\mathcal{D}$ by a probe brane representing an element in $H_2(\mathcal{D})$. This leads to well-defined *finite dimensional* off-shell deformation spaces associated with a particular parametrization by 'light' fields in the superpotential and this parametrization of off-shell deformations is also adapted to the topological string. By general arguments, different parametrizations are bound to fit together in an consistent way, as is explicitly demonstrated in one of the examples, where we parametrized the off-shell superpotentials by different choices of open-string deformation parameters. This means that by starting from a given supersymmetric configuration, we compare different off-shell deformation directions in the infinite-dimensional open-closed deformation space, and we find that the obtained on-shell tensions are independent from the chosen off-shell directions.[14] This is a gratifying result as the on-shell domain wall tensions should not depend on the details of integrating out the heavy modes.

The relative cohomology approach to open-closed deformations has successfully passed other non-trivial checks [56, 190]. In leading order the computed off-shell superpotentials are compatible with derivations of effective superpotentials using open-string worldsheet and matrix factorization techniques [191, 192, 193, 190, 140, 173, 194]. Beyond leading order, however, the discussed off-shell superpotentials predict in the context of type II theories higher order open-closed CFT correlators, which (at present) are difficult to compute by other means.

Turning to the effective four-dimensional $\mathcal{N} = 1$ supergravity theory we observed that the open-closed deformation space $\mathcal{M}$ is a fibration $\pi : \mathcal{M} \to \mathcal{M}_{CS}$ over the complex structure moduli and defines a Kähler manifold that is not of the most general form allowed by supergravity, but has a restricted "$\mathcal{N} = 1$ special geometry". By constructing a class of dual Calabi-Yau four-folds for F-theory compactification we derived an expression for the effective $\mathcal{N} = 1$ Kähler potential and the superpotential on $\mathcal{M}$ in terms of period integrals. The F-theory compactification on the dual four-fold provides a global embedding of the B-brane geometry on the three-fold and reduces to a local description in the decompactification/weak-coupling limit. The effective description obtained in this limit is in good agreement with the results obtained for D7-branes on orientifolds in the existing literature.

From this point of view some of the chain integrals we explicitly computed are off-shell brane superpotentials for the four-dimensional type II/F-theory compactifications depending on several open-closed deformations as well as their specialization to the on-shell values in the open-string direction. Mathematically the two potentials are therefore respectively related to

[14] See ref. [56] for an earlier example of this kind.

the integral period integrals on the (relative) cohomology group and the concept of normal functions, depending only on closed-string moduli [73, 54]. Both objects can therefore be studied Hodge theoretically by computing the variation of Hodge structure on the relevant (co-)homology fibers over the open-closed-string deformation space $\mathcal{M}$ as we have discussed. Ultimately, this determines the superpotential as a particular solution of a system of generalized GKZ type differential equations determined by the integral (relative) homology class of the brane.

From the phenomenological point of view, the superpotential determines the vacuum structure of four-dimensional F-theory compactifications. The complicated structure of the superpotential for our class of compactifications, described by infinite generalized hypergeometric series, should be contrasted with the simple structure of F-theory superpotentials in other classes of compactifications, as e.g. in [195, 196]. These hypergeometric series have sometimes a dual interpretation as D-instanton corrections and heterotic worldsheet corrections [57], and the rich structure of non-perturbative corrections to the brane superpotential should lead to interesting hierarchies of masses and couplings in the low-energy effective theory.

As shown in ref. [57], the solutions to the generalized GKZ system representing the F-theory superpotential do not only capture the superpotentials of dual Calabi-Yau three-fold compactifications, but more generally of type II and heterotic compactifications on generalized Calabi-Yau manifolds of complex dimension three.[15] This offers a powerful tool to study more generally the vacuum structure of phenomenologically interesting F-theory/type II/heterotic compactifications. It would be interesting to apply the Hodge theoretic approach described in these chapters to examples of phenomenologically motivated F-theory scenarios, as described e.g. in [200, 201, 202].[16] In the search for vacua, the step of passing from relative periods depending on open and closed-string deformations to normal functions depending only on closed-string moduli provides a natural split in the minimization process, which should be helpful in a regime of small string coupling. On the other hand, this distinction between closed and open-string moduli disappears away from this decoupling limit, for finite string coupling, where the two types of fields mix in a way determined by a certain degeneration of the F-theory four-fold.

However, the B-model results of these chapters, obtained predominantly from a Hodge theoretic approach, raise also a number of unanswered questions. The first is about the general meaning of mirror symmetry between open-closed deformation spaces in the presence of a non-trivial superpotential, which requires some sort of off-shell concept of mirror symmetry. As discussed above, a heuristic ansatz might be to define $\mathcal{M}$ first as the deformation space of an unobstructed family and then add in obstructions as a sort of perturbation, here represented by D5-brane charges. However, we feel that there should be a more fundamental answer to

[15] The first examples of dual compactifications of this type were given in ref. [197]. See also [198, 199] for related works and examples.

[16] See also ref. [203], for a recent review on this subject, and further references therein.

this important issue.

Another set of urgent questions concerns the A-model interpretation, such as a proper formulation of an A-model quantum ring that matches the ring structure observed on the B-model side and should include the Ooguri-Vafa invariants and Floer (co-)homology as essential ingredients. Similarly one would like to have a more explicit description of the target space geometry of the A-branes.

As we have seen the variation of mixed hodge structure is largely captured by the Hodge structure of the family of divisors $\mathcal{D}$. For examples with a single open-string deformation a detailed analysis of the Hodge structure of the K3 surface, equivalent to the subsystem defined by the Hodge structure on the surface $\mathcal{D}$, might be rewarding. In this work we explained how the analyzed supersymmetric domain wall tensions arise at enhancement points of the Picard lattice in the K3 moduli space. The leading term of the K3 periods near these specially symmetric points is a rational function in the deformations z and the roots α of the defining equations of the curves the chain connects. As argued in section 4.2, the global symmetry seems to be related to the discrete symmetry in the A-type brane in the mirror A-model configuration. It would be interesting to study in detail the structure of Picard lattice enhancement loci in order to systematically explore the web of $\mathcal{N}=1$ domain wall tensions in Calabi-Yau three-folds. Such an analysis potentially sheds light on the global structure of $\mathcal{N}=1$ superpotentials (see e.g. [204]).

In the examples we have focused on a single open-string deformation. Then the subsystem of the extended hypergeometric GKZ system, which governs the open deformations, describes the periods of an isogenic K3 surface. The presented techniques are directly applicable also to examples with several open deformations [205]. Then the subsystem geometry is not anymore governed by K3 periods but instead by the periods of a complex surface of a higher geometric genus. Exploring such examples is technically more challenging but new phenomena and interesting structures, like non-commutativity in the open-string sector, are likely to emerge. A related question in this context is the contribution from D-instanton corrections, which are also computed by the GKZ system for the F-theory compactification [57]. It would be very interesting to connect these results to the recent progress in computing D-brane instantons by different methods [206, 207, 208, 209, 210].

Part III

Wall-crossing, holomorphic anomaly and mock modularity of multiple M5-branes

5

Chapter 5

Mock modularity of multiple M5-branes

Contents

In this chapter, which is based on the original publication [4], we use wall-crossing formulae and the theory of mock modular forms to derive a holomorphic anomaly equation for the modified elliptic genus of two M5-branes wrapping a rigid divisor inside a Calabi-Yau manifold. The anomaly originates from restoring modularity of an indefinite theta-function capturing the wall-crossing of BPS invariants associated to D4-D2-D0 brane systems. We show the compatibility of this equation with anomaly equations previously observed in the context of $\mathcal{N} = 4$ topological Yang-Mills theory on $\mathbb{P}^2$ and E-strings obtained from wrapping M5-branes on a del Pezzo surface. The non-holomorphic part is related to a contribution originating from bound states of singly wrapped M5-branes on the divisor. We show in examples that the information provided by the anomaly is enough to compute the BPS degeneracies for certain charges. We further speculate on a natural extension of the anomaly to higher D4-brane charge.

5.1. Background dependence of multiple M5-branes

The study of background dependence of physical theories has been a rich source of insights. Understanding the change of correlators as the background parameters are varied supplemented by boundary data can be sufficient to solve the theory. A class of theories where the question of background dependence can be sharply stated are topological field theories. Correlators in topological theories typically have holomorphic expansions near special values of the background moduli. The expansion coefficients can be given precise mathematical meaning as topological invariants of the geometrical configuration contributing to the topological non-trivial sector of the path integral. Physically the expansion often captures information of the degeneracies of BPS states of theories related to the same geometry.

An example of this, that was analyzed in the preceding parts of this thesis, is the topological A-model [19], with a Calabi-Yau three-fold Z as target space, which in a large volume limit counts holomorphic maps from the worldsheet into $H_2(Z, \mathbb{Z})$ and physically captures the degeneracies of BPS states coming from an M-theory compactification on Z [28, 29]. Another example is the modified elliptic genus of an M5-brane wrapping a complex surface P,[1] which was related in ref. [211] to the partition function of topologically twisted $\mathcal{N} = 4$ Yang-Mills theory [30], which computes generating functions of Euler numbers of moduli spaces of instantons. This same quantity was shown in ref. [68] to capture the geometric counting of degeneracies of systems of D4-D2-D0 black holes associated to the MSW string [66].

In both cases the topological theories enjoy duality symmetries. T-duality acting on the Kähler moduli on Z in the topological string case and S-duality for the $\mathcal{N} = 4$ SYM theory acting on the gauge coupling $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$. The former symmetry extends by mirror symmetry and both might extend to U-duality groups. Both symmetries can be conveniently expressed in the language of modular forms.

The holomorphic expansions of the topological string correlators are given in the moduli spaces of families of theories. Fixing a certain background corresponding to a certain point in the moduli space, the topological correlators are expected to be holomorphic expansions. In refs. [64, 22] holomorphic anomaly equations governing topological string amplitudes were derived showing that this is not the case and hence the correlators suffer from background dependence.[2] In ref. [65] a background independent meaning was given to the correlators, stating that the anomaly merely reflects the choice of polarization if the partition function is considered as a wave function only depending on half of the variables of some phase space which has a natural geometric meaning in this context.

This anomaly is also manifest in a failure of target space duality invariance of the holomorphic expansion which can only be restored at the expense of holomorphicity as shown in ref. [219].[3] A similar story showed up in $\mathcal{N} = 4$ topological $U(2)$ SYM theory on $\mathbb{P}^2$ [30], where it was shown that different sectors of the partition function need a non-holomorphic completion which was found earlier in ref. [223] in order to restore S-duality invariance. An anomaly equation describing this non-holomorphicity was expected [30] in the cases where $b_2^+(P) = 1$. In these cases holomorphic deformations of the canonical bundle are absent. The non-holomorphic contributions were associated with reducible connections

[1] In the following we will use the terms surface, divisor and four-cycle (of a CY) interchangeably when the context is clear.

[2] The anomaly relates correlators at a given genus to lower genera thus providing a way to solve the theory. Using a polynomial algorithm [212, 213, 214] and boundary conditions [215] this can be used to compute higher genus topological string amplitudes on compact CY [216] manifolds and solve it on non-compact CY [217, 218].

[3] Following the anomaly reformulation of refs. [220, 221], see also [222].

$U(n) \to U(m) \times U(n-m)$ [30, 211]. In ref. [211] this anomaly was furthermore related to an anomaly appearing in the context of E-strings [224]. These strings arise from an M5-brane wrapping a del Pezzo surface $\mathcal{B}_9$, also called $\frac{1}{2}$K3. The anomaly in this context was related to the fact that n of these strings can form bound-states of m and $(n-m)$ strings. Furthermore, the anomaly could also be related to the one appearing in topological string theory.

The anomaly thus follows from the formation of bound-states. Although the holomorphic expansion would not know about the contribution from bound-states, the restoration of duality symmetry forces one to take these contributions into account. The non-holomorphicity can be understood physically as the result of a regularization procedure. The path integral produces objects like theta-functions associated to indefinite quadratic forms which need to be regularized to avoid divergences. This regularization breaks the modular symmetry, restoring the symmetry gives non-holomorphic objects. The general mathematical framework to describe these non-holomorphic completions is the theory of mock modular forms developed by Zwegers in ref. [61].[4] A mock modular form $h(\tau)$ of weight k is a holomorphic function which becomes modular after the addition of a function $g^*(\tau)$, at the cost of losing its holomorphicity. Here, $g^*(\tau)$ is constructed from a modular form $g(\tau)$ of weight $2-k$, which is referred to as shadow.

Another manifestation of the background dependence of the holomorphic expansions of the topological theories are wall-crossing phenomena associated to the enumerative content of the expansions. Mathematically, it is known that Donaldson-Thomas invariants jump on surfaces with $b_2^+(P) = 1$, see [31] and references therein, for related physical works see for example refs. [226, 227]. On the physics side wall-crossing refers to the jumping of the degeneracies of BPS states when walls of marginal stability are crossed. These phenomena were observed in the jumps of the soliton spectrum of two-dimensional theories [63] and were an essential ingredient of the work of Seiberg and Witten [7] in four-dimensional theories. Recent progress was triggered by formulae relating the degeneracies on both sides of the walls, which were given from a supergravity analysis in refs. [228, 229] and culminated in a mathematical rigorous formula of Kontsevich and Soibelman (KS) [72], which could also be derived from continuity of physical quantities in refs. [230, 231] (See also refs. [232, 233, 234]). The fact that the holomorphic anomaly describes how to transform the counting functions when varying the background moduli, which in turn changes the degeneracy of BPS states, suggests that non-holomorphicity and wall-crossing are closely related. In fact the failure of holomorphicity can be traced back to the boundary of the moduli space of the geometrical configuration, where the latter splits in several configurations with the same topological charges. Mock modularity was used in a physical context in the study of the wall-crossing of degeneracies of $\mathcal{N} = 4$ dyons[5] in ref. [236]. In the context of $\mathcal{N} = 2$ supersymmetric theories the application of ideas related

[4]See [60, 225] and Appendix H for an introduction and overview.
[5]See for example ref. [235] and references therein for more details.

to mock modularity was initiated in ref. [237] and further pursued in refs. [238, 239, 240].[6]

In the following we are going to study the relation between wall-crossing and non-holomorphicity and relate the appearance of the two. A central role is played by a wall-crossing formula by Göttsche [32], where the Kähler moduli dependence of a generating function of Euler numbers of stable sheaves is given in terms of an indefinite theta-function due to Göttsche and Zagier [31]. We show that this formula is equivalent to wall-crossing formulae of D4-D2-D0 systems in type IIA. The latter can be related to the (modified) elliptic genus of multiple M5-branes wrapping a surface. Rigid surfaces are subject to Göttsche's wall-crossing formula. Using ideas of Zwegers [61], we translate the latter into a holomorphic anomaly equation for two M5-branes wrapping the surface/divisor. We show that this anomaly equation is the equation which was found in the context of $\mathcal{N} = 4$ SYM [30] and E-strings [224, 211]. We further propose the generalization of the anomaly equation for higher wrappings and comment on its implications for the wall-crossing of multiple D4-branes.

5.2. Effective descriptions of wrapped M5-branes

We begin our discussion by reviewing the effective descriptions of M5-branes wrapping a complex surface P as well as previous appearances of the holomorphic anomaly which will be derived in the next section. The world-volume theory of M5-branes can have either a two-dimensional CFT description in terms of the so-called MSW-CFT [66] or a four-dimensional description giving the $\mathcal{N} = 4$ topologically twisted Yang-Mills theory of Vafa and Witten [30]. In the latter theory it was observed [30] that a non-holomorphicity [223] had to be introduced in order to restore S-duality, the resulting holomorphic anomaly was related in ref. [211] to an anomaly [224] appearing in the context of E-strings. The anomaly was conjectured to take into account contributions coming from reducible connections in $\mathcal{N} = 4$ SYM theory. In ref. [211] it was related to the curve counting anomaly [22] and was given the physical interpretation of taking into account the bound-state contribution of E-strings. Later we will show that the contributions from bound-states as a cause for non-holomorphicity will persist more generally for the class of surfaces we will be studying. In our work we investigate the (generalized/modified) elliptic genus which captures the content of the CFT description of the M5-branes [211, 69] and its relation to D4-D2-D0 systems [68, 71, 246, 229, 247, 248] and the associated counting of black holes which has been intensively studied (e.g. in ref. [249]). Our goal is to show that wall-crossing in D4-D2-D0 systems leads to an anomaly equation which coincides with the anomalies found before and hence our work complements in some sense this circle of ideas.

[6] Further physical appearances of mock modularity can be found for example in refs. [241, 242, 243, 244, 245].

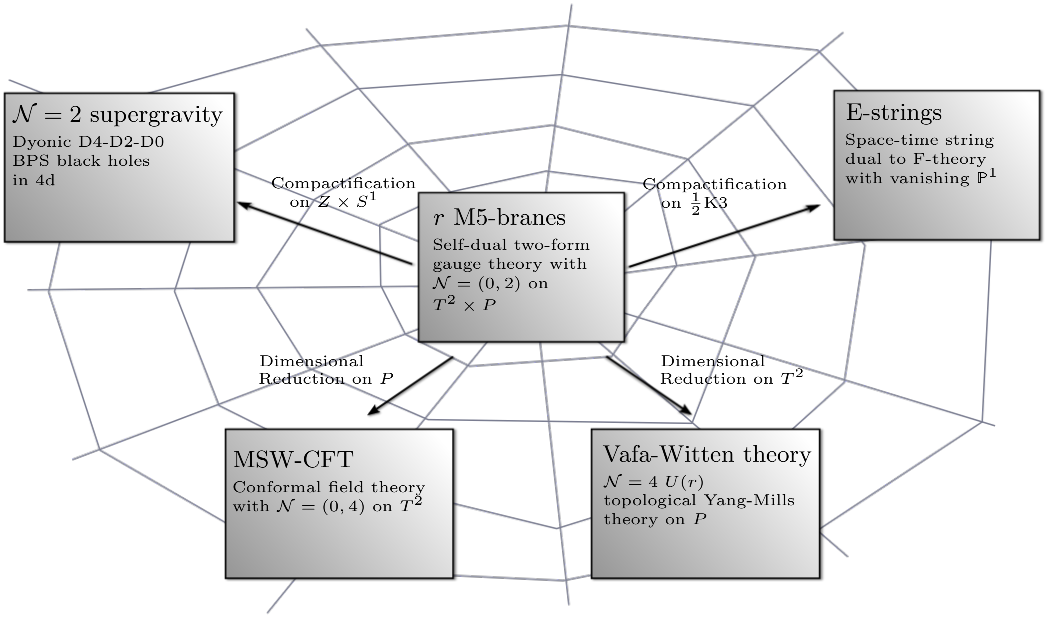

Figure 5.1.: Overview of the M5-brane duality web.

5.2.1. A Duality web for multiple M5-branes

Through dualities a system of multiple wrapped M5-branes can be viewed in various interrelated ways. For this we compactify time and view the world-volume of the M5-brane as $T^2 \times P$. The dual perspectives are then given by

1. $\mathcal{N} = (0, 2)$ world-volume theory living on the M5-brane:
   On the world-volume of r M5-branes lives a $\mathcal{N} = (0, 2)$ self-dual two-form gauge theory, which unfortunately admits no proper Lagrangian description (see e.g. [250] for a review). As the M5-brane wraps a curved manifold in this case, there appear also induced M2-brane charges on the world-volume of the M5-brane, described by non-trivial flux of the self-dual two-form gauge field on the M5-brane.

2. $\mathcal{N} = 4$ $SU(r)$ topological SYM on P:
   Reduction of the world-volume theory on the r M5-branes along the T^2 leads to a topologically twisted $\mathcal{N} = 4$ $U(r)$ Yang-Mills theory on P, called Vafa-Witten gauge theory [30]. The geometric SL$(2, \mathbb{Z})$ symmetry of the torus translates into the S-duality of the topological Yang-Mills theory on P. The M5-brane and induced M2-brane charges together with momentum around the circle direction correspond to the rank of the gauge group, the different flux sectors and instanton numbers.

3. $\mathcal{N} = (0, 4)$ SCFT living on T^2:
   By reducing along P one obtains a two-dimensional theory which lives on the T^2. Assuming the existence of a conformal fixed point this theory is expected to flow to a

non-trivial $\mathcal{N} = (0, 4)$ superconformal field theory, called MSW-CFT [66]. The degrees of freedom of the CFT are given by the deformations of the M5-branes inside Z and induced M2-brane charges on the M5-brane world-volume. This can also be considered as a special type of string theory obtained from the $(0, 2)$-world-volume theory of wrapped M5-branes [66, 251].

4. D4-D2-D0 black holes in $\mathcal{N} = 2$ supergravity in $d = 4$:
   Compactification of the theory on $S^1 \times Z$ leads to an effective $\mathcal{N} = 2$ supergravity theory in four dimensions. By M-theory – type IIA duality the M5-branes with induced M2-brane charges and momentum around the S^1 give rise to BPS-particles with D4-D2-D0-brane charges in four dimensions. This theory is supposed to capture the micro-states of dyonic D4-D2-D0 BPS-black holes in four dimensions and counts, in an appropriate sense, the number of bound-states of D4-D2-D0 branes.

5. Exceptional strings from F-theory:
   For a special type of surface P, called $\frac{1}{2}$K3 the M-theory compactification is dual to the heterotic string with an E_8 instanton of zero size and the resulting space-time string is called E-string [252, 224, 211]. This theory is furthermore dual to a F-theory compactification on a Calabi-Yau three-fold down to six dimensions, where the base $\mathbb{P}^1$ shrinks to zero size.

The above dualities allow to relate different physical quantities in each theory with each other. An example of this is the partition function of topological $\mathcal{N} = 4$ SYM, which is directly related to a particular BPS-index of the MSW-CFT, called the modified elliptic genus [211, 69]. The modified elliptic genus will be the central physical quantity which we are going to study in the following.

5.2.2. The elliptic genus and D4-D2-D0 branes

We now turn to a more detailed discussion of the $2d$ CFT perspective of the M5-brane world-volume theory. We want to study BPS states that arise in the context of an M-theory compactification on a Calabi-Yau manifold Z with r M5-branes wrapping a complex surface (or a four-cycle) P, and extended in $\mathbb{R}^{1,3} \times S^1$. Considering P to be small compared to the M-theory circle, the reduction of the world-volume theory of the M5-brane is described by a $(1 + 1)$-dimensional $(0, 4)$ MSW-CFT [66].[7] The BPS states associated to the string that remains after wrapping the M5-branes on P are captured by a further compactification on a circle. They are counted by the partition function of the world-volume theory of the M5-branes on $P \times T^2$ [211]. The effective CFT description will thus exhibit invariance under

[7] The target space sigma model description of which was given in ref. [251], for more details see ref. [253] and references therein. In the following we will be concerned with the natural extension of the analysis of the degrees of freedom to r M5-branes.

the full $\mathrm{SL}(2,\mathbb{Z})$ symmetry of the T^2. Furthermore, excitations of the M5-branes will induce M2-brane charges corresponding to the flux of the self-dual field strength of the M5-brane world-volume theory. In addition, the momentum of the M2-branes along the M-theory circle will give rise to a further quantum number. As a result BPS states of the effective two-dimensional description will be labeled by the class of the divisor the M5-branes wrap, the M2-brane charges and by the momentum along S^1. In a type IIA setup, r times the class of the divisor will correspond to D4-brane charge, the induced M2-brane charge corresponds to D2-brane charge and the momentum to D0-brane charge. Choosing a basis Σ_A , $A = 1, \dots, b_4(Z)$ of $H_4(Z,\mathbb{Z})$, the charge vector will be given by

$$\Gamma = (Q_6, Q_4, Q_2, Q_0) = r(0, p^A, q_A, q_0) \ , \tag{5.1}$$

where the Q_p are the Dp-brane charges and r is the number of coincident M5-branes wrapping the divisor specified by p^A. A priori the set of all possible induced D2-brane charges, or equivalently of $U(1)$ fluxes of the world-volume of the M5-brane would be in one-to-one correspondence with $\Lambda_P = H^2(P,\mathbb{Z})$ which is generically a larger lattice than $\Lambda = i^*H^2(Z,\mathbb{Z})$, where $i : P \hookrightarrow Z$, however the physical BPS states are always labeled by the smaller lattice Λ. The metric d_{AB} on Λ is given by

$$d_{AB} = -\int_P \alpha_A \wedge \alpha_B \ , \tag{5.2}$$

where α_A is a basis of two-forms in Λ, which is the dual basis to Σ_A of $H_4(Z,\mathbb{Z})$. In order to obtain a generating series of the degeneracies of those BPS states one has to sum over directions along $\Lambda^\perp$ which is the orthogonal complement to Λ in Λ_P w.r.t. d_{AB} [68]. [8]

The partition function of the MSW-CFT counting the BPS states is given by the modified elliptic genus[9] [211, 69]

$$Z_P'^{(r)}(\tau, z) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{RR}}} (-1)^{F_{\mathrm{R}}} F_{\mathrm{R}}^2 \, q^{L_0' - \frac{c_{\mathrm{L}}}{24}} \, \bar{q}^{\bar{L}_0' - \frac{c_{\mathrm{R}}}{24}} \, e^{2\pi i z \cdot Q_2} \ , \tag{5.3}$$

where the trace is taken over the RR Hilbert space. Furthermore, vectors are contracted w.r.t. the metric d_{AB}, i.e. $x \cdot y = x^A y_A = d_{AB} x^A y^B$. For a single M5-brane it was shown in ref. [71] that $Z_P'^{(1)}(\tau, z)$ transforms like a $\mathrm{SL}(2,\mathbb{Z})$ Jacobi form of bi-weight $(0, 2)$ due to the insertion of F_{R}^2, we demand that the same is true for all r.

Following ref. [71] the center of mass momentum $\vec{p}_{\mathrm{cm}}$ for the system of r M5-branes can be integrated out. In this way L_0' and $\bar{L}_0'$ can be written in the form

$$L_0' = \frac{1}{2}\vec{p}_{\mathrm{cm}}^{\,2} + L_0, \quad \bar{L}_0' = \frac{1}{2}\vec{p}_{\mathrm{cm}}^{\,2} + \bar{L}_0 \ . \tag{5.4}$$

[8] In general, the lattice $\Lambda \oplus \Lambda^\perp$ is only a sublattice of $H^2(P,\mathbb{Z})$, because $\det d_{AB} \neq 1$ in general, see for example ref. [251] and ref. [229] for a more recent exposition. However, we will only be concerned with divisors P with $b_2^+(P) = 1$, such that $\det d_{AB} = 1$.

[9] We follow the mathematics convention of not writing out explicitly the dependence on $\overline{\tau}$ which will be clear in the context. Moreover, we denote $q = e^{2\pi i \tau}$ and $\tau = \tau_1 + i\tau_2$. To avoid confusion without introducing new notation we will denote the charge vector of D2-brane charges by $\underline{q}$, its components by q_A.

This allows one to split up the center of mass contribution and rewrite formula (5.3) as

$$\begin{aligned} Z'^{(r)}_P(\tau,z) &= \int d^3p_{\rm cm}(q\bar{q})^{\frac{1}{2}\vec{p}^2_{\rm cm}} Z^{(r)}_P(\tau,z) \\ &\sim (\tau_2)^{-\frac{3}{2}}\, Z^{(r)}_P(\tau,z)\ , \end{aligned} \tag{5.5}$$

where $Z^{(r)}_P(\tau,z)$ is now a Jacobi form of weight $(-\frac{3}{2},\frac{1}{2})$ which we simply call elliptic genus for short in the following.

The decomposition of the elliptic genus

The elliptic genus $Z^{(r)}_P(\tau,z)$ and equivalently the generating function of D4-D2-D0 BPS degeneracies is subject to a theta-function decomposition, which has been studied in many places, see for example refs. [249, 68, 71, 246, 229]. This is ensured by two features of the superconformal algebra of the (0,4) CFT. One of these is that the $\overline{\tau}$ contribution entirely comes from BPS states $|\underline{q}\rangle$ satisfying

$$\left(\overline{L}_0 - \frac{c_{\rm R}}{24} - \frac{r}{2}q^2_{\rm R}\right)|\underline{q}\rangle = 0\ , \tag{5.6}$$

the other one is the spectral flow isomorphism of the $\mathcal{N}=(0,4)$ superconformal algebra, which we want to recall for r M5-branes here, building on refs. [71, 254], see also [249]. Proposition 2.9 of ref. [254] describes the spectral flow symmetry by an isomorphism between moduli spaces of vector bundles on complex surfaces. The complex surface here is the divisor P and the vector bundle configuration describes the bound-states of D4-D2-D0 branes. Within this setup the result of [254] translates for arbitrary r to a symmetry under the transformations

$$\begin{aligned} q_0 &\mapsto q_0 - k\cdot\underline{q} - \frac{1}{2}k\cdot k, \\ \underline{q} &\mapsto \underline{q} + k\ , \end{aligned} \tag{5.7}$$

where $k\in\Lambda$. Physically these transformations correspond to monodromies around the large radius point in the moduli-space of the Calabi-Yau manifold [249]. Denote by Λ^* the dual lattice of Λ with respect to the metric rd_{AB}. Keeping only the holomorphic degrees of freedom one can write

$$\begin{aligned} Z^{(r)}_P(\tau,z) &= \sum_{Q_0;Q_A} d(Q,Q_0)\, e^{-2\pi i\tau Q_0}\, e^{2\pi i z\cdot Q_2} \\ &= \sum_{q_0;\underline{q}\in\Lambda^*+\frac{[P]}{2}} d(r,\underline{q},-q_0)\, e^{-2\pi i\tau r q_0}\, e^{2\pi i r z\cdot\underline{q}}\ , \end{aligned} \tag{5.8}$$

where $d(r,\underline{q},-q_0)$ are the BPS degeneracies and the shift[10] $\frac{[P]}{2}$ originates from an anomaly [255, 256]. Now, spectral flow symmetry predicts [71]

$$d(r,\underline{q},-q_0) = (-1)^{rp\cdot k} d(r,\underline{q}+k,-q_0+k\cdot\underline{q}+\frac{k^2}{2})\ . \tag{5.9}$$

[10] In components, $[P]$ is given by $d_{AB}p^A$.

Making use of this symmetry and the following definition

$$\underline{q} = k + \mu + \frac{[P]}{2}, \qquad \mu \in \Lambda^*/\Lambda, \qquad k \in \Lambda \ , \tag{5.10}$$

one is led to the conclusion that the elliptic genus can be decomposed in the form

$$Z_P^{(r)}(\tau, z) = \sum_{\mu \in \Lambda^*/\Lambda} f_{\mu,J}^{(r)}(\tau)\theta_{\mu,J}^{(r)}(\tau, z) \ , \tag{5.11}$$

$$f_{\mu,J}^{(r)}(\tau) = \sum_{r\hat{q}_0 \geq -\frac{c_L}{24}} d_\mu^{(r)}(\hat{q}_0) e^{2\pi i \tau r \hat{q}_0} \ , \tag{5.12}$$

$$\theta_{\mu,J}^{(r)}(\tau, z) = \sum_{k \in \Lambda + \frac{[P]}{2}} (-1)^{rp\cdot(k+\mu)} e^{2\pi i \bar{\tau} r \frac{(k+\mu)_+^2}{2}} e^{2\pi i \tau r \frac{(k+\mu)_-^2}{2}} e^{2\pi i r z\cdot(k+\mu)} \ , \tag{5.13}$$

where $J \in \mathcal{C}(P)$, $\mathcal{C}(P)$ denotes the Kähler cone of P restricted to $\Lambda \otimes \mathbb{R}$ and $\hat{q}_0 = -q_0 - \frac{1}{2}\underline{q}^2$ is invariant under the spectral flow symmetry. The subscript $+$ refers to projection onto the sublattice generated by the Kähler form J and $-$ is the projection to its orthogonal complement, i.e.

$$k_+^2 = \frac{(k\cdot J)^2}{J \cdot J}, \quad k_-^2 = k^2 - k_+^2 \ . \tag{5.14}$$

There are two issues here for the case of rigid divisors with $b_2^+(P) = 1$ on which we want to comment. First of all note, that q_0 contains a contribution of the form $\frac{1}{2}\int_P F \wedge F$ where $F \in \Lambda_P$. Now, F can be decomposed into $F = \underline{q} + \underline{q}_\perp$ with $\underline{q}_\perp \in \Lambda^\perp$, which allows us to write

$$\hat{q}_0 = \tilde{q}_0 + \frac{1}{2}\underline{q}_\perp^2. \tag{5.15}$$

For $b_2^+(P) = 1$ and $r = 1$, the degeneracies $d(r, \mu, \tilde{q}_0)$ are independent of the choice of $\underline{q}_\perp$ and moreover it was shown by Göttsche [257] that

$$\sum_{\tilde{q}_0} d(1, \mu, \tilde{q}_0)\, e^{2\pi i \tau \tilde{q}_0} = \frac{1}{\eta^{\chi(P)}} \ . \tag{5.16}$$

Then, for $r = 1$ (5.12) becomes

$$f_{\mu,J}^{(1)}(\tau) = \frac{\vartheta_{\Lambda^\perp}(\tau)}{\eta^{\chi(P)}(\tau)}, \qquad \vartheta_{\Lambda^\perp}(\tau) = \sum_{\underline{q}_\perp \in \Lambda^\perp} e^{i\pi\tau \underline{q}_\perp^2} \ . \tag{5.17}$$

The second subtlety is concerned with the dependence on a Kähler class J. Due to wall-crossing phenomena we will find that $f_{\mu,J}^{(r)}(\tau)$ also depends on J. We expect that it has the following expansion ($\tilde{q}_0 = \frac{d}{r} - \frac{c_L}{24}$)

$$f_{\mu,J}^{(r)}(\tau) = (-1)^{rp\cdot\mu} \sum_{d \geq 0} \bar{\Omega}(\Gamma; J)\, q^{d - \frac{r\chi(P)}{24}} \ . \tag{5.18}$$

Here, the factor $(-1)^{rp\cdot\mu}$ is inserted to cancel its counterpart in the definition of $\theta^{(r)}_{\mu,J}$, which was only included to make the theta-functions transform well under modular transformations. The invariants $\bar{\Omega}(\Gamma; J)$ are rational invariants first introduced by Joyce [258, 259] and are defined as follows

$$\bar{\Omega}(\Gamma; J) = \sum_{m|\Gamma} \frac{\Omega(\Gamma/m; J)}{m^2} \ , \tag{5.19}$$

where $\Omega(\Gamma, J)$ is an integer-valued index of BPS degeneracies, given by [260]

$$\Omega(\Gamma, J) = \frac{1}{2}\mathrm{Tr}(2J_3)^2(-1)^{2J_3} \ , \tag{5.20}$$

where J_3 is a generator of the rotation group Spin(3). Note, that for a single M5-brane $\bar{\Omega}$ and Ω become identical and independent of J.

5.2.3. $\mathcal{N} = 4$ SYM, E-strings and bound-states

In the following we sketch the relation [211] of the elliptic genus of M5-branes to the $\mathcal{N} = 4$ topological SYM theory of Vafa and Witten [30]. Our goal is to relate the holomorphic anomaly equation which we will derive from wall-crossing in the next section to the anomalies appearing in the $\mathcal{N} = 4$ context. We review moreover the connection of the anomaly to the formation of bound-states given in ref. [211].

The $\mathcal{N} = 4$ topological SYM arises by taking a different perspective on the world-volume theory of n M5-branes on $P \times T^2$ considering the theory living on P which is the $\mathcal{N} = 4$ topologically twisted SYM theory described in ref. [30]. The gauge coupling of this theory is given by

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \tag{5.21}$$

and is geometrically realized by the complex structure modulus of the T^2. The partition function of this theory counts instanton configurations by computing the generating functions of the Euler numbers of moduli spaces of gauge instantons [30]. S-duality translates to the modular transformation properties of the partition function. The analogues of D4-D2-D0 charges are the rank of the gauge group, different flux sectors and the instanton number.

In ref. [211] the relation is made between this theory and the geometrical counting of BPS states of exceptional strings obtained by wrapping M5-branes around a del Pezzo surface $\mathcal{B}_9$, also called $\frac{1}{2}$K3. This string is dual to the heterotic string with an E_8 instanton of zero size [261, 262] and is therefore called E-string. In F-theory this corresponds to a $\mathbb{P}^1$ shrinking to zero size [263, 264, 265]. The geometrical study of the BPS states of this non-critical string was initiated in ref. [252] and further pursued in refs. [186, 266, 224]. In ref. [211] the counting of BPS states of the exceptional string with increasing winding n was related to the $U(n)$ $\mathcal{N} = 4$ SYM partition functions.

In the following we will use the geometry of ref. [252] which is an elliptic fibration over the Hirzebruch surface $\mathbb{F}_1$, which in turn is a $\mathbb{P}^1$ fibration over $\mathbb{P}^1$.[11] We will denote by t_E, t_F and t_D the Kähler parameters of the elliptic fiber, the fiber and the base of $\mathbb{F}_1$, respectively and enumerate these by $1,2,3$ in this order. We further introduce $\tilde{q}_a = e^{2\pi i \tilde{t}_a}$, $a = 1,2,3$ the exponentiated Kähler parameters appearing in the instanton expansion of the A-model at large radius, which are also the counting parameters of the BPS states.

Within this geometry we will be interested in the elliptic genus of M5-branes wrapping two different surfaces, one is a K3 corresponding to wrapping the elliptic fiber and the fiber of $\mathbb{F}_1$, the resulting string is the heterotic string. The other possibility is to wrap the base of $\mathbb{F}_1$ and the elliptic fiber corresponding to $\frac{1}{2}$K3 and leading to the E-string studied in refs. [252, 186, 266, 224, 211]. The two possibilities are realized by taking the limits $t_D, t_F \to i\infty$, respectively. The resulting surface in both cases is still elliptically fibered which allows one to identify the D4-D0 charges n and p with counting curves wrapping n-times the base and p-times the fiber of the elliptic fibration [211]. The multiple wrapping is hence encoded in the expansion of the prepotential $F_0(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)$ of the geometry. In order to get a parameterization inside the Kähler cone of the K3 in which the corresponding curves in $H_2(\text{K3}, \mathbb{Z})$ intersect with the standard metric of the hyperbolic lattice $\Gamma^{1,1}$, we define $t_1 = \tilde{t}_1\,, t_2 = \tilde{t}_2 - \tilde{t}_1$ and $t_3 = \tilde{t}_3$ as well as the corresponding $q_1 = \tilde{q}_1$, $q_2 = \tilde{q}_2/\tilde{q}_1$ and $q_3 = \tilde{q}_3$. Taking q_2 or $q_3 \to 0$, the multiple wrapping of the base is expressed by

$$F_0(t_1, t_a) = \sum_{n\geq 1} Z^{(n)}(t_1) q_a^n\,, \quad a = 2 \text{ or } 3\,. \tag{5.22}$$

The $Z^{(n)}$ can be identified with the elliptic genus of n M5-branes wrapping the corresponding surface after taking a small elliptic fiber limit [211]. In this limit the contribution coming from the theta-functions (5.13) reduce to $\tau_2^{-3/2}$ $\left(\tau_2^{-1/2}\right)$ for the K3($\frac{1}{2}$K3) cases, these are the contributions of 3(1) copies of the lattice $\Gamma^{1,1}$ appearing in the decomposition of the lattices of K3($\frac{1}{2}$K3). Omitting these factors gives the $Z^{(n)}$ of weight $(-2,0)$ in both cases. The elliptic genera of wrapping n M5-branes corresponding to n strings are in both cases related recursively to the lower wrapping. The nature of the recursion depends crucially on the ability of the strings to form bound-states.

The heterotic string, no bound-states

The heterotic string is obtained from wrapping an M5-brane on the K3 by taking the $q_3 \to 0$ limit. The heterotic string does not form bound-states and the recursion giving the higher wrappings in this case is the Hecke transformation[12] of $Z^{(1)}$ as proposed in ref. [211]. The formula for the Hecke transformation in this case is given by

$$Z^{(n)}(t) = n^{w_{\rm L}-1} \sum_{a,b,d} d^{-w_L} Z^{(1)}\left(\frac{at+b}{d}\right)\,, \tag{5.23}$$

[11] The toric data of this geometry is summarized in Appendix A.6.2.

[12] For a review on Hecke transformations see Zagier's article in [267].

with $ad = n$ and $b < d$ and $a, b, d \geq 0$. Which specializes for $w_{\rm L} = -2$ and $n = p$, where p is prime to

$$Z^{(p)}(t) = \frac{1}{p^3} Z^{(1)}(pt) + \frac{1}{p}\left[Z^{(1)}\left(\frac{t}{p}\right) + Z^{(1)}\left(\frac{t}{p} + \frac{1}{p}\right) + \cdots + Z^{(1)}\left(\frac{t}{p} + \frac{p-1}{p}\right)\right] . \tag{5.24}$$

For example the partition functions for $n = 1, 2$ obtained from the instanton part of the prepotential of the geometry read

$$Z^{(1)} = -\frac{2E_4E_6}{\eta^{24}}, \quad Z^{(2)} = -\frac{E_4E_6\left(17E_4^3 + 7E_6^2\right)}{96\eta^{48}} , \tag{5.25}$$

and are related by the Hecke transformation.[13] The fact that the partition functions of higher wrappings of the M5-brane on the K3, which correspond to multiple heterotic strings, are given by the Hecke transformation was interpreted [211] by the absence of bound-states. Geometrically, multiple M5-branes on a K3 can be holomorphically deformed off one another. This argument fails for surfaces with $b_2^+ = 1$ and in particular for $\frac{1}{2}$K3.

One reason that the higher $Z^{(n)}$ can be determined in such a simple way from $Z^{(1)}$ can be understood in topological string theory from the fact that the BPS numbers on K3 depend only on the intersection of a curve $\mathcal{C}^2 = 2g-2$ [269], and not on their class in $H_2(\mathrm{K3}, \mathbb{Z})$. This allows to prove (5.23) to all orders in the limit of the topological string partition function under consideration by slightly modifying the proof in [270]. Using the Picard-Fuchs system of the elliptic fibration one shows in the limit $q_3 \to 0$ the first equality in the identity

$$\begin{aligned} \frac{1}{2}\left(\frac{\partial}{\partial t_2}\right)^3 F_0|_{q_3\to 0} &= \frac{E_4(t_1)E_6(t_1)E_4(t_2)}{\eta(t_1)^{24}(j(t_1) - j(t_2))} \\ &= \frac{q_1}{q_1 - q_2} + E_4(t_2) - \sum_{d,l,k>0} l^3 c(kl) q_1^{kl} q_2^{ld} , \end{aligned} \tag{5.26}$$

where $j = E_4^3/\eta^{24}$ and $c(n)$ are defined as

$$-\frac{1}{2}Z^{(1)} = \sum_n c(n) q^n . \tag{5.27}$$

This equations shows two things. The BPS numbers inside the Kähler cone of K3 depend only on $\mathcal{C}^2 = kl$ and all $Z^{(n)}$ are given by one modular form. The second fact can be used as in [270] to establish that

$$\frac{1}{2}\left(\frac{\partial}{\partial t_2}\right)^3 F_0|_{q_3\to 0} = \sum_{n=0}^{\infty} F_n(t_1) q_2^n , \tag{5.28}$$

where F_n is the Hecke transform of F_1, i.e. $n^3 F_n = F_1|T_n$. Using Bol's identity and restoring the n^3 factors yields (5.23).

[13] Further examples of higher wrapping can be found in [4, 268].

E-strings and bound-states

The recursion relating the higher windings of the E-strings to lower winding, developed in [266, 224, 211] in contrast reads

$$\frac{\partial Z^{(n)}}{\partial E_2} = \frac{1}{24}\sum_{s=1}^{n-1} s(n-s)Z^{(s)}\,Z^{(n-s)}\,, \tag{5.29}$$

which becomes an anomaly equation, when E_2 is completed into a modular object $\widehat{E}_2$ by introducing a non-holomorphic part (see Appendix H). The anomaly reads:

$$\partial_{\bar{t}_1}\widehat{Z}^{(n)} = \frac{i(\mathrm{Im}\,t_1)^{-2}}{16\pi}\sum_{s=1}^{n-1} s(n-s)\widehat{Z}^{(s)}\widehat{Z}^{(n-s)}\,, \tag{5.30}$$

and was given the interpretation [211] of taking into account the contributions from bound-states. Starting from [252]

$$Z^{(1)} = \frac{E_4\sqrt{q}}{\eta^{12}}\,, \tag{5.31}$$

and using the vanishing of BPS states of certain charges one obtains recursively all $Z^{(n)}$ [266, 224, 211]. E.g. the $n=2$ the contribution reads:

$$\widehat{Z}^{(2)} = \frac{qE_4E_6}{12\eta^{24}} + \frac{q\widehat{E_2}E_4^2}{24\eta^{24}}\,, \tag{5.32}$$

where the second summand has the form $\widehat{E}_2\left(Z^{(1)}\right)^2$ and takes into account the contribution from bound-states of singly wrapped M5-branes.

A relation to the anomaly equations appearing in topological string theory [22] was pointed out in ref. [211] and proposed for arbitrary genus in refs. [271, 272]. The higher genus generalization reads [271, 272]:

$$\frac{\partial Z_g^{(n)}}{\partial E_2} = \frac{1}{24}\sum_{g_1+g_2=g}\sum_{s=1}^{n-1} s(n-s)Z_{g_1}^{(s)}\,Z_{g_2}^{(n-s)} + \frac{n(n+1)}{24}Z_{g-1}^{(n)}\,, \tag{5.33}$$

where the instanton part of the A-model free energies at genus g is denoted by $F_g(q_1,q_2,q_3)$, and $F_g(q_1,q_2\to 0,q_3) = \sum_{n\geq 1} Z_g^{(n)}q_3^n$. The $Z_g^{(n)}$ have the form [272]

$$Z_g^{(n)} = P_g^{(n)}(E_2,E_4,E_6)\frac{q_1^{n/2}}{\eta^{12n}}\,, \tag{5.34}$$

where $P_g^{(n)}$ denotes a quasi-modular form of weight $2g+6n-2$.

5.2.4. Generating functions from wall-crossing

In the last section we have argued that the partition function of $\mathcal{N}=4$ $U(r)$ Super-Yang-Mills theory suffers from a holomorphic anomaly for divisors with $b_2^+(P)=1$. In fact there exists another way to see the anomaly which is also intimately related to the computation of BPS degeneracies encoded in the elliptic genus and will be the subject of this section. This method relies on wall-crossing formulas and originally goes back to Göttsche and Zagier [31, 32]. In the physics context it has also been employed in [238, 239]. It will be used in section 5.3 to derive the elliptic genus for BPS states and their anomaly rigorously. In the following presentation we will be very sketchy as we merely want to stress the main ideas. We refer to section 5.3 for details.

The starting point is the Kontsevich-Soibelman formula [72] which describes the wall-crossing of bound-states of D-branes. Specifying to the case of two M5-branes and taking the equivalent D4-D2-D0 point of view the Kontsevich-Soibelman formula reduces to the primitive wall-crossing formula

$$\Delta\Omega(\Gamma; J \to J') = \Omega(\Gamma; J') - \Omega(\Gamma; J) = (-1)^{\langle\Gamma_1,\Gamma_2\rangle - 1}\langle\Gamma_1, \Gamma_2\rangle\, \Omega(\Gamma_1)\, \Omega(\Gamma_2)\ , \tag{5.35}$$

which describes the change of BPS degeneracies of a bound-state with charge vector $\Gamma = \Gamma_1 + \Gamma_2$, once a wall of marginal stability specified by J_W is crossed. The symplectic charge product $\langle\cdot,\cdot\rangle$ is defined by

$$\langle\Gamma_1, \Gamma_2\rangle = -Q_6^{(1)}Q_0^{(2)} + Q_4^{(1)}\cdot Q_2^{(2)} - Q_2^{(1)}\cdot Q_4^{(2)} + Q_0^{(1)}Q_6^{(2)}\ . \tag{5.36}$$

Hence, for D4-D2-D0 brane configurations $\langle\Gamma_1, \Gamma_2\rangle$ is independent of the D0-brane charge. Further, in eq. (5.35) Γ_1 and Γ_2 are primitive charge vectors such that $\Omega(\Gamma_i)$ do not depend on the moduli. Thus, the Γ_i can be thought of as charge vectors with $r = 1$ whereas Γ corresponds to a charge vector with $r = 2$. Assuming, that the wall of marginal stability does not depend on the D0-brane charge, formula (5.35) can be translated into a generating series $\Delta f^{(2)}_{\mu,J\to J'}$ defined by

$$\Delta f^{(2)}_{\mu,J\to J'} = \sum_{d\geq 0} \Delta\bar{\Omega}(\Gamma; J \to J')\, q^{d-\frac{\chi(P)}{12}}\ . \tag{5.37}$$

Assuming that there exists a reference chamber J' such that $\bar{\Omega}(\Gamma; J) = 0$, this gives us directly an expression for $f^{(2)}_{\mu,J}$.

As it will turn out in the next section, $\Delta f^{(2)}_{\mu,J\to J'}$ is given in terms of an indefinite theta-function $\Theta^{J,J'}_{\Lambda,\mu}$, which contains the information about the decays due to wall-crossing as one moves from J to J'. Indefinite theta-functions were analyzed by Zwegers in his thesis [61]. One of their major properties is that they are not modular as one only sums over a bounded domain of the lattice Λ specified by J and J'. However, Zwegers showed that by adding a non-holomorphic completion the indefinite theta-functions have modular transformation behavior and fall into the class of mock modular forms.[14] Every mock modular form h of weight k has a shadow g, which is a modular form of weight $2 - k$, such that the function

$$\hat{h}(\tau) = h(\tau) + g^*(\tau) \tag{5.38}$$

[14] We review some notions in Appendix H.

transforms as a modular form of weight k but is not holomorphic. Here, g^* is a certain transformation of the function g that introduces a non-holomorphic dependence. Taking the derivative of $\hat{h}$ with respect to $\bar{\tau}$ yields a holomorphic anomaly given by the shadow

$$\frac{\partial \hat{h}}{\partial \bar{\tau}} = \frac{\partial g^*}{\partial \bar{\tau}} = \tau_2^{-k}\overline{g(\tau)} \ , \tag{5.39}$$

where $\tau_2 = \mathrm{Im}(\tau)$.

As described in sections 5.2.2 and 5.2.3 the (MSW) CFT and the $\mathcal{N} = 4$ $U(r)$ Super-Yang-Mills partition functions should behave covariantly under modular transformations of the $\mathrm{SL}(2,\mathbb{Z})$ acting on τ. Thus, the modular completion outlined above will effect the generating functions $f^{(2)}_{\mu,J}$ through their relation to the indefinite theta-function $\Theta^{J,J'}_{\Lambda,\mu}$, which needs a modular completion to transform covariantly under modular transformations, i.e.

$$\Theta^{J,J'}_{\Lambda,\mu} \mapsto \widehat{\Theta}^{J,J'}_{\Lambda,\mu} \tag{5.40}$$

and consequently $f^{(2)}_{\mu,J}$ is replaced by $\hat{f}^{(2)}_{\mu,J}$. Due to eq. (5.38) the counting function of BPS invariants $\hat{f}^{(2)}_{\mu,J}$ and thus the elliptic genus $Z^{(2)}_P$ are going to suffer from a holomorphic anomaly, to which we turn next.

5.3. Wall-crossing and mock modularity

In this section we derive an anomaly equation for two M5-branes wound on a rigid surface/divisor P with $b_2^+(P) = 1$, inside a Calabi-Yau manifold Z. We begin by reviewing D4-D2-D0 bound-states in the type IIA picture and their wall-crossing in the context of the Kontsevich-Soibelman formula. Then we proceed by deriving a generating function for rank two sheaves from the Kontsevich-Soibelman formula which is equivalent to Göttsche's formula [32]. This generating function is an indefinite theta-function, which fails to be modular. As a next step we apply ideas of Zwegers to remedy this failure of modularity by introducing a non-holomorphic completion. This leads to a holomorphic anomaly equation of the elliptic genus of two M5-branes that we prove for rigid divisors P.

5.3.1. D4-D2-D0 wall-crossing

In the following we take on the equivalent type IIA point of view, adapting the discussion of refs. [273, 238, 239] to describe the relation to the Kontsevich-Soibelman wall-crossing formula [72]. We restrict our attention to the D4-D2-D0 system on the complex surface P and work in the large volume limit with vanishing B-field.

Let us recall that a generic charge vector with D4-brane charge r is given by

$$\Gamma = (Q_6, Q_4, Q_2, Q_0) = r\left(0,\, [P],\, i_*F(\mathcal{E}),\, \frac{\chi(P)}{24} + \int_P \frac{1}{2}F(\mathcal{E})^2 - \Delta(\mathcal{E})\right) , \tag{5.41}$$

where $\mathcal{E}$ is a sheaf on the divisor P. Further, we define

$$\Delta(\mathcal{E}) = \frac{1}{r(\mathcal{E})}\left(c_2(\mathcal{E}) - \frac{r(\mathcal{E})-1}{2r(\mathcal{E})}c_1(\mathcal{E})^2\right)\,, \quad \mu(\mathcal{E}) = \frac{c_1(\mathcal{E})}{r(\mathcal{E})}\,, \quad F(\mathcal{E}) = \mu(\mathcal{E}) + \frac{[P]}{2}\,. \tag{5.42}$$

We recall that in the large volume regime the notion of D-brane stability is equivalent to μ-stability, see e.g. [273]. Given a choice of $J \in \mathcal{C}(P)$, a sheaf $\mathcal{E}$ is called μ-semi-stable if for every sub-sheaf $\mathcal{E}'$

$$\mu(\mathcal{E}')\cdot J \leq \mu(\mathcal{E})\cdot J\,. \tag{5.43}$$

Moreover, a wall of marginal stability is a co-dimension one subspace of the Kähler cone $\mathcal{C}(P)$ where the following condition is satisfied

$$(\mu(\mathcal{E}_1) - \mu(\mathcal{E}_2))\cdot J = 0\,, \tag{5.44}$$

but is non-zero away from the wall. Across such a wall of marginal stability the configuration (5.41) splits into two configurations with charge vectors

$$\begin{aligned}
\Gamma_1 &= r_1\left(0,\, [P],\, i_*F_1,\, \frac{\chi(P)}{24} + \int_P \frac{1}{2}F_1^2 - \Delta(\mathcal{E}_1)\right)\,,\\
\Gamma_2 &= r_2\left(0,\, [P],\, i_*F_2,\, \frac{\chi(P)}{24} + \int_P \frac{1}{2}F_2^2 - \Delta(\mathcal{E}_2)\right)\,,
\end{aligned} \tag{5.45}$$

where $r_i = \mathrm{rk}(\mathcal{E}_i)$ and $\mu_i = \mu(\mathcal{E}_i)$. By making use of the identity

$$r\Delta = r_1\Delta_1 + r_2\Delta_2 + \frac{r_1 r_2}{2r}\left(\frac{c_1(\mathcal{E}_1)}{r_1} - \frac{c_1(\mathcal{E}_2)}{r_2}\right)^2\,, \tag{5.46}$$

one can show that $\Gamma = \Gamma_1 + \Gamma_2$. Therefore, charge-vectors as defined in (5.41) form a vector-space which will be essential for the application of the Kontsevich-Soibelman formula.

Before we proceed, let us note, that the BPS numbers and the Euler numbers of the moduli space of sheaves are related as follows. Denote by $\mathcal{M}_J(\Gamma)$ the moduli space of semi-stable sheaves characterized by Γ. Its dimension reads [274]

$$\dim_{\mathbb{C}}\mathcal{M}_J(\Gamma) = 2r^2 - r^2\chi(\mathcal{O}_P) + 1\,. \tag{5.47}$$

The relation between BPS invariants and the Euler numbers of the moduli spaces $\mathcal{M}_J(\Gamma)$ is then given by [273]

$$\Omega(\Gamma, J) = (-1)^{\dim_{\mathbb{C}}\mathcal{M}_J(\Gamma)}\chi(\mathcal{M}(\Gamma), J)\,. \tag{5.48}$$

Moreover, for the system of charges we have specified to, the symplectic pairing of charges simplifies to [273]

$$\langle\Gamma_1, \Gamma_2\rangle = r_1 r_2(\mu_2 - \mu_1)\cdot[P]\,. \tag{5.49}$$

The holomorphic function $f^{(r)}_{\mu,J}(\tau)$ appearing in eq. (5.11) can now be identified with the generating function of BPS invariants of moduli spaces of semi-stable sheaves. Its wall crossing will be described in the following.

Kontsevich-Soibelman wall-crossing formula

Kontsevich and Soibelman [72] have proposed a formula which determines the jumping behavior of BPS-invariants $\Omega(\Gamma; J)$ across walls of marginal stability. The wall-crossing formula is given in terms of a Lie algebra defined by generators e_Γ and a basic commutation relation

$$[e_{\Gamma_1}, e_{\Gamma_2}] = (-1)^{\langle\Gamma_1,\Gamma_2\rangle}\langle\Gamma_1, \Gamma_2\rangle e_{\Gamma_1+\Gamma_2} \ . \tag{5.50}$$

For every charge Γ an element U_Γ of the Lie group can be defined by

$$U_\Gamma = \exp\left(-\sum_{n\geq 1}\frac{e_{n\Gamma}}{n^2}\right) \ . \tag{5.51}$$

The Kontsevich-Soibelman wall-crossing formula states that across a wall of marginal stability the following formula holds

$$\prod^{\curvearrowright}_{\Gamma:Z(\Gamma;J)\in V} U_\Gamma^{\Omega(\Gamma;J_+)} = \prod^{\curvearrowright}_{\Gamma:Z(\Gamma;J)\in V} U_\Gamma^{\Omega(\Gamma;J_-)} \ , \tag{5.52}$$

where J_+ and J_- denote Kähler classes on the two sides of the wall. Further, V is a region in $\mathbb{R}^2$ bounded by two rays starting at the origin and $\curvearrowright$ denotes a clockwise ordering of the factors in the product with respect to the phase of the central charges $Z(\Gamma; J)$, that are given by mirror symmetry as

$$Z(\Gamma; J) = \quad -\textstyle\int e^{-(B+iJ)}\,\Gamma + (\text{instanton} - \text{corrections}) \ . \tag{5.53}$$

Restricting to the case $r = 2$ and $r_1 = r_2 = 1$, (5.52) can be truncated to

$$\prod_{Q_{0,1}} U_{\Gamma_1}^{\Omega(\Gamma_1)} \prod_{Q_0} U_\Gamma^{\Omega(\Gamma;J_+)} \prod_{Q_{0,2}} U_{\Gamma_2}^{\Omega(\Gamma_2)} = \prod_{Q_{0,2}} U_{\Gamma_2}^{\Omega(\Gamma_2)} \prod_{Q_0} U_\Gamma^{\Omega(\Gamma;J_-)} \prod_{Q_{0,1}} U_{\Gamma_1}^{\Omega(\Gamma_1)} \ , \tag{5.54}$$

where Q_0 is the D0-brane charge of Γ and the $Q_{0,i}$ are the D0-brane charges belonging to Γ_i, respectively. The above formula has been derived by setting all Lie algebra elements with D4-brane charge greater than two to zero. Therefore, the element e_Γ is central, using the Baker-Campbell-Hausdorff formula $e^X e^Y = e^Y e^{[X,Y]} e^X$ and the fact that the symplectic product is independent of the D0-brane charge, one finds the following change of BPS numbers across a wall of marginal stability [238, 230]

$$\Delta\Omega(\Gamma) = (-1)^{\langle\Gamma_1,\Gamma_2\rangle-1}\langle\Gamma_1, \Gamma_2\rangle \sum_{Q_{0,1}+Q_{0,2}=Q_0} \Omega(\Gamma_1)\,\Omega(\Gamma_2) \ . \tag{5.55}$$

Moreover, one can deduce that the rank one degeneracies $\Omega(\Gamma_1)$ and $\Omega(\Gamma_2)$ do not depend on the modulus J.

5.3.2. Relation of KS to Göttsche's wall-crossing formula

Göttsche has found a wall-crossing formula for the Euler numbers of moduli spaces of rank two sheaves in terms of an indefinite theta-function in ref. [32]. In this section we want to derive a modified version of this formula from the Kontsevich-Soibelman wall-crossing formula associated to D4-D2-D0 bound-states with D4-brane charge equal to two.

We use the short notation $\Gamma = (r, \mu, \Delta)$ to denote a rank r sheaf with the specified Chern classes that is associated to the D4-D2-D0 states. For rank one sheaves the generating function has no chamber dependence and we have already seen that it is given by (5.17). Following the discussion of our last section, higher rank sheaves do exhibit wall-crossing phenomena and therefore do depend on the chamber in moduli space, i.e. on $J \in \mathcal{C}(P)$.

Our aim now is to determine the generating function of the D4-D2-D0 system using the primitive wall-crossing formula derived from the KS wall-crossing formula. From now on we restrict our attention to rank two sheaves $\mathcal{E}$. They can split across walls of marginal stability into rank one sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ as outlined in section 5.3.1. Using relation (5.46) we can write

$$d = d_1 + d_2 + \xi \cdot \xi \ , \tag{5.56}$$

where $\xi = \mu_1 - \mu_2$ and $d = 2\Delta$. Further, a wall is given by (5.44), i.e. the set of walls given a split of charges ξ reads

$$W^\xi = \{ J \in \mathcal{C}(P) \,|\, \xi \cdot J = 0 \} \ . \tag{5.57}$$

Now, consider a single wall $J_W \in W^\xi$ determined by a set of vectors $\xi \in \Lambda + \mu$. Let J_+ approach J_W infinitesimally close from one side and J_- infinitesimally close from the other side. Thus, in our context the primitive wall-crossing formula (5.55) becomes

$$\bar\Omega(\Gamma; J_+) - \bar\Omega(\Gamma; J_-) = \sum_{Q_{0,1}+Q_{0,2}=Q_0} (-1)^{2\xi\cdot[P]}\, 2\, (\xi \cdot [P])\, \Omega(\Gamma_1)\, \Omega(\Gamma_2) \ , \tag{5.58}$$

where we have used the identity (5.49). Note, that $Q_{0,i}$ and Q_0 are determined in terms of Γ and Γ_i through (5.41) and (5.45). Now, we can sum over the D0-brane charges to obtain a generating series. This yields

$$\begin{aligned} &\sum_{d\geq 0} (\bar\Omega(\Gamma; J_+) - \bar\Omega(\Gamma; J_-)) q^{d - \frac{\chi(P)}{12}} \\ = &\sum_{d_1, d_2 \geq 0,\, \xi} (-1)^{2\xi\cdot[P]} (\xi \cdot [P])\, \Omega(\Gamma_1)\Omega(\Gamma_2) q^{d_1 + d_2 + \xi^2 - \frac{2\chi(P)}{24}} \\ = &\ (-1)^{2\mu\cdot[P]-1} \frac{\vartheta_{\Lambda^\perp}(\tau)^2}{\eta(\tau)^{2\chi(P)}} \sum_\xi (\xi \cdot [P])\, q^{\xi^2} \ , \end{aligned} \tag{5.59}$$

where for the first equality use has been made of the identities (5.56, 5.58), and for the second equality the identity (5.17) has been used. The last line can be rewritten as

$$(-1)^{2\mu\cdot[P]-1} \frac{1}{2} \frac{\vartheta_{\Lambda^\perp}(\tau)^2}{\eta(\tau)^{2\chi(P)}} \mathrm{Coeff}_{2\pi i y}(\Theta^{J_+,J_-}_{\Lambda,\mu}(\tau, [P]y)) \ , \tag{5.60}$$

where we have introduced the indefinite theta-function

$$\Theta^{J,J'}_{\Lambda,\mu}(\tau,x) := \frac{1}{2}\sum_{\xi\in\Lambda+\mu}(\mathrm{sgn}\langle J,\xi\rangle - \mathrm{sgn}\langle J',\xi\rangle)\, e^{2\pi i\langle\xi,x\rangle}\, q^{Q(\xi)}\ , \tag{5.61}$$

with the inner product[15] defined by $\langle x,y\rangle = 2d_{AB}x^Ay^B$ and the quadratic form $Q(\xi)=\frac{1}{2}\langle\xi,\xi\rangle$. As these theta-functions obey the cocycle condition [31]

$$\Theta^{F,G}_{\Lambda,\mu} + \Theta^{G,H}_{\Lambda,\mu} = \Theta^{F,H}_{\Lambda,\mu}\ , \tag{5.62}$$

we finally arrive at the beautiful relation between the BPS numbers in an arbitrary chamber J and those in a chamber J' first found by Göttsche in the case $\Lambda = H^2(P,\mathbb{Z})$:

$$f^{(2)}_{\mu,J'}(\tau) - f^{(2)}_{\mu,J}(\tau) = \frac{1}{2}\frac{\vartheta_{\Lambda^\perp}(\tau)^2}{\eta^{2\chi(P)}(\tau)}\,\mathrm{Coeff}_{2\pi iy}(\Theta^{J,J'}_{\Lambda,\mu}(\tau,[P]y))\ . \tag{5.63}$$

5.3.3. Holomorphic anomaly at rank two

In this subsection we discuss the appearance of a holomorphic anomaly at rank two and give a proof of it by combing our previous results with results of Zwegers [61].

Elliptic genus at rank two and modularity

An important datum in eq. (5.63) is the choice of chambers $J, J'\in\mathcal{C}(P)$, which are any points in the Kähler cone of P. As a consequence, the indefinite theta-series does not transform well under $\mathrm{SL}(2,\mathbb{Z})$ in general. However, from the discussion of sect. 5.2.2 we expect, that the generating series $f^{(r)}_{\mu,J}(\tau)$ transforms with weight $-\frac{r(\Lambda)+2}{2}$ in a vector-representation under the full modular group, where $r(\Lambda)$ denotes the rank of the lattice Λ. Hence, there is a need to restore modularity. The idea is as follows.

Following Zwegers [61], it turns out that the indefinite theta-function can be made modular at the cost of losing its holomorphicity. From the definition (5.61) Zwegers smoothes out the sign-functions and introduces a modified function as

$$\widehat{\Theta}^{c,c'}_{\Lambda,\mu}(\tau,x) = \frac{1}{2}\sum_{\xi\,\in\,\Lambda+\mu}\left(E\left(\frac{\langle c,\xi+\frac{\mathrm{Im}\,(x)}{\tau_2}\rangle\sqrt{\tau_2}}{\sqrt{-Q(c)}}\right) - E\left(\frac{\langle c',\xi+\frac{\mathrm{Im}\,(x)}{\tau_2}\rangle\sqrt{\tau_2}}{\sqrt{-Q(c')}}\right)\right)e^{2\pi i\langle\xi,x\rangle}q^{Q(\xi)}\ , \tag{5.64}$$

where E denotes the incomplete error function

$$E(x) = 2\int_0^x e^{-\pi u^2}du\ . \tag{5.65}$$

Note, that if c or c' lie on the boundary of the Kähler cone, one does not have to smooth out the sign-function. Zwegers shows, that the non-holomorphic function $\widehat{\Theta}^{c,c'}_{\Lambda,\mu}(\tau,x)$ satisfies the correct transformation properties of a Jacobi form of weight $\frac{1}{2}r(\Lambda)$. Due to the non-holomorphic pieces it contains mock modular forms, that we want to identify in the following.

[15]Note, that this is not the symplectic product of D-brane charges defined before.

In order to separate the holomorphic part of $\widehat{\Theta}^{c,c'}_{\Lambda,\mu}(\tau,x)$ from its shadow we recall the following property of the incomplete error function

$$E(x) = \mathrm{sgn}(x)(1-\beta_{\frac{1}{2}}(x^2)) \ , \tag{5.66}$$

which enables us to split up $\widehat{\Theta}^{c,c'}_{\Lambda,\mu}(\tau,x)$ into pieces. Here, β_k is defined by

$$\beta_k(t) = \int_t^\infty u^{-k} e^{-\pi u} du \ . \tag{5.67}$$

Hence, one can write eq. (5.64) as

$$\widehat{\Theta}^{c,c'}_{\Lambda,\mu}(\tau,x) = \Theta^{c,c'}_{\Lambda,\mu}(\tau,x) - \Phi^c_\mu(\tau,x) + \Phi^{c'}_\mu(\tau,x) \ , \tag{5.68}$$

with

$$\Phi^c_\mu(\tau,x) = \frac{1}{2}\sum_{\xi\,\in\,\Lambda+\mu}\left[\mathrm{sgn}\langle\xi,c\rangle - E\left(\frac{\langle c,\xi+\frac{\mathrm{Im}\,(x)}{\tau_2}\rangle\sqrt{\tau_2}}{\sqrt{-Q(c)}}\right)\right] e^{2\pi i\langle\xi,x\rangle} q^{Q(\xi)} \ . \tag{5.69}$$

If c belongs to $\mathcal{C}(P)\cap\mathbb{Q}^{r(\Lambda)}$, we may write

$$\Phi^c_\mu(\tau,x) = R(\tau,x)\theta(\tau,x) \ , \tag{5.70}$$

where we decomposed the lattice sum into contributions along the direction of c and perpendicular to c given by R and θ, respectively. Hence, θ is a usual theta-series associated to the quadratic form $Q|\langle c\rangle^\perp$, i.e. of weight $(r(\Lambda)-1)/2$. R is the part which carries the non-holomorphicity. It transforms with a weight $\frac{1}{2}$ factor and therefore $\mathrm{Coeff}_{2\pi iy}(R(\tau,[P]y))$ is of weight $\frac{3}{2}$. Following the general idea of Zagier [60] we should encounter the $\beta_{\frac{3}{2}}$ function in the $2\pi iy$-coefficient of Φ. From the above expressions one can indeed prove the following identity [275]

$$\begin{aligned}\mathrm{Coeff}_{2\pi iy}\Phi^c_\mu(\tau,[P]y) = \quad & -\frac{1}{4\pi}\frac{\langle c,[P]\rangle}{\langle c,c\rangle}\sum_{\xi\,\in\,\Lambda+\mu}|\langle c,\xi\rangle|\,\beta_{\frac{3}{2}}\left(\frac{\tau_2\langle c,\xi\rangle^2}{-Q(c)}\right)q^{Q(\xi)} + \\ & \sum_{\xi\,\in\,\Lambda+\mu}\frac{\mathrm{sgn}(\langle c,\xi\rangle)}{2}\langle\xi_-,[P]\rangle\,\beta_{\frac{1}{2}}\left(\frac{\tau_2\langle c,\xi\rangle^2}{-Q(c)}\right)q^{Q(\xi)} \ .\end{aligned} \tag{5.71}$$

The first term in (5.71) has the expected form, however there is also a second contribution. One has to note that the second term vanishes whenever $[P]$ can be chosen to lie in the Kähler cone. For the examples we are going to consider this is possible and thus we are going to drop the second contribution in the following. For rational ruled surfaces of higher genus, this is however not possible and it would be interesting to understand the meaning of this term in more detail [275].

Taking the derivative with respect to $\bar\tau$ in order to obtain the shadow we arrive at the following final expression for surfaces with $[P]\in\mathcal{C}(P)$

$$\partial_{\bar\tau}\mathrm{Coeff}_{2\pi iy}\Phi^c_\mu(\tau,[P]y) = -\frac{\tau_2^{-\frac{3}{2}}}{8\pi i}\frac{c\cdot[P]}{\sqrt{-c^2}}\,(-1)^{4\mu^2}\,\theta^{(2)}_{\mu-\frac{[P]}{2},c}(\tau,0) \ , \tag{5.72}$$

where we define the Siegel-Narain theta-function $\theta^{(r)}_{\mu,c}(\tau,z)$ as in eq. (5.13). For more details on the transformation properties of the indefinite theta-functions we refer the reader to appendix H.

Now, these results can be used to compute the elliptic genus for two M5-branes wrapping the divisor P. Consider

$$f^{(2)}_{\mu,J}(\tau) = f_{\mu,J'}(\tau) - \frac{1}{2}\frac{\vartheta_{\Lambda^\perp}(\tau)^2}{\eta^{2\chi(P)}}\,\mathrm{Coeff}_{2\pi iy}\Theta^{J,J'}_{\Lambda,\mu}(\tau,[P]y)\ , \tag{5.73}$$

where $f_{\mu,J'}(\tau)$ is a holomorphic ambiguity given by the generating series in a reference chamber J', which we choose to lie at the boundary of the Kähler cone $J' \in \partial\mathcal{C}(P)$. In explicit computations it may be possible to choose J' such that the BPS numbers vanish. In general, however, such a vanishing chamber might not always exist, but since J' is at the boundary of the Kähler cone, $f_{\mu,J'}(\tau)$ has no influence on the modular transformation properties, nor on the holomorphic anomaly. We write the full M5-brane elliptic genus as

$$Z^{(2)}_P(\tau,z) = \sum_{\mu\,\in\,\Lambda^*/\Lambda} \hat{f}^{(2)}_{\mu,J}(\tau)\theta^{(2)}_{\mu,J}(\tau,z)\ , \tag{5.74}$$

where $\hat{f}^{(2)}_{\mu,J}$ denotes the modular completion as outlined above. We can show using Zwegers' results [61], that the M5-brane elliptic genus transforms like a Jacobi form of bi-weight $(-\frac{3}{2},\frac{1}{2})$.

Proof of holomorphic anomaly at rank two

Now, we are in position to prove the holomorphic anomaly at rank two for general surfaces P with $b_2^+(P)=1$ and $[P]\in\mathcal{C}(P)$. The holomorphic anomaly takes the following form

$$\mathcal{D}_2 Z^{(2)}_P(\tau,z) = \tau_2^{-3/2}\frac{1}{16\pi i}\frac{J\cdot[P]}{\sqrt{-J^2}}\left(Z^{(1)}_P(\tau,z)\right)^2\ , \tag{5.75}$$

where the derivative $\mathcal{D}_k$ is given as

$$\mathcal{D}_k = \partial_{\bar\tau} + \frac{i}{4\pi k}\partial^2_{z_+}\ , \tag{5.76}$$

and z_+ refers to the projection of z along a direction $J\in\mathcal{C}(P)$. For the proof, $\mathcal{D}_2 Z^{(2)}_P$ can be computed explicitly. Using (5.72) we obtain directly

$$\mathcal{D}_2 Z^{(2)}_P(\tau,z) = \tau_2^{-3/2}\frac{1}{16\pi i}\frac{J\cdot[P]}{\sqrt{-J^2}}\frac{\vartheta_{\Lambda^\perp}(\tau)^2}{\eta(\tau)^{2\chi}}\sum_{\mu\,\in\,\Lambda^*/\Lambda}(-1)^{4\mu^2}\theta^{(2)}_{\mu-\frac{[P]}{2},J}(\tau,0)\theta^{(2)}_{\mu,J}(\tau,z)\ . \tag{5.77}$$

Since the following identity among the theta-functions $\theta_{\mu,J}$ holds

$$\left(\theta^{(1)}_{0,J}(\tau,z)\right)^2 = \sum_{\mu\,\in\,\Lambda^*/\Lambda}(-1)^{4\mu^2}\theta^{(2)}_{\mu-\frac{[P]}{2},J}(\tau,0)\theta^{(2)}_{\mu,J}(\tau,z)\ , \tag{5.78}$$

we have proven the holomorphic anomaly equation at rank two.

5.4. Applications and extensions

In the following we want to apply the previous results to several selected examples. Before doing so, we explain two mathematical facts which will help to fix the ambiguity $f_{\mu,J'}(\tau)$, which are the blow-up formula and the vanishing lemma. After discussing the examples, we turn our attention to a possible extension to higher rank. This leads us to speculations about mock modularity of higher depth and wall-crossing having its origin in a meromorphic Jacobi form.

5.4.1. Blow-up formulae and vanishing chambers

There is a universal relation between the generating functions of stable sheaves on a surface P and on its blow-up $\tilde{P}$ [30, 276, 277, 278, 32]. Let P be a smooth projective surface and $\pi : \tilde{P} \to P$ the blow-up at a non-singular point with E the exceptional divisor of π. Let $J \in \mathcal{C}(P)$, r and μ such that $\gcd(r, r\mu \cdot J) = 1$. Then, the generating series $f^{(r)}_{\mu,J}(\tau; P)$ and $f^{(r)}_{\mu,J}(\tau; \tilde{P})$ are related by the blow-up formula

$$f^{(r)}_{\pi^*(\mu)-\frac{k}{r}E,\pi^*(J)}(\tau; \tilde{P}) = B_{r,k}(\tau) f^{(r)}_{\mu,J}(\tau; P) \ , \tag{5.79}$$

with $B_{r,k}$ given by

$$B_{r,k}(\tau) = \frac{1}{\eta^r(\tau)} \sum_{a \in \mathbb{Z}^{r-1}+\frac{k}{r}} q^{\sum_{i\leq j} a_i a_j} \ . \tag{5.80}$$

The second fact states that for a class of semi-stable sheaves on certain surfaces the moduli space of the sheaves is empty. We refer to this fact as the vanishing lemma [32]. For this let P be a rational ruled surface $\pi : P \to \mathbb{P}^1$ and J be the pullback of the class of a fiber of π. Picking a Chern class μ with $r\mu \cdot J$ odd, we have

$$\mathcal{M}((r, \mu, \Delta), J) = \emptyset \tag{5.81}$$

for all d and $r \geq 2$.

5.4.2. Applications to surfaces with $b_2^+ = 1$

The surfaces we are going to consider are $\mathbb{P}^2$, the Hirzebruch surfaces $\mathbb{F}_0$ and $\mathbb{F}_1$, and the del Pezzo surfaces $\mathcal{B}_8$ and $\mathcal{B}_9$.

Projective plane $\mathbb{P}^2$

The projective plane $\mathbb{P}^2$ has been discussed quite exhaustively in the literature. The rank one result was obtained by Göttsche [257]

$$Z^{(1)}_{\mathbb{P}^2} = \frac{\vartheta_1(-\bar{\tau}, -z)}{\eta^3(\tau)} \ . \tag{5.82}$$

The generating functions of the moduli space of rank two sheaves or $SO(3)$ instantons of super-Yang-Mills theory on $\mathbb{P}^2$ were written down by [279, 276, 30] and are given by

$$\begin{aligned} f_0(\tau) &= \sum_{n=0}^{\infty} \chi(\mathcal{M}((2,0,n),J))q^{n-\frac{1}{4}} = \frac{3h_0(\tau)}{\eta^6(\tau)} \ , \\ f_1(\tau) &= \sum_{n=0}^{\infty} \chi(\mathcal{M}((2,1,n),J))q^{n-\frac{1}{2}} = \frac{3h_1(\tau)}{\eta^6(\tau)} \ . \end{aligned} \tag{5.83}$$

Here, $h_j(\tau)$ are mock modular forms given by summing over Hurwitz class numbers $H(n)$

$$h_j(\tau) = \sum_{n=0}^{\infty} H(4n+3j)q^{n+\frac{3j}{4}}, \qquad (j=0,1) \ . \tag{5.84}$$

Their modular completion is denoted by $\hat{h}_j(\tau)$, where the shadows are given by $\vartheta_{3-j}(2\tau)$ [223]. Explicitly, we have

$$\partial_{\bar{\tau}} \hat{h}_j(\tau) = \frac{\tau_2^{-\frac{3}{2}}}{16\pi i} \vartheta_{3-j}(-2\bar{\tau}) \ . \tag{5.85}$$

Note, that these results are valid for all Kähler classes $J \in H^2(\mathbb{P}^2, \mathbb{Z})$ as there is no wall crossing in the Kähler moduli space of $\mathbb{P}^2$. This leads directly to the following elliptic genus of two M5-branes wrapping the $\mathbb{P}^2$ divisor

$$Z^{(2)}_{\mathbb{P}^2}(\tau,z) = \hat{f}_0(\tau)\vartheta_2(-2\bar{\tau},-2z) - \hat{f}_1(\tau)\vartheta_3(-2\bar{\tau},-2z) \ . \tag{5.86}$$

Denoting by $\mathcal{D}_2 = \partial_{\bar{\tau}} + \frac{i}{8\pi}\partial_z^2$ one finds the expected holomorphic anomaly equation at rank two, given by[16]

$$\mathcal{D}_2 \, Z^{(2)}_{\mathbb{P}^2}(\tau,z) = -\frac{3}{16\pi i}\tau_2^{-\frac{3}{2}} \left(Z^{(1)}_{\mathbb{P}^2}(\tau,z)\right)^2 \ , \tag{5.87}$$

which can be derived directly from the simple fact that

$$\vartheta_1(\tau,z)^2 = \vartheta_2(2\tau)\vartheta_3(2\tau,2z) - \vartheta_3(2\tau)\vartheta_2(2\tau,2z) \ . \tag{5.88}$$

Further note, that the q-expansion of f_0, eq. (5.83), has non-integer coefficients. It was explained in [238] that this is due to the fact that the generating series involves the fractional BPS invariants $\bar{\Omega}(\Gamma)$, which we encountered before.

Hirzebruch surface $\mathbb{F}_0$

Our next example is the Hirzebruch surface $P = \mathbb{F}_0$. We denote by F and B the fiber and the base $\mathbb{P}^1$'s respectively. For an embedding of this surface into a compact Calabi-Yau manifold one may consult the Appendix A.6.2. Let us choose $J = F + B$, $J' = B$ and Chern class $\mu = F/2$. The choice $\mu = B/2$ can be treated analogously and leads to the same results.

[16] This result has already been derived in [240].

The other sectors corresponding to $\mu = 0$ and $\mu = (F+B)/2$ require a knowledge of the holomorphic ambiguity at the boundary and will not be treated here. One obtains

$$\begin{aligned} f^{(2)}_{\mu,F+B}(\tau) &= \frac{1}{2\eta^8(\tau)}\mathrm{Coeff}_{2\pi i y}(\Theta^{F+B,B}_{\Lambda,\mu}(\tau,[P]y)) \\ &= q^{-\frac{1}{3}}\left(2q + 22q^2 + 146q^3 + 742q^4 + \dots\right), \end{aligned} \tag{5.89}$$

where we denote by μ either $B/2$ or $F/2$. This exactly reproduces the numbers obtained in [280].

We want to compute the shadow of the completion given by adding Φ^{F+B}_μ and Φ^B_μ to the indefinite theta-series $\Theta^{F+B,B}_{\Lambda,\mu}$. Since B is chosen at the boundary, Φ^B_μ vanishes for $\mu = F/2, B/2$. The only relevant contribution has a shadow proportional to $\vartheta_2(\tau)$. Precisely, we obtain

$$\partial_{\bar\tau} f^{(2)}_{\mu,F+B}(\tau) = -\tau_2^{-3/2}\frac{1}{4\pi i\sqrt{2}}\frac{\overline{\vartheta_2(\tau)}\vartheta_2(\tau)}{\eta^8(\tau)} \qquad \left(\mu = \frac{F}{2},\frac{B}{2}\right). \tag{5.90}$$

Hirzebruch surface $\mathbb{F}_1$

The next example is the Hirzebruch surface $\mathbb{F}_1$, which is a blow-up of $\mathbb{P}^2$. Again we denote by F and B the fiber and base $\mathbb{P}^1$'s. The $\mathbb{P}^2$ hyperplane is given by the pullback of $F+B$ and B is the exceptional divisor. This example is particularly nice, since we can check our results against the blow-up formula (5.79) or use the results known from $\mathbb{P}^2$ to write generating functions in sectors which are not accessible through the vanishing lemma. Notice, that the holomorphic expansions have been already discussed in ref. [239]. From the general discussion one sees that there are four different choices for the Chern class $\mu \in \{\frac{B}{2}, \frac{F+B}{2}, \frac{F}{2}, 0\}$.

First, we choose $J = F+B$, $J' = F$ and Chern class $\mu = B/2$. We then obtain

$$\begin{aligned} f^{(2)}_{\mu,F+B}(\tau) &= \frac{1}{2\eta^8(\tau)}\mathrm{Coeff}_{2\pi i y}(\Theta^{F+B,F}_{\Lambda,B}(\tau,[P]y)) \\ &= q^{-\frac{1}{12}}\left(-\frac{1}{2} - q + \frac{15}{2}q^2 + 91q^3 + 558q^4 + \dots\right). \end{aligned} \tag{5.91}$$

A check of this result against the blow-up formula (5.79) applied to $\mathbb{P}^2$ yields

$$\frac{3h_0(\tau)}{\eta^6(\tau)}\frac{\vartheta_2(2\tau)}{\eta^2(\tau)} = q^{-\frac{1}{12}}\left(-\frac{1}{2} - q + \frac{15}{2}q^2 + 91q^3 + 558q^4 + \dots\right) = f^{(2)}_{\mu,F+B}(\tau). \tag{5.92}$$

Further, we calculate the shadow by differentiating $\hat f^{(2)}$ with respect to $\bar\tau$

$$\partial_{\bar\tau}\hat f^{(2)}_{\mu,F+B}(\tau) = \frac{3}{16\pi i}\tau_2^{-3/2}\frac{\overline{\vartheta_3(2\tau)}\vartheta_2(2\tau)}{\eta^8(\tau)}, \tag{5.93}$$

which also is in accord with the blow-up formula. Note, that (5.91) has half-integer expansion coefficients, since $J = B+F$ lies on a wall for the Chern class $\mu = B/2$.

As a second case we choose $J = F + B$, $J' = F$ and Chern class $\mu = (F + B)/2$ and obtain

$$\begin{aligned} f^{(2)}_{\mu,F+B}(\tau) &= \frac{1}{2\eta^8(\tau)}\mathrm{Coeff}_{2\pi i y}(\Theta^{F+B,F}_{\Lambda,F+B}(\tau,[P]y)) \\ &= q^{-\frac{7}{12}}\left(q + 13q^2 + 93q^3 + 496q^4 + \dots\right) \ , \end{aligned} \tag{5.94}$$

which we again can check against the blow-up formula (5.79) for $\mathbb{P}^2$

$$\frac{3h_1(\tau)}{\eta^6(\tau)}\frac{\vartheta_3(2\tau)}{\eta^2(\tau)} = q^{-\frac{7}{12}}\left(q + 13q^2 + 93q^3 + 496q^4 + \dots\right) = f^{(2)}_{\mu,F+B}(\tau) \ . \tag{5.95}$$

Calculating the shadow yields

$$\partial_{\bar\tau}\hat f^{(2)}_{\mu,F+B}(\tau) = \frac{3}{16\pi i}\tau_2^{-3/2}\frac{\overline{\vartheta_2(2\tau)}\vartheta_3(2\tau)}{\eta^8(\tau)} \ , \tag{5.96}$$

which is also in accord with the blow-up formula.

The last two sectors $\mu = F/2, 0$ are not accessible via the vanishing lemma. However, using a blow-down to $\mathbb{P}^2$ we observe, that the above two cases reproduce correctly the two Chern classes in the cases of rank two sheaves on $\mathbb{P}^2$. Using the blow-up formulas once more we finally arrive at

$$\begin{aligned} f^{(2)}_{(0,0),J}(\tau) &= \frac{3h_0(\tau)}{\eta^6(\tau)}\frac{\vartheta_3(2\tau)}{\eta^2(\tau)} \ , \\ f^{(2)}_{(\frac{1}{2},0),J}(\tau) &= \frac{3h_1(\tau)}{\eta^6(\tau)}\frac{\vartheta_2(2\tau)}{\eta^2(\tau)} \ , \\ f^{(2)}_{(0,\frac{1}{2}),J}(\tau) &= \frac{3h_0(\tau)}{\eta^6(\tau)}\frac{\vartheta_2(2\tau)}{\eta^2(\tau)} \ , \\ f^{(2)}_{(\frac{1}{2},\frac{1}{2}),J}(\tau) &= \frac{3h_1(\tau)}{\eta^6(\tau)}\frac{\vartheta_3(2\tau)}{\eta^2(\tau)} \ , \end{aligned} \tag{5.97}$$

where $J = F + B$ and $\mu = (a, b) = aF + bB$. Note, that in the cases $f^{(2)}_{(0,0),J}$ and $f^{(2)}_{(0,\frac{1}{2}),J}$ the blow-up formula is not valid since we violate the gcd-condition, as $\pi_*\mu = 0$ in these cases. However, for rank two sheaves on $\mathbb{F}_1$ the blow-up formula seems to work anyway, since the generating series using the blow-up procedure and the indefinite theta-function description coincide for the Chern class $\mu = (0, \frac{1}{2})$.

Del Pezzo surface $\mathcal{B}_8$

As in [68] we embed the surface $\mathcal{B}_8$ in a certain free $\mathbb{Z}_5$ quotient[17] of the Fermat quintic $\tilde X = \{\sum_{i=1}^5 x_i^5 = 0\}$ in $\mathbb{P}^4$. The action of the group $G = \mathbb{Z}_5$ on the projective coordinates of the ambient space is given by $x_i \sim \omega^i x_i$, where $\omega = e^{2\pi i/5}$. For the hyperplane section, denoted P, we observe that $P^3 = 1$, as for the Fermat quintic the five points of intersection of three hyperplanes $\{x_i = x_j = x_k = 0\}$ are identified under the action of the group G. The

[17] The only freely acting group actions for the quintic are a $\mathbb{Z}_5^2$ and the above $\mathbb{Z}_5$.

Euler character of the hyperplane is given by $\chi(P) = 11$. It can be shown that the divisor P is rigid and has $b_2^+ = 1$. We observe that $H^2(P,\mathbb{Z}) = \mathbb{Z} \oplus (-E_8)$ as is explained in [68]. The elliptic genus of a single M5-brane is then fixed by the modular weights

$$Z_P^{(1)}(\tau,z) = \frac{E_4(\tau)}{\eta^{11}(\tau)}\,\vartheta_1(-\bar\tau,-z)\ . \tag{5.98}$$

The form of $Z_P^{(2)}$ can now be calculated as for $\mathbb{P}^2$ and is given by

$$Z_P^{(2)}(\tau,z) \sim \frac{E_4(\tau)^2}{\eta(\tau)^{22}}\,(\hat h_0(\tau)\,\vartheta_2(-2\bar\tau,-2z) - \hat h_1(\tau)\,\vartheta_3(-2\bar\tau,-2z))\ . \tag{5.99}$$

The holomorphic anomaly equation fulfilled by $Z_P^{(2)}(\tau,z)$ can be obtained as in the $\mathbb{P}^2$ case

$$\mathcal{D}_2\,Z_P^{(2)}(\tau,z) \sim \frac{\tau_2^{-\frac{3}{2}}}{16\pi i}\left(Z_P^{(1)}(\tau,z)\right)^2\ . \tag{5.100}$$

Del Pezzo surface $\mathcal{B}_9$, the $\frac{1}{2}$K3

We end our examples by returning and commenting on $\frac{1}{2}$K3 or $\mathcal{B}_9$ which was the example of section (5.2.3), as M5-branes wrapping on it give rise to the multiple E-strings. The $\mathcal{B}_9$ surface can be understood as a $\mathbb{P}^2$ blown up at nine points (see appendix A.6 for details) or a rational elliptic surface. This case is interesting as one can map via T-duality along the elliptic fibration the computation of the modified elliptic genus to the computation of the partition function of topological string theory on the same surface [211]. The middle dimensional cohomology lattice of $\mathcal{B}_9$ is given by $H^2(\mathcal{B}_9,\mathbb{Z}) = \Gamma^{1,1} \oplus E_8$ and the Euler number can be computed to $\chi(\mathcal{B}_9) = 12$. Modularity then fixes the form of the elliptic genus at rank one to

$$Z_{\mathcal{B}_9}^{(1)}(\tau,z) = \frac{E_4(\tau)}{\eta(\tau)^{12}}\,\theta_{0,J}^{(1)}(\tau,z)\ , \tag{5.101}$$

where $\theta_{0,J}^{(1)}(\tau,z)$ is the theta-function associated to the lattice $\Gamma^{1,1}$ with standard intersection form

$$(-d_{AB}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ . \tag{5.102}$$

Choosing the Kähler form $J = (R^{-2},1)^T$, where $(1,0)^T$ is the class of the elliptic fiber, one can show that

$$\theta_{0,J}^{(1)}(\tau,0) \to \frac{R}{\sqrt{\tau_2}} \quad \text{as} \quad R \to \infty\ . \tag{5.103}$$

In this limit of small elliptic fiber one recovers the results of sect. 5.2.3. The factor $E_4(\tau)$ is precisely the theta-function of the E_8 lattice. The results obtained from the anomaly for higher wrappings of refs. [224, 211] were proven mathematically for double wrapping in ref. [281]. In this analysis the Weyl group of the E_8 lattice was used to perform the theta-function decomposition.

5.4.3. Extensions to higher rank and speculations

In the following sections we want to discuss the extension of our results to higher rank. Partial results for rank three can be found already in the literature [280, 239, 254, 282, 283]. Thereafter, we discuss a possible generalization of mock modularity and speculate about a contour description which stems from a relation to a meromorphic Jacobi form.

Higher rank anomaly and mock modularity of higher depth

We want to focus on the holomorphic anomaly equation at general rank as conjectured in [211]. We recall that its form is given by

$$\mathcal{D}_r Z_P^{(r)}(\tau,z) \sim \sum_{n=1}^{r-1} n(r-n) Z_P^{(n)}(\tau,z) Z_P^{(r-n)}(\tau,z)\ , \tag{5.104}$$

where $Z_P^{(r)}(\tau,z)$ can be decomposed into Siegel-Narain theta-functions as described in section 5.2.2. One may thus ask the question what it implies for the functions $\hat{f}_{\mu,J}^{(r)}(\tau)$ for general r. In order to extract this information we want to compare the coefficients in the theta-decomposition on both sides of (5.104). For this we need a generalization of the identity (5.78). A computation shows that

$$\theta_{\nu,J}^{(n)}(\tau,z)\,\theta_{\lambda,J}^{(r-n)}(\tau,z) = \sum_{\mu\,\in\,\Lambda^*/\Lambda} c_{\nu\lambda}^{\mu}(\tau)\,\theta_{\mu,J}^{(r)}(\tau,z)\ , \tag{5.105}$$

where $c_{\nu\lambda}^{\mu}$ are Siegel-Narain theta-functions themselves given by

$$c_{\nu\lambda}^{\mu}(\tau) = \delta_g(\mu) \sum_{\xi\,\in\,\Lambda+\mu+\frac{g}{r}(\nu-\lambda)} \bar{q}^{-\frac{rn(r-n)}{2g^2}\xi_+^2} q^{\frac{rn(r-n)}{2g^2}\xi_-^2} \tag{5.106}$$

with $g = \gcd(n, r-n)$ and $\delta_g(\mu)$ yields one if $r\mu$ is divisible by g and vanishes otherwise. With this input one finds

$$\partial_{\bar{\tau}} \hat{f}_{\mu,J}^{(r)}(\tau) \sim \sum_{n=1}^{r-1} n(r-n) \sum_{\nu,\lambda\,\in\Lambda^*/\Lambda} \hat{f}_{\nu,J}^{(n)}(\tau) \hat{f}_{\lambda,J}^{(r-n)}(\tau) c_{\nu\lambda}^{\mu}(\tau)\ , \tag{5.107}$$

which sheds some light onto the question about the modular properties of generating functions at higher rank as follows.

The structure of eq. (5.107) indicates, that an appropriate description of the generating function $\hat{f}_{\mu,J}^{(r)}$ needs a generalization of the usual notion of mock modularity. This results from the fact, that on the right hand side of the anomaly equation (5.107), mock modular forms appear, such that the shadow of $\hat{f}_{\mu,J}^{(r)}$ is a mock modular form itself. Therefore, it is also subject to a holomorphic anomaly equation. This would lead to the notion of mock modularity of higher depth [62], similar to the case of almost holomorphic modular forms of higher depth. These are functions like $\widehat{E}_2(\tau)$ and powers thereof, which can be written as a polynomial in τ_2^{-1} with coefficients being holomorphic functions.

A further motivation for this comes from the observation that the generating functions $\hat{f}^{(r)}_{\mu,J}$ could be obtained from an indefinite theta-function as in the case of two M5-branes. The lattice, however, that is summed over in these higher rank indefinite theta-functions will be of higher signature. In the case of r M5-branes one would expect a signature $(r-1,(r-1)(r(\Lambda)-1))$ due to the $r-1$ relative D2-brane charges of the possible r decay products of D4-D2-D0 bound-states [237, 238]. However, a complete discussion of the modular properties of such functions and their relation to mock modular forms of depth is beyond the scope of this work.

The contour description

The elliptic genus of r M5-branes wrapping P is denoted by $Z^{(r)}_P(\tau,z)$, where we don't indicate any dependence of $Z^{(r)}_P$ on a Kähler class/ chamber $J \in \mathcal{C}(P)$. The basic assumption is that the elliptic genus does not depend on such a choice. We simply think about $Z^{(r)}_P$ as being a *meromorphic* Jacobi form, which has poles as a function of the elliptic variable z. We assume, that it is of bi-weight $(-\frac{3}{2},\frac{1}{2})$. In the following we want to exploit the implications of this statement.

It is known that a Jacobi form has an expansion into theta-functions with coefficients being modular forms. Since Zwegers [61], we also know that a meromorphic Jacobi form with one elliptic variable has a similar expansion, where the coefficients are mock modular. Using our Siegel-Narain theta-function $\theta^{(r)}_{\mu,J}(\tau,z)$, eq. (5.13), we conjecture the following expansion

$$Z^{(r)}_P(\tau,z) = \sum_{\mu\,\in\,\Lambda^*/\Lambda} f^{(r)}_{\mu,J}(\tau)\theta^{(r)}_{\mu,J}(\tau,z) + \mathrm{Res}\ , \tag{5.108}$$

with J a point in the Kähler cone which is related to a point $z_J \in \Lambda_{\mathbb{C}}$ where the decomposition is carried out. Note, that in eq. (5.108) the term "Res" should be given as a finite sum over the residues of $Z^{(r)}_P(\tau,z)$ in the fundamental domain $z_J + e\tau + e$ with $e = [0,1]^{r(\Lambda)}$.

Let's see how the dependence on J comes about. Doing a Fourier transform we can write

$$f^{(r)}_{\mu,J}(\tau) = (-1)^{r\mu\cdot[P]}\bar{q}^{\frac{r}{2}\mu_+^2}q^{-\frac{r}{2}\mu_-^2}\int_{\mathcal{C}_J} Z^{(r)}_P(\tau,z)e^{-2\pi i r(\mu+\frac{[P]}{2})\cdot z}dz\ , \tag{5.109}$$

where $\mathcal{C}_J$ is a contour which has to be specified since $Z^{(r)}_P$ is meromorphic. Due to the periodicity in the elliptic variable $\mathcal{C}_J$ can be given as $z_J + e$ for some point z_J. Now, suppose we have a parallelogram $\mathcal{P} = z_J + e z_{J'} + e$ and that there is a single pole of $Z^{(r)}_P$ inside $\mathcal{P}$, say at $z = z_0$. Then, we obtain by integrating over the boundary of $\mathcal{P}$

$$f^{(r)}_{\mu,J}(\tau) - f^{(r)}_{\mu,J'}(\tau) = 2\pi i\,\alpha_\mu(\tau)\operatorname*{Res}_{z=z_0}\left(Z^{(r)}_P(\tau,z)e^{-2\pi i r(\mu+\frac{[P]}{2})\cdot z}\right)\ , \tag{5.110}$$

where we abbreviate

$$\alpha_\mu(\tau) = (-1)^{r\mu\cdot[P]}\bar{q}^{\frac{r}{2}\mu_+^2}q^{-\frac{r}{2}\mu_-^2}\ . \tag{5.111}$$

That is, the coefficients of the Laurent expansion of the elliptic genus encode the jumping of the BPS numbers across walls of marginal stability and the walls are in one-to-one correspondence with the positions of the poles of $Z_P^{(r)}$. An analogous dependence on a contour of integration for wall-crossing of $\mathcal{N}=4$ dyons was introduced in refs. [284, 285].

Moreover, the shadow of $f_{\mu,J}^{(r)}$ should be determined in terms of the residues of $Z_P^{(r)}$, since a generalizations of the ideas of [61] should show, that it is contained in the factor "Res" of eq. (5.108). Thus, combining this result with the interpretation of eq. (5.110) one expects, that the shadow not only renders $f_{\mu,J}^{(r)}$ modular, but also encodes the decay of bound-states and hence knows about the jumping of BPS invariants across walls of marginal stability.

It is tempting to speculate even further. When comparing our results to the case of dyon state counting in $\mathcal{N}=4$ theories [235, 236] one might suspect that there is an analog of the Igusa cusp form ϕ_{10} in our setup. In the $\mathcal{N}=4$ dyon case there are meromorphic Jacobi forms, often denoted ψ_m, which are summed up to give ϕ_{10}. In analogy, it may be useful to introduce another parameter $\rho\in\mathcal{H}$ and to study the object

$$\phi_P^{-1}(\tau,\rho,z)=\sum_{r\geq 1} Z_P^{(r)}(\tau,z)e^{2\pi i r\rho}\ . \tag{5.112}$$

Summary and outlook of Part III

In this part of the thesis we investigated background dependence of theories that originate from r M5-branes wrapping a smooth (semi-)rigid divisor P in a Calabi-Yau three-fold background. Such divisors P have $b_2^+ = 1$ and (semi-)positive anti-canonical class. In this case the wrapped M5-brane can be studied locally in the Calabi-Yau manifold using an effective description of the M5-brane theory on $P \times T^2$ by a twisted $U(r)$ $\mathcal{N} = 4$ Super-Yang-Mills theory on P.

The main object of interest was the partition function $Z_P^{(r)}$ of the twisted gauge theory and its modular and holomorphic properties. This partition function can be related to the modified elliptic genus of the $\mathcal{N} = (0,4)$ sigma model description of the M5-brane. Using the spectral flow symmetry one establishes for all r a decomposition of the partition function into vector-valued modular forms $\hat{f}_{\mu,J}^{(r)}(\tau)$ w.r.t. the S-duality group of $\mathcal{N} = 4$ super-Yang-Mills theory and Siegel-Narain theta-functions $\theta_{\mu,J}^{(r)}(\tau, z)$.

Our main result is a rigorous proof of a holomorphic anomaly equation of the partition function valid for rank two on all P described above. The proof in section 5.3.3 relies on the large volume wall-crossing formula of Göttsche [32] for invariants associated to sheaves on P, which are related to integer BPS invariants. By summing the change of the invariants across all intermediate walls one can express the difference of the generating function of the invariants $f_{\mu,J}^{(r)}(\tau)$ in two arbitrary chambers J and J' in the Kähler cone in terms of an indefinite theta-function $\Theta_{\Lambda,\mu}^{J,J'}(\tau, z)$ [31]. This theta-function is regularized by cutting out the negative directions of the quadratic form on the homology lattice, a procedure which renders the result in general non-modular. The spoiled S-duality invariance can be regained following the work of Zwegers by smoothing out the cutting procedure with the non-holomorphic error function. The non-holomorphicity introduced by this procedure completes the mock modular forms $f_{\mu,J}^{(r)}(\tau)$ to non-holomorphic modular forms $\hat{f}_{\mu,J}^{(r)}(\tau)$. The non-holomorphicity of the Siegel-Narain theta-functions on the other is trivial since it is annihilated by the non-holomorphic heat operator. This allows to write a concise holomorphic anomaly equation for the partition function (5.75).

We checked this holomorphic anomaly equation and its implications for the counting of invariants of sheaves on $\mathbb{P}^2$, $\mathbb{F}_0$, $\mathbb{F}_1$ and $\mathcal{B}_8$ in section 5.4.2. The anomaly equation (5.75) is in particular compatible with the form of a holomorphic anomaly that has been conjectured in the context of E-strings on $\frac{1}{2}$K3 for all r and checked for certain classes using the duality to the genus zero topological string partition function [211]. Since the non-holomorphicity of the $\hat{f}_{\mu,J}^{(r)}(\tau)$ for $r > 1$ is related in an intriguing way to mock modularity and wall-crossing, we analyzed the decomposition for arbitrary rank and give a general form of the conjectured general anomaly equation at the level of the $\hat{f}_{\mu,J}^{(r)}(\tau)$ in equation (5.107), which indicates a theory of mock modular forms of higher depth [62]. The holomorphic limit of the $\hat{f}_{\mu,J}^{(r)}(\tau)$ yield generating functions for invariants associated to sheaves of rank r. However, it is in general difficult to provide boundary conditions, which fix the holomorphic ambiguity.

The wall-crossing formula of Göttsche, which induces in the steps described above the non-holomorphicity of the $\hat{f}^{(2)}_{\mu,J}(\tau)$, can be rederived using the Kontsevich-Soibelman wall-crossing formula, as we did in section 5.3.2. As the wall-crossing formula takes a primitive form at rank two, one can rewrite the generating function of BPS differences in terms of an indefinite theta-function. The Kontsevich-Soibelman formula can be used for arbitrary rank to determine the counting functions $f^{(r)}_{\mu,J}(\tau)$ for all sectors μ in all chambers, if it is known in one chamber for all μ, e.g. by a vanishing lemma or use of the blow-up formula. This was studied for rank 3 by [239], where it was also shown that the rank three wall-crossing formula is primitive. In general if the wall-crossing formula is primitive, the sum over walls induce lattice sums of signature $(r-1)(b_2^+, b_2^-)$ with similar regularization requirements as for the rank two case. It is an interesting question if the program of Zwegers to build modular objects can be extended to the higher rank situation and leads upon non-holomorphic modular completion to the conjectured form of holomorphic anomaly equation and a precise notion of the mock modular forms of higher depth.

The problem of providing boundary conditions at least in one chamber for the del Pezzo surfaces (except for the Hirzebruch surface $\mathbb{F}_0$) can in principle be solved by using the blow formula in both directions in connection with the wall-crossing formula before and after the blow-up. However, the blow-up formulae in the literature apply only if r and $c_1 \cdot J$ have no common divisor. This restriction forbids in general to provide boundary conditions for all sectors.

The higher genus information discussed in equation (5.33) gives finer information about the cohomology of moduli spaces of sheaves than its Euler number. Namely, an elliptic genus obtained by tracing over the right j_R^3 quantum numbers of the Lefshetz decomposition in the cohomology of the moduli space. On rigid surfaces it can be further refined to include the general Ω background parameters of Nekrasov [286], which capture the individual (j_L^2, j_R^3) quantum numbers [287]. For rank two such refined partition functions have been considered in [288] and it should be possible to extend the consideration above to the refined BPS numbers. Furthermore, the relation between D6-D2-D0 brane systems as counted by topological string theory and D4-D2-D0 brane systems associated to black hole state counting is the hallmark of the OSV conjecture [289], which has been intensively studied. Wall-crossing issues in combination with this conjecture have been studied in ref. [229] and more recently from an M-theory perspective for example in ref. [290]. It would be interesting to examine the implications of the anomaly equation in these contexts.

A conceptually very interesting but at this point more speculative approach is to consider the elliptic genus as a J independent meromorphic Jacobi form, as we did in section 5.4.3. As shown by Zwegers such meromorphic Jacobi forms have an expansion in theta-functions whose coefficients are mock modular forms, just as holomorphic Jacobi forms have an expansion in theta-functions with holomorphic modular forms as coefficients. This formalism relates the changes in the BPS numbers across walls of marginal stability to the different choices of the contour in the definition of $f^{(r)}_{\mu,J}(\tau)$ as a Fourier integral of $Z_P^{(r)}$, i.e. to the poles in $Z_P^{(r)}$, like in the $\mathcal{N} = 4$ case [291, 285].

Part IV

Appendices

Appendix A

A Aspects of toric geometry

Contents

In this appendix we want to give a short account of toric geometry and the related techniques used for the computations in the main text. There are a lot of good resources on toric varieties available, see e.g. [35, 292, 293, 294, 295, 296, 297] for physics oriented reviews and [59, 298, 27] for mathematical treatments.

Toric varieties are useful for two reasons, which we will both discuss in the following. Mathematically a large part of their geometry and topology can be encoded in certain combinatorial data, which makes it easier to do explicit computations. Physically toric varieties appear as the vacuum manifolds of certain two-dimensional supersymmetric gauge theories, called gauged linear σ-models (GLSM). This viewpoint allows for a physical construction of mirror manifolds [114] which parallels the mathematical constructions of dual polyhedra [109].

We will first review certain aspects of the mathematical theory of toric geometry and then turn to the physical GLSM description.

A.1. Toric varieties

Abstractly a toric variety X is a complex algebraic variety which contains an open torus $T = (\mathbb{C}^*)^r$ as a dense open subset together with an appropriate action of T on X. Thus a toric variety can be understood as a generalization of a (complex) torus, which is allowed to degenerate in particular ways. Examples of such varieties include (weighted) projective spaces $\mathbb{P}^n$ or orbifolds thereof.

There are two basic ways to encode the geometry of a toric variety. The first uses so-called fans and yields normal toric varieties, while the second way uses lattice points in polytopes and produces so-called projective toric varieties. We will start by constructing a toric variety via its fan.

A.1.1. The fan description

First we have to clarify what a fan is. For this let N be an integral lattice $N \simeq \mathbb{Z}^r$ and $N_{\mathbb{R}} = N \otimes \mathbb{R}$ its real extension. We will need the notion of a strongly convex rational polyhedral cone σ, which is defined by

$$\sigma = \left\{ \sum_{i=1}^{k} a_k v_k | a_k \geq 0 \right\} \subset N_{\mathbb{R}}, \qquad \text{with} \qquad \sigma \cap (-\sigma) = \{0\} \ , \tag{A.1}$$

for some vectors $v_k \in N$. A fan Σ is then given by a collection of cones σ such that each face of a cone in Σ is itself a cone in Σ and the intersection of two cones is a face of both cones.

In the following we will also need the dual of the lattice N, which is denoted by $M = \mathrm{Hom}(N, \mathbb{Z})$ and the associated natural pairing $\langle ., . \rangle$ between the lattices N and M.

The toric variety X can now be given as a quotient

$$X = \frac{(\mathbb{C}^n \setminus Z(\Sigma))}{G} \ , \tag{A.2}$$

where all the necessary data can be extracted from the fan Σ. This can be obtained as follows

- Let $\Sigma(1)$ be the set of one-dimensional cones of Σ then $n = |\Sigma(1)|$. To each $\rho \in \Sigma(1)$ one can associate a divisor D_ρ of X and a local coordinate $z_\rho \in \mathbb{C}$ such that $D_\rho = \{z_\rho = 0\}$. In the following we will choose an ordering of $\Sigma(1)$ and denote the corresponding coordinates as z_i.

- The set $Z(\Sigma)$ is called exceptional set and consists of all the loci where some or all the coordinates z_i are not allowed to vanish simultaneously. To extract it from the fan one takes the union of all subsets of $\Sigma(1)$ which do themselves not generate a cone.

- The group G by which one quotients needs a bit of explanation. First of all there is short exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow A_{r-1}(X) \longrightarrow 0 \ , \tag{A.3}$$

  where the first map is given by $m \mapsto \sum_i \langle m, v_i \rangle D_i$ and the second map is given by taking the linear equivalence class of the divisor. The group $A_{r-1}(X)$ is called the Chow group of X and is given by the Weil divisors modulo linear equivalence. For a toric variety it can be shown that the Chow group is actually generated by the torus invariant irreducible divisors [59]. By applying $\mathrm{Hom}(-, \mathbb{C}^*)$ and noting that $\mathrm{Hom}(M, \mathbb{C}^*) = N$ one obtains

$$0 \longrightarrow \mathrm{Hom}(A_{r-1}(X), \mathbb{C}^*) \longrightarrow \mathrm{Hom}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*) \longrightarrow \mathrm{Hom}(M, \mathbb{C}^*) \simeq N \longrightarrow 0 \ , \tag{A.4}$$

where the second map is given by $(Q_1, \ldots, Q_n) \mapsto \sum_i Q_i v_i$. The group G is now given by $\mathrm{Hom}(A_{r-1}(X), \mathbb{C}^*) \simeq \ker\big(\mathrm{Hom}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*) \longrightarrow \mathrm{Hom}(M, \mathbb{C}^*)\big)$. From this one can see that

$$G \simeq \mathbb{C}^{n-r} \times H, \qquad \text{where} \qquad H = \times_i \mathbb{Z}_{a_i} \ . \tag{A.5}$$

Here H denotes a possible torsion part. One can also give a more direct characterization of G by using local coordinates. The group G can then be written as

$$G = \{(t_i) \in (\mathbb{C}^*)^{\Sigma(1)} | \prod_{v_j \in \Sigma(1)} t_j^{\langle m, v_j \rangle} = 1 \quad \text{for all} \quad m \in M\} \tag{A.6}$$

or by decomposing the $v_j = \sum v_j^k e_k$ w.r.t. an appropriate basis e_k of M

$$G = \{(t_i) \in (\mathbb{C}^*)^{\Sigma(1)} | \prod_{v_j \in \Sigma(1)} t_j^{v_j^k} = 1\} \ . \tag{A.7}$$

As $G \subset \mathrm{Hom}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*)$ this induces also an action on the coordinates z_i. This action can be seen to be given by

$$(z_1, \ldots, z_n) \sim (\lambda_a^{Q_1^a} z_1, \ldots, \lambda_a^{Q_n^a} z_n), \quad \text{where} \qquad \lambda_a \in \mathbb{C}^* \tag{A.8}$$

with $\sum_j Q_j^a v_j = 0$. The numbers Q_j^a are sometimes called charges, because they are physical charges in the GLSM description.

We will also need two important properties which can be read off directly from the fan.

1. Compactness: The variety obtained from the above quotient construction is compact iff its fan Σ spans the whole lattice N.

2. Calabi-Yau condition: A section of the anti-canonical bundle of a smooth variety is given by $\Omega = \prod \frac{dz_i}{z_i}$ and thus one can show that the anti-canonical class is $K_X = \sum_i [D_i]$. Furthermore one can show that X is Calabi-Yau when the generators of $\Sigma(1)$ all lie in a common affine hyperplane. Thus any toric Calabi-Yau variety is necessarily non-compact. In terms of the charges Q_i^a the Calabi-Yau condition can be stated as $\sum_i Q_i^a = 0$.

Example: $\mathcal{O}(-3) \to \mathbb{P}^2$
The vertices for this geometry are given by

$$v_0 = (1,0,0), \quad v_1 = (1,1,0), \quad v_2 = (1,0,1), \quad v_3 = (1,-1,-1) \tag{A.9}$$

and they fulfill $-3v_0 + v_1 + v_2 + v_3 = 0$. From these vertices one can build two different fans

1. The top dimensional cones are given by $\{\langle 012\rangle, \langle 013\rangle, \langle 023\rangle\}$. In this case $Z(\Sigma)$ turns out to be given by $Z(\Sigma) = \{z_1 = z_2 = z_3 = 0\}$. From the relation between the vertices one finds for the action of G:

$$(z_0, z_1, z_2, z_3) \sim (\lambda^{-3} z_0, \lambda z_1, \lambda z_2, \lambda z_3) \ . \qquad \text{(A.10)}$$

   Note that the divisor D_0 is a $\mathbb{P}^2$. By studying the transition functions one can show that the total space is actually the bundle $\mathcal{O}(-3) \to \mathbb{P}^2$. As the charges Q_i sum up to zero in this case, one sees that this space is a Calabi-Yau space. Furthermore the fan does not span the whole lattice $N \simeq \mathbb{Z}^3$ and thus the variety is non-compact.

2. One can leave out the vertex v_0 and then is left with a single top dimensional cone $\{\langle 123\rangle\}$. The exceptional set is now empty and there are only three coordinates z_1, z_2, z_3. The action of G can now for example be seen from the kernel description.[1] One has

$$\left.\begin{aligned} t_1 t_2 t_3 &\overset{!}{=} 1 \\ t_1 t_3^{-1} &\overset{!}{=} 1 \\ t_2 t_3^{-1} &\overset{!}{=} 1 \end{aligned}\right\} \quad \Longrightarrow \quad G = \{(\lambda, \lambda, \lambda) : \lambda \in \mathbb{C}^* \wedge \lambda^3 = 1\} \simeq \mathbb{Z}_3 \ . \qquad \text{(A.11)}$$

   The toric variety is therefore given by an orbifold $X = \mathbb{C}^3 \setminus \mathbb{Z}_3$, which is singular at the origin. By adding the vertex v_0 and sub-dividing the fan one can smooth this singularity by inserting a $\mathbb{P}^2$ at the origin.

A.1.2. Divisors and intersections

A very nice feature of toric varieties is that the intersection theory is also encoded in the combinatorial data of the fan. Already from the exceptional set $Z(\Sigma)$ one sees that certain divisors might have no intersection at all, as the corresponding coordinates might not be allowed to vanish simultaneously. The leads to non-linear relations in the intersection ring of torus invariant divisors. A collection of all these relations is called the Stanley-Reisner ideal R. Together with the linear relations coming from the linear equivalence relations one can fix the intersection ring up to a normalization. The linear relations can be obtained from the fact that the principal divisors are given by $(m) = \sum \langle m, v_j \rangle D_j \sim 0$ and are trivial in the Chow group. For a maximal dimensional cone σ, spanned by vertices $v_{i_1}, \dots v_{i_r}$ the corresponding intersection number is given by

$$D_{i_1} \cdot \ldots \cdot D_{i_r} = \frac{1}{\mathrm{Vol}(\sigma)} \ , \qquad \text{(A.12)}$$

where $\mathrm{Vol}(\sigma)$ is the standard volume divided by $r!$.

It turns out that the charges Q_i^a also have an interpretation in terms of intersections. They are given by

$$Q_i^a = D_i \cdot C^a \ , \qquad \text{(A.13)}$$

[1] Alternatively one can also note that $v_1 + v_2 + v_3 = 0 \mod 3$.

where C^a denotes a particular basis of $H_2(X,\mathbb{Z})$. The C^a can be represented as complete intersection of $(r-1)$ torus-invariant divisors, e.g. $C^a = (D_{a_1} \cdot \ldots \cdot D_{a_{r-1}})$, where one can pin down the particular representation by using the intersection ring and the linear and non-linear relations among the divisors. By Poincaré duality the above can be used to compute Kähler volumes of cycles. The Kähler form can be expanded as $J = t^a \alpha_a$ where $\alpha_a \in H^2(X,\mathbb{Z})$ is Poincare dual to $D_a \in H_4(X,\mathbb{Z})$. Thus the volume of a curve for example given by

$$\mathrm{Vol}(C^a) = \int_{C^a} t^b \alpha_b = t^b (D_b \cdot C^a) = t^b Q^a_b \tag{A.14}$$

and analogous for higher-dimensional cycles.

As we have seen above toric Calabi-Yau spaces are necessarily non-compact. To build compact Calabi-Yau varieties one can look at zero sections of bundles in toric non-Calabi-Yau spaces. For this it is useful to give a representation of the global sections of line bundles on toric varieties. A general Weil-divisor is given by a formal linear combination of irreducible subvarieties in X, e.g. $D = \sum_i a_i D_i$. First we note that the torus invariant globally defined coordinates are given by $t_i = \prod_j z_j^{\langle e_i, v_j \rangle}$ for some basis $e_i \in M$. The torus action acts as

$$t_i \to \prod_j (\lambda_a^{Q^a_j} z_j)^{\langle e_i, v_j \rangle} = \lambda_a^{\sum_j \langle e_i, Q^a_j v_j \rangle} z_j^{\langle e_i, v_j \rangle} = t_i \ , \tag{A.15}$$

because $\sum_j Q^a_j v_j = 0$. A general coordinate is then given by the character $\chi^m(t) = \prod_i t_i^{m_i} = \prod_j z_j^{\langle m, v_j \rangle}$ for $m \in M$. One associates to every (Cartier) divisor D the so-called SUPPORTING POLYHEDRON

$$\Delta_D = \{ u \in M_{\mathbb{R}} \,|\, \langle u, v_i \rangle \geq -a_i \} \ , \tag{A.16}$$

which gives the holomorphic sections of the bundle $\mathcal{O}(D)$

$$\Gamma(X, \mathcal{O}(D)) = \bigoplus_{m \in \Delta_D \cap M} \mathbb{C} \cdot \chi^{m+a} \ , \tag{A.17}$$

where $a = (a_1, \dots)$. For the description of mirror symmetry one needs a second polyhedron, which encodes which kinds of blow-ups are consistent with a given toric polynomial. It is called the POLAR POLYHEDRON and given by

$$\Delta^* = \{ v \in N_{\mathbb{R}} \,|\, \langle m, v \rangle \geq -1, \quad \text{for all}\, w \in \Delta_K \} \ , \tag{A.18}$$

where $K = \sum_i D_i$ denotes the anti-canonical bundle. By the adjunction formula sections of the anti-canonical bundle lead to Calabi-Yau varieties and thus are of primary interest to us. If the vertices of $\Delta^* \subset N_{\mathbb{R}}$ lie again in N the polyhedron is called REFLEXIVE.

A.1.3. Hodge numbers

The Hodge numbers of a generic hypersurface in a toric ambient space are also encoded in the combinatorial data of the fan. Denoting by Δ a reflexive polyhedron, which defines the ambient toric variety $P_{\Sigma(\Delta)}$ (see the next section), it was shown in [109, 299, 300] that the Hodge numbers $h^{p,1}(X)$ of a generic anti-canonical hypersurface X of dimensions d can be computed as

$$\begin{aligned} h^{1,1}(X) &= l(\Delta^*) - (d+2) - \sum_{\operatorname{codim} S^*=1} l'(S^*) + \sum_{\operatorname{codim} S^*=2} l'(S^*) \cdot l'(S) , && \text{(A.19)} \\ h^{p,1}(X) &= \sum_{\operatorname{codim} S^*=p+1} l'(S) \cdot l'(S^*), \quad 1 < p < d-1 , && \text{(A.20)} \\ h^{d-1,1}(X) &= l(\Delta) - (d+2) - \sum_{\operatorname{codim} S=1} l'(S) + \sum_{\operatorname{codim} S=2} l'(S) \cdot l'(S^*) , && \text{(A.21)} \end{aligned}$$

where S denotes the faces of Δ and S^* the dual face of S. Furthermore l and l' are the number of integral points on a face and in the interior of a face, respectively.

A.2. Batyrev's mirror construction

In [109, 126] it was shown how to construct mirror pairs of compact Calabi-Yau manifold which are given by hypersurfaces in toric varieties. It turns out that this construction is equivalent to the one coming from the GLSM description, to be reviewed below.

For this the ambient toric variety W is given by a reflexive polytope Δ, which is the convex hull of p integral vertices, which all lie in a hyperplane at distance one from the origin. The toric variety is then given by the set of cones over the faces of Δ and one writes $W = P_{\Sigma(\Delta)}$. From the dual or polar polytope Δ^* one obtains analogously a toric variety $W^* = P_{\Sigma(\Delta^*)}$. [109] has shown that the hypersurfaces (Z, Z^*), given as the zero set of sections of the anti-canonical bundles of (W, W^*), are mirror to each other.

Example: The quintic in $\mathbb{P}^4$

The toric polyhedron Δ and its dual Δ^* describing the anti-canonical bundle over $\mathbb{P}^4$ are given by

$$\begin{array}{|c|lcrrrrr|} \hline \Delta & \nu_0 & = & (& 0 & 0 & 0 & 0 \;) \\ & \nu_1 & = & (& 0 & 0 & 0 & 1 \;) \\ & \nu_2 & = & (& 0 & 0 & 1 & 0 \;) \\ & \nu_3 & = & (& 0 & 1 & 0 & 0 \;) \\ & \nu_4 & = & (& 1 & 0 & 0 & 0 \;) \\ & \nu_5 & = & (& -1 & -1 & -1 & -1 \;) \\ \hline \end{array} , \quad \begin{array}{|c|lcrrrrr|} \hline \Delta^* & \tilde\nu_0 & = & (& 0 & 0 & 0 & 0 \;) \\ & \tilde\nu_1 & = & (& 4 & -1 & -1 & -1 \;) \\ & \tilde\nu_2 & = & (& -1 & 4 & -1 & -1 \;) \\ & \tilde\nu_3 & = & (& -1 & -1 & 4 & -1 \;) \\ & \tilde\nu_4 & = & (& -1 & -1 & -1 & 4 \;) \\ & \tilde\nu_5 & = & (& -1 & -1 & -1 & -1 \;) \\ \hline \end{array} , \quad \text{(A.22)}$$

where the vertices are understood to lie in a hyperplane at distance one from the origin. These polytopes lead to the following manifolds

$$\begin{aligned} \Delta: &\qquad \mathbb{P}^4 \cap \{x \in \mathbb{P}^4 : \sum_{\{\sum_i n_i = 5\}} a_{\{n_i\}} \prod_{i=1}^{5} x_i^{n_i} = 0\} \ , \\ \Delta^*: &\qquad \mathbb{P}^4/\mathbb{Z}_5^3 \cap \{x \in \mathbb{P}^4 : \sum_{i=1}^{5} b_i x_i^5 + b_0 \prod_{i=1}^{5} b_i = 0\} \ , \end{aligned} \tag{A.23}$$

which gives the known quintic and mirror quintic varieties.

A.3. The gauged linear σ-model

A gauged linear σ-model (GLSM) is an auxiliary two dimensional $\mathcal{N} = 2$ supersymmetric gauge theory [5]. Its usefulness stems from the fact that its vacuum manifold is given by a toric variety for an appropriate choice of the parameters. We just give a schematic overview and refer for further details to the literature [5, 301].

The gauge group of the GLSM is taken to be $G = U(1)^r$ and the field content is given by n chiral superfields Φ_i and r vector superfields V_a. The Lagrangian density can then be written in superspace as

$$\mathcal{L} = \int d^4\theta \left[\sum_i \bar{\Phi}_i e^{\sum_a Q_i^a V_a} \Phi_i - \frac{1}{4e^2} \sum_a \bar{\Sigma}_a \Sigma_a \right] + \mathrm{Re} \int d^2\tilde{\theta} \left[- \sum_a \tau^a \Sigma_a \right] + \mathrm{Re} \int d^2\theta \, W(\Phi) \ . \tag{A.24}$$

Here the parameters

$$\tau^a = r^a - i\theta^a \tag{A.25}$$

are linear combinations of the FayetIliopoulos parameters r^a and the theta angles θ^a for the a-th gauge group and $\Sigma_a = \bar{D}_+ D_- V_a$ is the twisted chiral field-strength. The last term denotes a possible superpotential, which is a holomorphic gauge-invariant polynomial. Integrating out the auxiliary fields one obtains the following bosonic potential

$$U = \frac{e^2}{2} \sum_{a=1}^{r} D_a^2 + \sum_{i=1}^{n} |F_i|^2 + \sum_{i=1}^{n} \sum_{a,b=1}^{r} Q_i^a Q_i^b \, \bar{\sigma}_a \sigma_b \, |\phi_i|^2 \ , \tag{A.26}$$

where the D- and F-terms are given by

$$D_a = \sum_{i=1}^{n} Q_i^a |\phi_i|^2 - r^a, \qquad F_i = \frac{\partial W}{\partial \phi_i}(\phi) \ . \tag{A.27}$$

The field σ_a denotes the bosonic component of the a-th vector multiplet in the above. For certain values of the parameters r^a the vacuum manifold $\mathcal{M}$ is then given by $\mathcal{M} = U^{-1}(0)/G$ subjected to the F-term equations.

A.3.1. The phases

Depending on the values of the parameters one may obtain different solutions to the equation $U = 0$. Thus the space of the Fayet-Iliopoulos parameters is divided into chambers. These chambers are called phases and the boundaries between them phase boundaries. One can encounter several possibilities

- Geometric phase: The gauge group is completely broken for any solution to $U = 0$ and all the modes transverse to $U = 0$ are massive. In this phase the theory is described by a non-linear σ-model with target space the vacuum manifold $\mathcal{M} = U^{-1}(0)/G$ subjected to the F-term equations $F_i = 0$. Geometrically the quotient can be understood as

$$\mathcal{M} = \frac{\mathbb{C}^n \setminus Z}{G_{\mathbb{C}}} \,, \tag{A.28}$$

  where Z denotes the deleted set, which consists of all points whose orbits under the gauge group do not pass through the solutions to the D-term equations. We thus recover the mathematical description of a smooth toric variety.

- Landau-Ginzburg orbifold phase: It might happen that the solutions to the D-term equations consists of isolated points and that all transverse modes are massless. In the limit of large r^a all the tangent modes are expected to decouple and one is left with an (orbifold) Landau-Ginzburg model with superpotential given by the F-term equations.

- Hybrid phases: In such a phase one might find a fibration of a geometric model over a Landau-Ginzburg model or reverse. In this sense these are mixtures of the two above phases.

The phase boundaries can be found by looking for the values of parameters where a non-compact Coulomb branch emerges, as the wave functions on these branches are not normalizable. This leads to the following parametrization of the phase boundaries [5, 301]:

$$\prod_{i=1}^{n} \left(\sum_{b=1}^{r} Q_i^b \sigma_b \right)^{Q_i^a} = e^{-t^a} \,. \tag{A.29}$$

This part of the singular locus coming from a non-compact Coulomb-branch is sometimes called Principal discriminant. There are also other singular loci coming from mixed Coulomb-Higgs branches [301].

A very useful pictorial description is given by the so-called secondary fan. It can be obtained by drawing the n vectors $(Q_i)^a$ in the t^a plane. This partitions the t^a-plane in the above mentioned chamber structure. The good parameters for a given chamber can then be obtained by demanding that the respective chamber is spanned by positive linear combinations of these parameters.

Example: $\mathcal{O}(-3) \to \mathbb{P}^2$

The gauge group is given by a single $U(1)$ and the field content by four chiral superfields with charges

$$\begin{array}{c|ccc} \phi_0 & \phi_1 & \phi_2 & \phi_3 \\ \hline -3 & 1 & 1 & 1 \end{array} .$$

The superpotential is $W = 0$. The bosonic potential then reads

$$U = \frac{e}{2}\left(-3|\phi_0|^2 + \sum_{i=1}^{3}|\phi_i|^2 - r\right)^2 + |\sigma|^2\left(9|\phi_0|^2 + \sum_{i=1}^{3}|\phi_i|^2\right) . \tag{A.30}$$

One finds the following phases:

1. $r \gg 0$: Not all ϕ_i are allowed to vanish simultaneously by the first term in (A.30) and thus $\sigma = 0$ by the second term. The deleted set is given by $Z = \{\phi_1 = \phi_2 = \phi_3 = 0\}$. Thus the vacuum manifold is given by

$$\mathcal{M} = \left\{-3|\phi_0|^2 + \sum_{i=1}^{3}|\phi_i|^2 = r\right\} / \left\{(\phi_0, \phi_1, \phi_2, \phi_3) \sim (\lambda^{-3}\phi_0, \lambda\phi_1, \lambda\phi_2, \lambda\phi_3)\right\} , \quad \lambda \in U(1) . \tag{A.31}$$

   This form can be shown to be equivalent to the toric description given before.

2. $r = 0$: At this point a solution with $\phi_0 = \phi_i = 0$ and unbroken gauge group is possible. The value of σ is unconstrained and thus one finds a non-compact field branch. This phase is called the Coulomb branch.

3. $r \ll 0$: In this case the deleted set is $Z = \{\phi_0 = 0\}$ and thus $\sigma = 0$ again. The gauge group is broken to a $\mathbb{Z}_3$ by the non-vanishing ϕ_0. Thus the vacuum manifold is given by

$$\mathcal{M} \simeq \{(\phi_1, \phi_2, \phi_3) \sim (\lambda\phi_1, \lambda\phi_2, \lambda\phi_3) \,|\, \lambda^3 = 1\} . \tag{A.32}$$

A.3.2. Calabi-Yau hypersurfaces

To describe Calabi-Yau hypersurfaces in toric varieties. One takes the superpotential to be $W = \phi_0 G(\phi_i)$ for a transverse and homogeneous polynomial G. The charges of the field ϕ_0 are given by $Q_0^a = -\sum_i Q_i^a$ in order to satisfy the Calabi-Yau condition $\sum_i Q_i^a = 0$.[2] The F-term contribution to (A.26) is then given by

$$\sum_{i=0}^{n}|F_i|^2 = |G(\phi_i)|^2 + |\phi_0|^2\sum_i|\partial_i G(\phi_i)|^2 . \tag{A.33}$$

Example: A Calabi-Yau hypersurface in $\mathbb{P}^4_{(1,1,2,2,2)}$

To describe the ambient weighted projective space the charges are chosen to be

[2] Which is the condition that the FI-parameters do not run under RG-flow and that the axial and vector $U(1)$ R-charges are non-anomalous.

ϕ_0	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6
-4	1	1	1	1	0	0
0	0	0	0	-2	1	1

.

The superpotential is given by $W = \phi_0 \, G(\phi_i)$ where G is a homogeneous polynomial of bidegree $(6,0)$. The D-terms are given by

$$D_1 = -4|\phi_0|^2 + |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 + |\phi_4|^2 - r_1 \, , \quad \text{(A.34)}$$

$$D_2 = -2|\phi_4|^2 + |\phi_5|^2 + |\phi_6|^2 - r_2 \, . \quad \text{(A.35)}$$

One finds then the following phases:

1. $r_1 \geq 0$, $r_2 \geq 0$: The deleted set in this phase is given by $Z_1 = \{\phi_1 = \phi_2 = \phi_3 = \phi_4 = 0\} \cup \{\phi_5 = \phi_6 = 0\}$ and the gauge group is completely broken for every solution of $U = 0$. Thus the target space of the GLSM is a the zero section $G = 0$ of the space $\mathcal{O}(-4,0)$ over the toric base specified by the charges.

2. $r_2 \leq 0$, $2r_1 + r_2 \geq 0$: The deleted set is given by $Z_2 = \{\phi_1 = \phi_2 = \phi_3 = \phi_5 = \phi_6 = 0\} \cap \{\phi_4 = 0\}$. One finds a unbroken $\mathbb{Z}_2$ gauge group along $\phi_5 = \phi_6 = 0$ and the total space is $\mathcal{O}(-8) \to \mathbb{P}^4_{(1,1,2,2,2)}$. Incorporating the F-term one finds in this phase the hypersurface $G = 0$ in the unresolved ambient space $\mathbb{P}^4_{(1,1,2,2,2)}$, which has a $\mathbb{Z}_2$ singularity.

3. $r_2 \leq 0$, $2r_1 + r_2 \leq 0$: The deleted set is $Z_3 = \{\phi_0 = 0\} \cup \{\phi_4 = 0\}$. The vevs of the fields ϕ_0 and ϕ_4 are then fixed up to an $\mathbb{Z}_4 \times \mathbb{Z}_8 \simeq \mathbb{Z}_8$ residual gauge symmetry at $\{\phi_1 = \phi_2 = \phi_3 = \phi_5 = \phi_6 = 0\}$. The total space is thus $\mathbb{C}^5/\mathbb{Z}_8$ subjected to $G = 0$. Therefore one obtains a LG-orbifold model with superpotential G.

4. $r_1 \leq 0$, $r_2 \geq 0$: The deleted set is $Z_4 = \{\phi_0 = 0\} \cup \{\phi_5 = \phi_6 = 0\}$ and the residual gauge symmetry is $\mathbb{Z}_4$. Thus one obtains a non-linear LG-orbifold with target $\mathcal{O}_{\mathbb{P}^1}(-2) \times \mathbb{C}^3/\mathbb{Z}_4$ and superpotential G. This phase is a hybrid phase.

The principal discriminant can be computed from (A.29) to

$$\Delta = -1 + 512 \, e^{-t_1} + 65536(-1 + 4 \, e^{-t_2}) \, e^{-2t_2} \, . \quad \text{(A.36)}$$

Another singularity is at $e^{-t_2} = 1/4$. At this point the model becomes birational equivalent to the one-parameter model $\mathbb{P}^5[2,4]$ [116].

A.4. Mirror symmetry for the GLSM

In [114] a physical proof of mirror symmetry was given for the models which are described by a GLSM. To a given GLSM a mirror LG-model is associated, which can be deformed to a non-linear σ-model by Kähler-deformations to which the B-model side is insensitive.

Given a GLSM with gauge group $U(1)^r$, chiral fields ϕ_i of charge Q_i^a and superpotential $W = P_\beta\, G_\beta(\phi_i)$, where the fields P_β have charges $-d_{\beta a}$, one can describe a complete intersection manifold in a toric variety. [114] associates to this a mirror model with mirror variables Y_i and a twisted chiral superpotential $\widetilde{W}$ given by

$$\widetilde{W} = \sum_{a=1}^{r} \Sigma_a \left(\sum_i Q_i^a Y_i - \sum_\beta d_{\beta a} P_\beta - t^a \right) + \sum_i e^{-Y_i} + \sum_\beta e^{-Y_\beta} \ . \tag{A.37}$$

The variables Y_i are periodic and satisfy

$$\mathrm{Re}\, Y_i = |\phi_i|^2 \ . \tag{A.38}$$

The periods of the compact mirror model can then be computed from the following ordinary integral

$$\Pi = \int \prod_a d\Sigma_a \int \prod_i dY_i \int \prod_\beta dY_\beta \left(\prod_\gamma \delta_\gamma \right) e^{-\widetilde{W}} \ , \tag{A.39}$$

where $\delta_\gamma = \sum_a d_{\gamma a} \Sigma_a$. For a non-compact model the factor $\left(\prod_\gamma \delta_\gamma \right)$ is absent [114].

Non-compact models

For non-compact toric models there exists a mirror description in terms of a non-compact Calabi-Yau manifold which is a fibration over a Riemann surface [38, 39, 164]. To see how this works one can use the expression for the periods of the non-compact mirror (A.39). We will specialize to the case of a single field ϕ_0 with charges $Q_0^a = -\sum_{i=1}^n Q_i^a$. The periods are then given by [302, 164]

$$\Pi = \int \prod_a d\Sigma_a \int \prod_i dY_i \int dY_0\, e^{-\left(\sum_a \Sigma_a \left[\sum_{i=0}^n Q_i^a Y_i - t^a \right] + \sum_{i=0}^n e^{-Y_i} \right)} \tag{A.40}$$

$$= \int \prod_{i=0}^n dY_i \prod_a \delta \left(\sum_{i=0}^n Q_i^a Y_i - t^a \right) e^{-\sum_{i=0}^n e^{-Y_i}} \ . \tag{A.41}$$

By a change of coordinates $x_i = e^{-Y_i}$ this can be written as

$$\Pi \sim \int \prod_{i=0}^n \frac{dx_i}{x_i} \prod_a \delta \left(\prod_{i=1}^n x_i^{Q_i^a} - e^{-t^a} x_0^{\sum_i Q_i} \right) e^{-\sum_{i=0}^n x_i} \tag{A.42}$$

$$= \int \prod_{i=1}^n \frac{d\hat{x}_i}{\hat{x}_i} \frac{dx_0}{x_0} \prod_a \delta \left(\prod_{i=1}^n \hat{x}_i^{Q_i^a} - e^{-t^a} \right) e^{-x_0 \left(1 + \sum_{i=1}^n \hat{x}_i \right)} \ , \tag{A.43}$$

where we introduced $\hat{x}_i = x_i / x_0$. By noting that $1/x_0 = \int du\, dv\, e^{x_0 uv}$ one can simplify further

$$\Pi = \int \prod_{i=1}^n \frac{d\hat{x}_i}{\hat{x}_i} du\, dv\, dx_0 \prod_a \delta \left(\prod_{i=1}^n \hat{x}_i^{Q_i^a} - e^{-t^a} \right) e^{-x_0 \left(1 + \sum_{i=1}^n \hat{x}_i - uv \right)} \tag{A.44}$$

$$= \int \prod_{i=1}^n \frac{d\hat{x}_i}{\hat{x}_i} du\, dv \prod_a \delta \left(\prod_{i=1}^n \hat{x}_i^{Q_i^a} - e^{-t^a} \right) \delta \left(1 + \sum_{i=1}^n \hat{x}_i - uv \right) \ . \tag{A.45}$$

This means that as far as the BPS-geometry is concerned the mirror model we started with is equivalent to the model

$$uv = F(\hat{x}_i), \qquad F(\hat{x}_i) = 1 + \sum_{i=1}^{n} \hat{x}_i, \qquad \text{subjected to} \quad \prod_{i=1}^{n} \hat{x}_i^{Q_i^a} = e^{-t^a} \ . \tag{A.46}$$

For a Calabi-Yau three-fold one has $n = 3 + r - 1$ and thus via the delta-functions one can eliminate r of the $\hat{x}_i$ leading to two independent variables $\hat{z}_1, \hat{z}_2$. Thus (A.46) gives the mirror geometry as a complete intersection $\{uv - F(\hat{z}_1, \hat{z}_2) = 0\} \subset \mathbb{C}^2 \times (\mathbb{C}^*)^2$ and $\{F(\hat{z}_1, \hat{z}_2) = 0\} \subset (\mathbb{C}^*)^2$ defines a Riemann surface. The holomorphic $(3,0)$-form can then be written as [38, 39, 164]

$$\Omega = \frac{\prod_{i=1}^{2} \frac{d\hat{z}_i}{\hat{z}_i} du\, dv}{dF} = \frac{\prod_{i=1}^{2} \frac{d\hat{z}_i}{\hat{z}_i} du}{u} \ . \tag{A.47}$$

Compact geometries: The Quintic in $\mathbb{P}^4$

In this section we show how the mirror of the quintic hypersurface in $\mathbb{P}^4$ can be recovered from the above [164]. The charges of the GLSM are given by

ϕ_0	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
-5	1	1	1	1	1

,

The periods are in this case given by

$$\Pi = \int d\Sigma \int \prod_i dY_i \int dY_0 \, (5\Sigma) \, e^{-\left(\Sigma\left[\sum_{i=0}^n Y_i - 5Y_0 - t\right] + \sum_{i=0}^n e^{-Y_i}\right)} \tag{A.48}$$

$$\overset{p.i.}{\sim} \int d\Sigma \int \prod_i dY_i \int dY_0 \, e^{-Y_0} \, e^{-\left(\Sigma\left[\sum_{i=0}^n Y_i - 5Y_0 - t\right] + \sum_{i=0}^n e^{-Y_i}\right)} \tag{A.49}$$

$$= \int \prod_i \frac{dy_i}{y_i} \int dy_0 \, \delta\left(\prod_{i=0}^n y_i - z\, y_0^5\right) e^{-\sum_{i=0}^n y_i} \tag{A.50}$$

$$\sim \int \prod_i dx_i \, e^{-\sum_{i=1}^n x_i^5 - z^{-1/5} \prod_{i=1}^5 x_i} \ , \tag{A.51}$$

where $z = e^{-t}$ and $e^{-Y_i} = y_i = x_i^5$ and $e^{-Y_0} = y_0 = z^{-1/5} x \prod_{i=1}^5 x_i$. One has to note that the coordinate change is not one-to-one and this leads to the Greene-Plesser orbifold group. It can be shown that the last expression for the periods is equivalent to the residue expression which comes from the large volume perspective [302].[3]

A.5. Toric branes

In [142, 303] a class of special Lagrangian branes in toric varieties was defined. These are called toric or Harvey-Lawson branes. Given a toric variety W as above with M constraints

$$\sum_i l_i^a |\phi_i|^2 = t^a \ , \tag{A.52}$$

[3] See also Appendix E.

where $a = 1 \ldots M$, one considers the following further N constraints

$$\sum_i \hat{l}_i^b |\phi_i|^2 = c^b, \qquad \sum_i v_b^i \theta^i = 0, \qquad \sum_i \hat{l}_i^a v_b^i = 0 \ , \tag{A.53}$$

where $b = M+1, \ldots N+M$ and $\phi_i = |\phi_i| e^{i\theta^i}$. This defines a Lagrangian cycle and in order to obtain a special Lagrangian cycle one furthermore has to require that $\sum_i \hat{l}_i^b = 0$.

To obtain the mirror of the above brane in the mirror variety W^* one can use the techniques of [109, 114]. This leads to the following constraint for the mirror manifold

$$\sum_i y_i = 0, \qquad \prod_i y_i^{l_i^a} = z_a, \qquad a = 1, \ldots, M \tag{A.54}$$

and the brane [142]

$$\prod_i y_i^{\hat{l}_i^a} = \hat{z}_a, \qquad \hat{z}_a = \epsilon_a \, e^{-c^a}, \qquad a = M+1, \ldots M+N \ , \tag{A.55}$$

where ϵ_a denotes a phase which is dual to the gauge configuration on the A-model side.

A.6. Divisors in Calabi-Yau spaces

In this section we recapitulate some facts about the geometry of smooth divisors P in a Calabi-Yau three-fold Z.

A.6.1. General facts about rigid divisors

We start with some facts about complex surfaces. The Riemann Roch formula relates the signature σ and arithmetic genus χ_0 to Chern class integrals

$$\sigma = \sum_i (b_{2i}^+ - b_{2i}^-) = \frac{1}{3} \int_P (c_1^2 - 2c_2), \qquad \chi_0 = \sum_i (-1)^i h_{i,0} = \frac{1}{12} \int_P (c_1^2 + c_2) \ . \tag{A.56}$$

Regarding the embedding one has the distinction whether P is very ample or not, i.e. if the line bundle $\mathcal{L}_P$ is generated by its global sections or not. In the former case P has $h^0(Z, \mathcal{L}_P) - 1$ deformations and there exists an embedding $j : Z \to \mathbb{P}^n_P$ so that $\mathcal{L}_P = j^*(\mathcal{O}(1))$, i.e. P can be described by some polynomial. This situation has been considered in [66], where the deformations and b^+, b^- have been given. Generically one has $h^{2,0}(P) = \frac{1}{2}(b_2^+ - 1)$, which is positive in the very ample case.

In this work we consider mainly rigid smooth divisors. In this case one has no deformations and locally the Calabi-Yau manifold can be written as the total space of the canonical line bundle $\mathcal{O}(K_P) \to P$ and the latter can be globalized to a elliptic fibration over P, see section A.6.2, for $P = \mathbb{F}_n$. In this case $\Lambda_P = \Lambda$, compare sec. 5.2.2.

As Z is a Calabi-Yau manifold and to allow no section, P has to have a positive $D^2 > 0$ anti-canonical divisor class $D = -K_P$, which is also required to be nef, i.e. $D.C \geq 0$ for any irreducible curve C. This defines a weak del Pezzo surface. If $D.C > 0$, then D is ample and P is a del Pezzo surface [304]. Del Pezzo surfaces are either $\mathcal{B}_n$, which are blow-ups of $\mathbb{P}^2$ in $n \leq 8$ points or $\mathbb{P}^1 \times \mathbb{P}^1$. We can also allow the Hirzebruch surface $\mathbb{F}_2$ which is weak del Pezzo.

As $h_{1,0} = h_{2,0} = 0$ one has $\chi_0(\mathcal{B}_n) = 1$ for all surfaces under consideration. As the Euler number $\chi(\mathcal{B}_n) = 3 + n$ one has by (A.56) that $\int_P c_1^2 = 9 - n$, which implies that $n = 9$ is the critical case for positive anti-canonical class, and $(b_2^+, b_2^-) = (1, n)$. The case $n = 9$ is called $\frac{1}{2}$K3. We include this semi-rigid situation.

In more detail the homology of $\mathcal{B}_n$ is generated by the hyperplane class h of $\mathbb{P}^2$ and the exceptional divisors of the blow-ups e_i, with the non-vanishing intersections $h^2 = 1 = -e_i^2$. The anti-canonical class is given by $-K_{\mathcal{B}_n} = 3h - \sum_{i=1}^n e_i$. Defining the lattice generated by this element in $H_2(P, \mathbb{Z})$ as $\mathbb{Z}_{K_{\mathcal{B}_n}}$ and $E_n^* = (\mathbb{Z}_{K_{\mathcal{B}_n}})^\perp$ one sees that E_1^* is trivial and E_n^* are the lattices of the Lie algebras $(A_1, A_1 \times A_2, A_4, D_5, E_6, E_7, E_8)$ for $n = 2, \ldots, 8$. The corresponding basis in terms of (h, e_i) is worked out in [304]. The homology lattice for B_9 is $\Gamma^{1,1} \oplus E_8$, where $\Gamma^{1,1}$ is the hyperbolic lattice with standard metric.

In order to study topological string theory in Calabi-Yau backgrounds realized in simple toric ambient spaces, one has to consider situations in which $\Lambda \subset \Lambda_P$, which is the case for the $\frac{1}{2}$K3 realized in the toric ambient space discussed in the next section.

A.6.2. Toric data of a CY containing Hirzebruch surfaces $\mathbb{F}_n$

Let Z be an elliptic fibration over $\mathbb{F}_n$ for $n = 0, 1, 2$ given by a generic section of the anti-canonical bundle of the ambient spaces specified by the following vertices

$$\begin{array}{c|cccccccc}
\Delta & \nu_0 & = & (& 0 & 0 & 0 & 0 &) \\
 & \nu_1 & = & (& 0 & 0 & 0 & 1 &) \\
 & \nu_2 & = & (& 0 & 0 & 1 & 0 &) \\
 & \nu_3 & = & (& 0 & 0 & -2 & -3 &) \\
 & \nu_4 & = & (& 0 & -1 & -2 & -3 &) \\
 & \nu_5 & = & (& 0 & 1 & -2 & -3 &) \\
 & \nu_6 & = & (& 1 & 0 & -2 & -3 &) \\
 & \nu_7 & = & (& -1 & -n & -2 & -3 &)
\end{array}$$

One finds large volume phases with the following Mori-vectors

$$\begin{array}{c|cccccccc|c}
 & D_0 & D_1 & D_2 & D_3 & D_4 & D_5 & D_6 & D_7 & \\
\hline
l^1 = & -6 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & C^1 \\
l^2 = & 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & C^2 \\
l^3 = & 0 & 0 & 0 & n-2 & -n & 0 & 1 & 1 & C^3
\end{array} .$$

We choose a basis $\{C^A, A = 1, 2, 3\}$ of $H_2(X, \mathbb{Z})$. Let K_A be a Poincaré dual basis of the Chow group of linearly independent divisors of X, i.e. $\int_{C^A} K_B = \delta_B^A$. The divisors $D_i = l_i^A K_A$ have intersections with the cycles C^A given by $D_i.C^A = l_i^A$. We have the following non-vanishing intersections of the divisors given by

$$K_1.K_2.K_3 = 1, \quad K_1.K_2^2 = n, \quad K_1^2.K_2 = n+2, \quad K_1^2.K_3 = 2, \quad K_1^3 = 8 \ . \tag{A.57}$$

The divisor giving the Hirzebruch surface inside the Calabi-Yau manifold corresponds to

$$[\mathbb{F}_n] = D_3 = K_1 - 2K_2 - (2-n)K_3 \ . \tag{A.58}$$

Thus, the metric on $H^2(\mathbb{F}_n, \mathbb{Z})$ coming from the intersections in the Calabi-Yau manifold is

$$(K_A.K_B.[\mathbb{F}_n]) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & n & 1 \\ 0 & 1 & 0 \end{pmatrix} . \tag{A.59}$$

Projecting out the direction corresponding to the elliptic fiber we reduce the problem to the Hirzebruch surface itself. We denote by $F = K_3$ and $B = K_2 - nK_3$ the class of the fiber and base, respectively. Thus, the canonical class reduces to $[\mathbb{F}_n] = -(2+n)F - 2B$. The intersection numbers are given as follows

$$\begin{pmatrix} F.F & F.B \\ B.F & B.B \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -n \end{pmatrix} . \tag{A.60}$$

Hence, the Kähler cone is spanned by the two vectors F and $2B + nF$, i.e.

$$\mathcal{C}(\mathbb{F}_n) = \{ J \in H^2(\mathbb{F}_n, \mathbb{R}) \,|\, J = t_1 F + t_2(2B + nF),\ t_1, t_2 > 0 \} \ . \tag{A.61}$$

For $n = 1$ the geometry admits also an embedding of a K3 and a $\mathcal{B}_9$ surface.

B

Appendix B

Central charges for B-branes

In this appendix we give a short overview of the computation of central charges of B-branes. Near a large volume point of a Calabi-Yau d-fold X mirror symmetry predicts the following expression for the central charges of a B-brane described by a coherent sheaf $\mathcal{E}$ of rank r [305, 306]

$$\vec{Q}(\mathcal{E}) \cdot \vec{\Pi}(t) = Z(t) = -\int_X e^{-J}\mathrm{ch}(\mathcal{E})\sqrt{\mathrm{Td}(X)}, \qquad \text{for} \quad t \to \infty \ , \tag{B.1}$$

where

$$\vec{Q}(\mathcal{E}) = (Q_0, Q_2, Q_4, \ldots, Q_{2d}) \ , \tag{B.2}$$

$$\vec{\Pi}(t)^T = (\Pi_0(t), \Pi_2(t), \Pi_4(t), \ldots, \Pi_{2d}(t)) \ . \tag{B.3}$$

Using the expressions for the Todd-class and the Chern-character

$$\begin{aligned}
\mathrm{Td}(X) &= 1 + \frac{c_1}{2} + \frac{1}{24}c_1c_2 + \frac{1}{12}\left(c_1^2 + c_2\right) + \frac{1}{720}\left(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4\right) + \ldots \ , \\
\mathrm{ch}(\mathcal{E}) &= r + c_1(\mathcal{E}) + \frac{c_1(\mathcal{E})^2}{2} + c_2(\mathcal{E}) + \frac{c_1(\mathcal{E})^3}{6} - \frac{c_1(\mathcal{E})c_2(\mathcal{E})}{2} + \frac{c_3(\mathcal{E})}{2} + \ldots \ ,
\end{aligned}$$

together with the Grothendieck-Hirzebruch-Riemann-Roch-theorem for a divisor $i : D \to X$

$$i_* \left(\mathrm{ch}(\mathcal{E})\mathrm{Td}(D)\right) = \mathrm{ch}(i_*\mathcal{E})\mathrm{Td}(X) \ , \tag{B.4}$$

one finds the following central charges:

- $d = 3$ [306, 129]:

$$Z_6 = \frac{J^3}{3!} + \frac{J \cdot c_2}{24}, \qquad Z_4 = -\frac{J^2 \cdot [D]}{2} + \frac{1}{2}J \cdot i_*c_1(D) \ . \tag{B.5}$$

- $d = 4$:[1]

$$\begin{aligned}
Z_8 &= -\frac{J^4}{4!} - \frac{J^2 \cdot c_2}{48} - \frac{7c_2^2}{5760} + \frac{c_4}{1440} \ , \\
Z_6 &= \frac{J^3 \cdot [D]}{3!} - \frac{J^2 \cdot i_*c_1(D)}{4} + J \cdot \left(\frac{i_*c_2(D)}{12} + \frac{[D] \cdot c_2}{24}\right) - \\
&\quad \frac{1}{12}i_*c_1(D) \cdot i_*c_1(D) - \frac{1}{48}i_*c_1(D) \cdot c_2 \ .
\end{aligned} \tag{B.6}$$

[1]Implicitly given in [123].

Appendix C

Analytic continuation via fractional branes

In this appendix we apply the techniques of [129] to the example of the sextic Calabi-Yau four-fold discussed in section 2.6.2. For the notation and further explanations of the method we refer to [129].
The target space geometry Y is given by the zero locus of a generic section of the bundle $\mathcal{O}(-6) \to \mathbb{P}^5$. The basis $\{R_a\}$ of geometric bundles is therefore

$$\{R_a\} = \{\mathcal{O}(-5), \mathcal{O}(-4), \mathcal{O}(-3), \mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}\} \ . \tag{C.1}$$

The inverse of the index $\chi^{\mathbb{Z}_6}_{ab}$ which describes the intersections of the basis of fractional branes $\{S^a\}$ is given by

$$\chi^{\mathbb{Z}_6,ab} = \begin{pmatrix} 1 & -6 & 15 & -20 & 15 & -6 \\ 0 & 1 & -6 & 15 & -20 & 15 \\ 0 & 0 & 1 & -6 & 15 & -20 \\ 0 & 0 & 0 & 1 & -6 & 15 \\ 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{C.2}$$

Using the central charge formula (B.1) and the following form of the period vector[1]

$$\vec{\Pi}(t) = \begin{pmatrix} 1 \\ t \\ 3t^2 + 3t - \frac{15}{4} \\ t^3 + \frac{3t^2}{2} + \frac{3t}{4} - \frac{15}{8} \\ -\frac{t^4}{4} - \frac{15t^2}{8} + \frac{75}{64} \end{pmatrix} \ , \tag{C.3}$$

[1] Compare to (2.166), noting that $\frac{3375\zeta(4)}{32\pi^4} = \frac{75}{64} = -\frac{7c_2^2}{5760} + \frac{c_4}{1440} + 1$ in view of (B.6). Note also that the 'loop'-terms proportional to $\zeta(3)$ in (2.166) cannot be obtained from the open string picture of [129].

leads to the following matrix Q_S of D-brane charges for the bundles $V^a = S^a|_Y$

$$\begin{array}{c|ccccc} & Q_0 & Q_2 & Q_4 & Q_6 & Q_8 \\ \hline V^1 & -5 & -4 & 0 & -1 & 1 \\ V^2 & 10 & 13 & -1 & 4 & -5 \\ V^3 & -10 & -21 & 3 & -6 & 10 \\ V^4 & 5 & 19 & -3 & 4 & -10 \\ V^5 & -1 & -7 & 1 & -1 & 5 \\ V^6 & 1 & 0 & 0 & 0 & -1 \end{array} \tag{C.4}$$

where $\vec{Q}(V^6) = -\sum_{a=1}^{5} \vec{Q}(V^a)$. The monodromy matrix M_0 around the orbifold point expressed in the large volume basis (C.3) fulfills

$$Q_S \cdot M_0 = g \cdot Q_S \ , \tag{C.5}$$

where $g = h + \delta_{6,1}$ and h is the 6×6 shift matrix with unit entries above the diagonal and zeroes otherwise [129]. Choosing the bundles V^a, $a = 1, \dots 5$ as a basis near the LG-point the analytic continuation matrix m between the basis $\vec{\Pi}$ (C.3) and $\omega_a = Z(V^a)$ can be obtained from

$$\Pi_a = m_{ab} \cdot \omega_a, \qquad Q_S \cdot m = D \cdot (1\!\!1_{6\times 6} - g) \cdot \begin{pmatrix} 1\!\!1_{5\times 5} \\ R \end{pmatrix} , \tag{C.6}$$

where $R = (-1,-1,-1,-1,-1,-1)$ and $D_{ij} = -\delta_{7-i,j}$, with $i, j = 1, \dots 6$. Explicitly the matrix m is given by

$$m = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & -\frac{2}{3} & -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{6} \\ \frac{5}{2} & 0 & -\frac{3}{2} & -2 & -\frac{3}{2} \\ \frac{7}{3} & \frac{2}{3} & 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix} . \tag{C.7}$$

Using the relation $M_1 = M_\infty \cdot M_0^{-1}$ between the conifold monodromy M_1 and the large volume monodromy M_∞, obtained from $t \to t+1$, one gets the monodromy matrices (2.163) for the large volume basis (C.3)

$$M_\infty = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 6 & 6 & 1 & 0 & 0 \\ 7 & 3 & 1 & 1 & 0 \\ -4 & -4 & 0 & -1 & 1 \end{pmatrix}, \ M_0 = \begin{pmatrix} -4 & -4 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 6 & 6 & 1 & 0 & 0 \\ 7 & 3 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \ M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.8}$$

This reproduces the results from mirror symmetry and analytic continuation given in section 2.6.2.

D

Appendix D

Four-fold computation in the quintic example

In the following, we hand in some technical details of the computation of the superpotential (D.9) and the Kähler potential (3.103) from the F-theory four-fold X_4 presented in section 3.8 and substantiate some of the claims made there. This includes an explicit illustration of the relation between the four-fold flux superpotential (D.9) and the three-fold superpotential (3.96) from RR and NS fluxes in the weak coupling limit as well as the derivation of the Kähler potential (3.103). To avoid excessive repetitions, we refer to [75, 99, 100] for the details on mirror symmetry for Calabi-Yau four-folds and to [178] for the application of toric geometry to analyze geometry of the relevant F-theory four-folds.

The A-model four-fold X_4 is defined by the polyhedron Δ with vertices given in Table 3.1 and (3.95). We consider a phase of the Kähler cone described by the charge vectors given in the text, which we reproduce here for convenience:

$$\begin{array}{ccccccccccc} \tilde{l}^1 & (& -4 & 0 & 1 & 1 & 1 & 1 & -1 & 1 & 0 &) \\ \tilde{l}^2 & (& -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 &) \\ \tilde{l}^3 & (& 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 &) \end{array} . \tag{D.1}$$

The Kähler form is $J = \sum_a t_a J_a$, where J_a, $a = 1, 2, 3$ denotes the basis of $H^{1,1}(X_4)$ dual to the Mori cone defined by (D.1). The basic topological data are the intersections

$$K_{abcd} = \int_{X_4} J_a \wedge J_b \wedge J_c \wedge J_d \ ,$$

which can be concisely summarized in terms of the generating function[1]

$$\begin{aligned} \mathcal{F}_4 &= \frac{1}{4!}\int_{X_4} J^4 = \frac{1}{4!}\sum_{a,b,c,d} K_{\alpha\beta\gamma\delta} t^\alpha t^\beta t^\gamma t^\delta \\ &= \frac{1}{4}t_1^4 + \frac{5}{3}t_1^3(t_2 + \frac{1}{2}t_3) + \frac{5}{2}t_1^2 t_2(t_2 + t_3) + \frac{5}{3}t_1 t_2^2(t_2 + \frac{3}{2}t_3) + \frac{5}{12}t_2^3(t_2 + 2t_3) \ . \end{aligned} \tag{D.2}$$

[1] Details on the computation of the following data from toric geometry and many sample computations for three-fold fibered four-folds can be found in [99, 100].

From these intersections one obtains the following topological invariants of X_4[2]

$$\begin{aligned} \chi &= \int_{X_4} c_4 = 1860 = 12 \bmod 24\,, \qquad\qquad R_4 = \int_{X_4} c_2^2 = 1100\ , \\ R_2 &= \int_{X_4} J\wedge J\wedge c_2 = 90J_1^2+110J_2^2+220J_1J_2+100J_3(J_1+J_2)\,, \\ R_3 &= \int_{X_4} J\wedge c_3 = -330J_1-410J_2-200J_3\,, \end{aligned} \tag{D.3}$$

where c_k denote the Chern classes of X_4. The independent Hodge numbers $h^{1,1}=3$, $h^{1,2}=0$, $h^{1,3}=299$ can be read off from the toric polyhedra [99, 100] and fix $h^{2,2}=12+\frac{2}{3}\chi+2h^{1,2}=1252$; we refer to [307] for more details and the meaning of the mod 24 condition on χ.

After a linear change of coordinates, the form of the generating function $\mathcal{F}_4$ simplifies to

$$\mathcal{F}_4 = S\ \frac{5}{6}t^3\ +\ \left(\frac{5}{12}t^4-\frac{1}{6}t_1^4\right)\ , \tag{D.4}$$

where $t_3=S$ is the Kähler modulus of the $\mathbb{P}^1$ base, $t_1+t_2=t$ is the modulus of the generic quintic three-fold fiber and $t_2=\hat{t}$ will be related to the open string modulus. Note that the fibration structure becomes explicit in the coordinates (S,t) and that the leading term in large S contains the intersection form on the quintic fiber Z. To simplify some of the following expressions, we use the linear combinations

$$t_1'=t=t_1+t_2, \qquad t_2'=\hat{t}-t=-t_1, \qquad t_3'=S=t_3\ .$$

To compute the superpotential and the Kähler potential we have to determine the integral periods of the elements of the vertical subspace $H^{2q}_{\rm ver}(X_4)$ generated by wedge products of the elements $J_a\in H^2(X_4)$ over topological cycles in $H_{2q}(X_4,\mathbb{Z})$, for $q=0,...,4$. Except for $q=2$, there is a canonical basis for $H_{2q}(X_4,\mathbb{Z})$ given by the class of a point, the class of X_4, the divisors dual to the generators J_a and the curves dual to these divisors, respectively.[3] In an expansion near the large volume point $\mathrm{Im}\,t_a\to\infty\ \forall a$, the leading part of the periods, denoted by $\tilde{\Pi}_{q,.}$, is, up to a sign, the Kähler volume of the cycle as measured by the volume form $\frac{1}{k!}J^k$, explicitly

$$\tilde{\Pi}_0=1, \qquad \tilde{\Pi}_{1,a}=t_a', \qquad \tilde{\Pi}_{3,a}=-\frac{\partial}{\partial t_a'}\mathcal{F}_4, \qquad \tilde{\Pi}_4=\mathcal{F}_4\ . \tag{D.5}$$

As for the mid-dimensional part, $k=2$, we choose basis elements defined by intersections of the toric divisors $D_i=\{x_i=0\}$

$$\gamma_1=D_1\cap D_2, \qquad \gamma_2=D_2\cap D_8, \qquad \gamma_3=D_2\cap D_6\ .$$

[2] Contrary to first appearances, our choice of X_4 is not at all related to a preference for a particular soccer club.

[3] To be precise, some of these basis elements may actually be integral multiples of the generators of $H_{2q}(X_4,\mathbb{Z})$ on the hypersurface.

with intersection form and inverse

$$(\tilde{\eta})_{kl} = \gamma_k \cap \gamma_l = \begin{pmatrix} -10 & 5 & 0 \\ 5 & 0 & 0 \\ 0 & 0 & -4 \end{pmatrix}, \qquad \left(\tilde{\eta}^{-1}\right)^{kl} = \begin{pmatrix} 0 & \frac{1}{5} & 0 \\ \frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 0 & -\frac{1}{4} \end{pmatrix}. \tag{D.6}$$

The leading parts of the $q = 2$ periods are then

$$\tilde{\Pi}_{2,1} = 5t'_1 t'_3\,, \qquad \tilde{\Pi}_{2,2} = \frac{5}{2} t'^2_1\,, \qquad \tilde{\Pi}_{2,3} = 2t'^2_2\,. \tag{D.7}$$

The subleading terms of the exact periods $\Pi_{q,\cdot}$, correcting the leading parts $\tilde{\Pi}_{q,\cdot}$ away from the large $\mathrm{Im}\, t$ limit, come in two varieties. Firstly polynomial corrections in the t_a of lower degree, which involve the topological invariants (D.3).[4] For simplicity, we ignore these terms in the following, as they can be interesting for other applications[5] but do not matter for the present purpose.

Secondly, there are instanton corrections $\sim q_a = \exp 2\pi i t_a$, which will be important in the match between the open string superpotential (3.96) and the F-theory superpotential (D.9) claimed in section 3.8. These instanton corrections can be computed by mirror symmetry of the four-folds (X_4, X_4^*) as in [75, 99, 100] which can be summarized as follows. To determine the instanton corrections, we may simply replace the leading periods $\tilde{\Pi}_q$ in (D.5), (D.7) with the exact solutions to the Picard-Fuchs equations in (3.64) with the same leading part when rewritten in t_a coordinates, using the mirror map (3.67).[6] The result for the exact periods $\Pi_{q,k} = \tilde{\Pi}_{q,k} + \mathcal{O}(q_a)$ is of the form

$$\begin{aligned}
&\Pi_0 = 1, \\
&\Pi_{1,1} = t'_1, && \Pi_{1,2} = t'_2, && \Pi_{1,3} = \boldsymbol{S}\cdot 1, \\
&\Pi_{2,1} = \boldsymbol{S}\cdot, 5t'_1 + \pi_{2,1}, && \Pi_{2,2} = -\tilde{F}_t, && \Pi_{2,3} = -\tilde{W}, \\
&\Pi_{3,1} = \boldsymbol{S}\cdot, \tilde{F}_t + \pi_{3,1}, && \Pi_{3,2} = \tilde{T}, && \Pi_{3,3} = -\mathrm{F}_0, \\
&\Pi_4 = \boldsymbol{S}\cdot, \tilde{F}_0 + \pi_4\ ,
\end{aligned} \tag{D.8}$$

where we have again indicated the fibration structure by explicitly indicating powers in the base volume $S = t_3 = t'_3$. The other functions $\pi_{q,k}$ denote subleading corrections whose precise form does not matter for the moment.

The leading terms in weak coupling limit $\mathrm{Im}\, S \to \infty$ are characterized by the Kähler moduli $t, \hat{t}$ and the four functions F_t, F_0 and W, T. The first two functions are equal to the periods $F_t = \partial_t \mathcal{F}(t)$ and $F_0 = 2\mathcal{F}(t) - t\partial_t \mathcal{F}(t)$ of the quintic three-fold up to instanton corrections in

[4] The coefficients should be determined by a computation of the anomalous charges of wrapped D-branes, similarly as in [129].

[5] See e.g. [308].

[6] A more formal account of this relation between A- and B-models is elaborated on in [99, 100], using Frobenius method as in [309, 110].

$q_S = \exp(2\pi i S)$, where $\mathcal{F}(t)$ denotes the exact instanton corrected prepotential of the quintic

$$\tilde{F}_t = F_t(t) + \mathcal{O}(q_S) = -\frac{5}{2}t^2 + ..., \qquad \tilde{F}_0 = F_0(t) + \mathcal{O}(q_S) = \frac{5}{6}t^3 + ... \ .$$

Similarly, the remaining functions $\tilde{W}$, $\tilde{T}$ agree with the domain wall tension and the top period computed in the main text, up to instanton corrections in S

$$\mathcal{W} = W + \mathcal{O}(q_S) = -2t_2'^2 + ... \ , \qquad \tilde{T} = T + \mathcal{O}(q_S) = \frac{2}{3}t_2'^3 + ... \ .$$

Indeed, the function W is of the form

$$W = -2t_1^2 + \frac{1}{4\pi^2}\sum_{k=1}^{\infty}\sum_{n_a\geq 0}\frac{1}{k^2}\,N_{n_1,n_2}(q_1^{n_1}q_2^{n_2})^k \ ,$$

where the instanton numbers N_{n_1,n_2} and the classical term are exactly the same as in the superpotential $\mathcal{T}^{(1)}$ in (4.19) computed for the brane compactification (Z, L) (see Table 4.3). This remarkable correspondence between disc instantons in a three-fold Z and sphere instantons of a dual four-fold X_4 [48] is possible due to the coincidence of the multicover factor $\sim k^{-2}$ for discs in a three-fold [52] and spheres in a four-fold [75, 99].

From the above it follows that the exact instanton corrected four-fold periods on X_4 can be informally written as[7]

$$\underline{\Pi}^{\Sigma}(X_4) = \begin{cases} (1, S) \times \Pi^{\Sigma}(Z^*) \\ \\ W(t, \hat{t}) \end{cases} + \ldots$$

up to subleading corrections in S denoted by the dots. Thus the closed-string superpotential of the four-fold, (D.9), agrees with the open-closed superpotential computed for the brane compactification (Z^*, E) in eq. (3.96), up to subleading corrections in S:

$$\mathcal{W}(z, \hat{z}, S) = \sum_{\gamma_\Sigma\in H_3(Z^*)}(N_\Sigma + S\ M_\Sigma)\int_{\gamma_\Sigma}\underline{\Omega}^{(3,0)} + \sum_{\substack{\gamma_\Sigma\in H_3(Z^*,\mathcal{D})\\ \partial\gamma_\Sigma\neq 0}}\hat{N}_\Sigma\int_{\gamma_\Sigma}\underline{\Omega}^{(3,0)} + \ldots \ . \tag{D.9}$$

The first term includes the three-fold flux superpotential on Z^* from both RR and NS fluxes, in agreement with our identification of the modulus S with the type IIB string coupling.[8] The last term describes the disc instanton corrected domain wall tension. This concludes the derivation of our claims in sect. 3.8 concerning the superpotential.

Eventually we can obtain the Kähler potential from (3.102), using the topological metric

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \mathbb{1}_3 & 0 \\ 0 & 0 & \tilde{\eta}^{-1} & 0 & 0 \\ 0 & \mathbb{1}_3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \ .$$

[7] A similar relation holds in a more general form for any three-fold fibration over $\mathbb{P}^1$, see sect. 2 of [99].

[8] This has already been observed earlier in a related context in [198].

It is not very illuminating to display the exact result and we restrict to discuss the leading order in an expansion for large S and near large volume. Inserting the periods (D.8) into (3.102), and keeping only the leading order terms one obtains the result in (3.103). It is straightforward to check, that this can be rewritten, to leading orders in S, in the form (3.99) with a pairing matrix $(\underline{\eta})$ of the form (3.101) with blocks

$$\eta_{Z^*} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} , \qquad \tilde{\eta}_{D^\sharp} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{4} & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

To leading order the flat relative periods are given by

$$\underline{\tilde{\Pi}} = \left(1, t, \tilde{F}_t, \tilde{F}_0, t_1, \tilde{W}, \tilde{T}\right) = \left(1, t, -\frac{5}{2}t^2, \frac{5}{6}t^3, t_1, -2t_1^2, -\frac{2}{3}t_1^3\right) .$$

This substantiates our claim in section 3.8 concerning the Kähler potential.

Appendix E

Three-fold – four-fold correspondence

In the following we will clarify the relation between the compact open/closed geometry (Z, D) on the three-fold Z^* and the non-compact four-fold $X_4^{\sharp *}$ which can be associated to it via the open/closed duality of [48]. We show how the periods of the four-fold can be mapped to the relative periods of the open/closed system [48, 56]. This will also show how to obtain explicit expressions for the periods of the surface D.

Mapping on the level of periods

We would like to compute the periods of the fourfold directly by taking the contour integral over the four cycles of the Calabi-Yau four-fold. As has been discussed in [48], there are two types of four cycles $\gamma^{(4)}$ in the non-compact fourfold. The first type corresponds to those four cycles which are made from a three cycle $\gamma^{(3)}$ in Z and a compact small cycle from the base cylinder, e.g. $\gamma^{(4)} = \gamma^{(3)} \times S^1$. The period associated with this type of cycles should reduce to the pure threefold periods modulo an irrelevant constant. The second type of periods of the fourfold come from the integration between branch points of the cylinder, where a 3-cycle in the fiber collapses. We study both types of cycles in the following.
There are two convenient ways of representing the period integral for bulk geometries which are mirror to a complete intesection in a toric variety. One can represent them either by a residue integral[1] or in terms of an integral in a LG-orbifold model [109, 114]. For the case at hand the second representation was already skillfully used in [48, 56]. The advantages of the LG-model representation are that the orbifold actions which are introduced to obtain the mirror model in our examples are automatically taken care of and that the integrals involved are simpler than their residue counterparts. However the geometric embedding of the divisor into the bulk geometry is not directly transparent in the LG-model. For this it is more convenient to study the relative periods in terms of their residue representations directly on the three-fold. In the following we will also show how the two representations can be transformed into each other.[2]
We exemplify the map between the periods in the example of the mirror quintic with the

[1] This representations is also available for general complete intersection Calabi-Yau manifolds.
[2] See also [302].

divisor

$$Q = x_5^4 - \xi x_1 x_2 x_3 x_4 \ . \tag{E.1}$$

The non-compact A-model four-fold $X_4^\sharp$ which can be associated to this geometry has the following GLSM charges in a large volume phase

	ν_0	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6	ν_7
l^1	-4	1	1	1	1	1	-1	1
l^2	-1	0	0	0	0	1	1	-1

In terms of the complex structure variables of the mirror corresponding to the above mori vectors we find

$$z_c = z_1 z_2, \qquad \xi = z_2 z_c^{-1/5} \ , \tag{E.2}$$

where z_c denotes the usual bulk complex structure variable of the mirror quintic. The Hori-Vafa construction of the four-fold mirror leads to the following expression for the four-fold periods

$$\Pi_\gamma = \int_{\gamma_i} \prod_{i=1}^{5} \frac{dw_i}{w_i} \int_{\gamma_0} dw_0 \int_{\gamma_v} \frac{du}{u}\, \delta\left(z_c w_0^5 - \prod_{i=1}^{5} w_i \right) e^{-\sum_{i=0}^{5} w_i - u(w_5 + z_2 w_0)} \ , \tag{E.3}$$

where we already integrated out w_6 and w_7. By a change of coordinates

$$w_0 = z_c^{-1/5} \prod_{i=1}^{5} x_i, \qquad w_i = x_i^5 \quad i = 1, \dots 5, \qquad u = \frac{v}{x_5} \tag{E.4}$$

this can also be written as

$$\Pi_\gamma \sim \int_{\gamma^{(3)} \times \gamma_v} \Big(\prod_{i=1}^{5} dx_i \Big) \frac{dv}{v}\, e^{-P - v\,Q} \ , \tag{E.5}$$

where $\gamma^{(3)} = \times_{i=1}^{5} \gamma_i$ and

$$P = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\psi\, x_1 x_2 x_3 x_4 x_5 \ , \tag{E.6}$$

$$Q = x_5^4 - \xi\, x_1 x_2 x_3 x_4 \ . \tag{E.7}$$

with $5\psi = -z_c^{-1/5}$ and $\xi = -z_2 z_c^{-1/5}$. Now we can look at the two types of cycles mentioned above.

1. $\gamma = \gamma^{(3)} \times S^1$ where $\gamma^{(3)}$ denotes a three-cycle on the three-fold. This type of cycle should reproduce the closed string periods of the mirror quintic. The period is given by

$$\Pi_\gamma \sim \int_{\gamma^{(3)} \times S^1} \Big(\prod_{i=1}^{5} dx_i \Big) \frac{dv}{v}\, e^{-P - v\,Q} \ , \tag{E.8}$$

and by integrating around $v=0$, we obtain

$$\Pi_\gamma \sim \int_{\gamma^{(3)}} \Big(\prod_{i=1}^{5} dx_i\Big)\, e^{-P} \ . \tag{E.9}$$

Now, we introduce the following change of variables

$$y_5 = x_5^5 \qquad , \qquad y_i = \frac{x_i}{x_1}\,, \quad i=1,2,3,4 \ . \tag{E.10}$$

Note that the Jacobian of (E.10) is one and this is linked with the fact that the degree of polynomial P is five (CY condition). Having this, the period becomes

$$\begin{aligned}\Pi_\gamma &\sim \int \Big(\prod_{i=1}^{5} dy_i\Big) e^{-y_5 F}\ ,\ F \equiv 1 + y_1^5 + y_2^5 + y_3^5 + y_4^5 - 5\psi\, y_1 y_2 y_3 y_4 \\ &= \int \Big(\prod_{i=1}^{4} dy_i\Big) \delta(F) = \int \Big(\prod_{i=1}^{4} dy_i\Big) \frac{1}{\partial_{y_4} F}\Big|_{F=0} \\ &= \int_\gamma \frac{1}{P}\,\Delta \ .\end{aligned} \tag{E.11}$$

In the last line, in addition to the integral over the 3-cycle of the three-fold, we have a residue integral as well. Thus we recover the familiar residue expression of the bulk periods of the mirror quintic.

2. $\gamma = \gamma^{(3)} \times \gamma_v$ where we integrate v between branch points of the cylinder, where a 3-cycle in the fiber collapses. The periods associated with these type of 4-cycles is evaluated from the derivative of the period with respect to the open modulus of three-fold.[3] One obtains

$$\begin{aligned}\partial_\xi \Pi_\gamma &\sim \int \Big(\prod_{i=1}^{5} dx_i\Big) \frac{dv}{v}(-v\,\partial_\xi Q)\, e^{-P-v\,Q} = \int \Big(\prod_{i=1}^{5} dx_i\Big) dv (x_1x_2x_3x_4)\, e^{-P-v\,Q} \\ &= \int \Big(\prod_{i=1}^{5} dx_i\Big) (x_1x_2x_3x_4) \delta(Q)\, e^{-P} = \int \Big(\prod_{i=1}^{4} dx_i\Big) \frac{x_1x_2x_3x_4}{\partial_5 Q}\, e^{-P}\Big|_{Q=0} \\ &= \frac{1}{4\,\xi^{3/4}} \int \Big(\prod_{i=1}^{4} dx_i\Big) (x_1x_2x_3x_4)^{1/4}\, e^{-G}\ ,\end{aligned} \tag{E.12}$$

where in the last line G is defined as

$$G \equiv x_1^5 + x_2^5 + x_3^5 + x_4^5 - u\,(x_1x_2x_3x_4)^{5/4} \quad , \quad u = \xi^{1/4}(5\psi - \xi) \ . \tag{E.13}$$

[3] Alternatively one can also directly use the expression (E.3) to quickly derive the GLSM charges of the subsystem and the functional dependence of its algebraic complex structure parameters on the four-fold parameters by first taking the derivative w.r.t. the open modulus and then integrating out v.

Note that u is the algebraic modulus of the K3 subsystem in the threefold language. Now, we change the variables by introducing

$$y_i = x_i^{5/4} \quad , \quad i = 1,2,3,4 \; , \tag{E.14}$$

and under (E.14) transforms to

$$\partial_\xi \Pi \sim \frac{1}{\xi^{3/4}} \int \Big(\prod_{i=1}^{4} dy_i \Big)\, e^{-(y_1^4+y_2^4+y_3^4+y_4^4-u\, y_1y_2y_3y_4)} \; . \tag{E.15}$$

This is exactly the LG-period expression for the mirror of a K3-surface given as $\mathbb{P}^3_{1,1,1,1}[4]$. Note however that the coordinate change (E.14) is not one-to-one. Thus the surface $\{P=0\}\cap\{Q=0\}$ is actually a branched cover of a K3-surface in the three-fold Z^*. We can now perform the same trick as we did for evaluating the previous type of periods by introducing new variables s_i as

$$s_4 = y_4^4 \quad , \quad s_i = \frac{y_i}{y_4} \; , \; i = 1,2,3 \; . \tag{E.16}$$

The Jacobian of (E.16) is unity and this originates from the fact that the subsystem is a K3. The period (E.15) under (E.16) goes to

$$\begin{aligned}
\partial_\xi \Pi &\sim \frac{1}{\xi^{3/4}} \int \Big(\prod_{i=1}^{4} ds_i \Big)\, e^{-s_4 H} \; , \; H = 1 + s_1^4 + s_2^4 + s_3^4 - u\, s_1 s_2 s_3 \\
&\sim \frac{1}{\xi^{3/4}} \int \Big(\prod_{i=1}^{4} ds_i \Big) \delta(H) = \frac{1}{\xi^{3/4}} \int \Big(\prod_{i=1}^{3} ds_i \Big) \frac{1}{\partial_{s_4} H}\Big|_{H=0} \\
&\sim \frac{1}{\xi^{3/4}} \int \frac{\Delta_{K3}}{x_1^4+x_2^4+x_3^4+x_4^4 - u\, x_1x_2x_3x_4} \; ,
\end{aligned} \tag{E.17}$$

where in the last line we also have a residue integral over the tubular neighborhood of $H=0$ in $\mathbb{P}^3_{1,1,1,1}$. This is exactly what we compute by $\partial_\xi \Pi \sim \int \frac{1}{PQ}\Delta$ as the starting point. Now, we can even push the computation further and compute the periods explicitly. Let us take the limit where the modulus of K3 subsystem is very large. Then we can compute the residue integrals by expanding the denominator of (E.17)

$$\begin{aligned}
\partial_\xi \Pi &\sim \frac{1}{\xi(5\psi-\xi)} \sum_{n=0}^{\infty} \int \frac{\Delta}{x_1x_2x_3x_4} \Big(\frac{x_1^4+x_2^4+x_3^4+x_4^4}{u\, x_1x_2x_3x_4} \Big)^n \\
&\sim \frac{1}{\xi(5\psi-\xi)} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \Big(\frac{1}{u^4} \Big)^n \\
&\sim \frac{1}{\xi(5\psi-\xi)}\, {}_3F_2\Big(\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, 1; \frac{256}{u^4} \Big) \; .
\end{aligned} \tag{E.18}$$

If we analytically continue this result, we exactly get the semi-periods (5.21) in [103] in the limit of small u.

Therefore we have seen, that the two types of 4-cycles that were introduced in [48] capture all periods of the open/closed geometry.

F

Appendix F

Normalization of the domain wall tension in the quintic example

In section 4.1 we found the expression (4.12) for the domain wall tension associated with the involution brane of [53]. In this appendix we give a derivation of its normalization, as in [53, 54], by an analysis of the open/closed moduli space and analytic continuation techniques, similar as in the closed string case. For the notation we refer to the section 2.6.

The Picard-Fuchs operator of the bulk geometry for the quintic is given by

$$\mathcal{L} = \theta^4 - 5z \prod_{i=1}^{4} (5\theta - i) \ , \qquad \text{(F.1)}$$

where z denotes the complex structure coordinate near the point of maximal unipotent monodromy and $\theta = z\partial_z$. The solutions to (F.1) for small z can be generated from the Frobenius generating function (2.84), which in this case becomes

$$B(z;\rho) = \sum_{n\geq 0} \frac{\Gamma(1+5n+5\rho)\ z^{n+\rho}}{\Gamma(1+n+\rho)} \ . \qquad \text{(F.2)}$$

This leads to a vector $\bar{\omega}$ of solutions[1]

$$\bar{\omega}(z) = \begin{pmatrix} 1 \\ \frac{\log(z)}{2\pi i} \\ \frac{\log^2(z)}{(2\pi i)^2} - \frac{5}{6} \\ \frac{\log^3(z)+10\pi^2\log(z)-240\zeta(3)}{(2\pi i)^3} \\ 60\sqrt{z} \end{pmatrix} + \mathcal{O}(z) \ . \qquad \text{(F.3)}$$

Note that the domain wall solution can be represented as $W_{\text{crit}}(z) = B(z,1/2)$ and fulfills

$$\mathcal{L}W_{\text{crit}}(z) = \frac{15\sqrt{z}}{4} \ . \qquad \text{(F.4)}$$

To compute the integral period vector we compare $\bar{\omega}$ to the solutions near the orbifold point $\psi = (5^5 z)^{-1/5} = 0$. This can be achieved by analytic continuation of the generating function (F.2) via a Barnes-type integral for $\rho \geq 0$

$$B(z;\rho) = -\oint_{C_+} \frac{(-1)^s\Gamma(-s)\Gamma(s+1)\Gamma(5s+5\rho+1)\ z^{s+\rho}}{\Gamma(s+\rho+1)^5}\, ds \ , \qquad \text{(F.5)}$$

[1] In addition we chose a convenient normalization for the solutions.

where C_+ is a contour which encircles the roots at $s = n$, for $n \in \mathbb{N}_{\geq 0}$. By closing the contour to the left along a contour C_- one picks poles at $s = -n - \rho - \frac{j}{5}$, for $j = 1, \ldots 4$ and at $s = -n - 1$, where $n \in \mathbb{N}_{\geq 0}$ and arrives at[2]

$$B(z(\psi); \rho) = \frac{\pi}{5} \sum_{j=1}^{4} \sum_{n \geq 0} \frac{e^{-i\pi(\frac{6}{5}j + 5n + \rho)}}{\Gamma(-\frac{j}{5} - n + 1)^5 \Gamma(j + 5n) \, \sin\left(\frac{\pi i}{5} + \pi\rho\right)} (5\psi)^{5n+j} + \sum_{n \geq 0} \frac{\Gamma(-5n + 5\rho - 4)}{\Gamma(\rho - n)^5} (5\psi)^{5(n-\rho+1)} . \tag{F.6}$$

Upon differentiating with respect to ρ and evaluating at $\rho = 0$ and $\rho = 1/2$ one can express the large volume basis of solutions $\bar{\omega}(z)$ in terms of a basis of solutions $\omega^{\text{orb}}(\psi)$ to (F.1) near the orbifold point. The basis $\omega^{\text{orb}}(\psi)$ is given by

$$\omega_j^{\text{orb}}(\psi) = \frac{\pi}{5} \sum_{n \geq 0} \frac{e^{-i\pi(\frac{6}{5}j + 5n)}}{\Gamma(-\frac{j}{5} - n + 1)^5 \Gamma(j + 5n) \, \sin\left(\frac{\pi i}{5}\right)} (5\psi)^{5n+j} , j = 1, \ldots, 4 , \tag{F.7}$$

$$\omega_5^{\text{orb}}(\psi) = \pi^2 \sum_{n \geq 0} \frac{\Gamma(-5n - 3/2)}{\Gamma(1/2 - n)^5} (5\psi)^{5(n+1/2)} . \tag{F.8}$$

This leads to an analytic continuation matrix m

$$\bar{\omega}(z) = m \cdot \omega^{\text{orb}}(\psi(z)) . \tag{F.9}$$

The basis of solutions $\omega^{\text{orb}}(\psi)$ has a monodromy around the the orbifold point given by

$$\omega^{\text{orb}}(\alpha\psi) = \tilde{M}_0 \cdot \omega^{\text{orb}}(\psi), \qquad \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \alpha^2 & 0 & 0 & 0 \\ 0 & 0 & \alpha^3 & 0 & 0 \\ 0 & 0 & 0 & \alpha^4 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} , \tag{F.10}$$

where $\alpha^5 = 1$. The large volume basis is related to the extended integral period vector $\Pi(z)$ by a change of basis $\Pi(z) = B \cdot \bar{\omega}(z)$. Using the relation[3] $M_1 = M_0^{-1} \cdot M_\infty$ for the conifold monodromy and the explicit expression for the analytic continuation matrix m the change of basis can be fixed by demanding integrality of the involved monodromy matrices. This leads to[4]

$$M_\infty = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -3 & -5 & 1 & 0 & 0 \\ 5 & -8 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}, \; M_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; M_0 = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ -3 & -5 & 1 & 3 & 0 \\ 5 & -8 & 1 & -4 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix} \tag{F.11}$$

[2]Note that the second set of poles at $s = -n - 1$, for $n \in \mathbb{N}_{\geq 0}$, gives effectively only a contribution for $\rho \neq 0$.

[3]Note that the relation between the monodromy matrices is different as compared to the choices made in the rest of this work.

[4]This monodromy matrices are consistent with the closed string result of [37].

and

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{11}{2} & \frac{5}{2} & 0 & 0 \\ 0 & -\frac{25}{6} & 0 & -\frac{5}{6} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & 0 & -\frac{1}{4} \end{pmatrix} . \tag{F.12}$$

Using the mirror-map

$$2\pi i\, t(z) = \frac{\bar{\omega}_1(z)}{\bar{\omega}_0(z)} = \log(z) + \dots \ ,$$

the extended integral period vector $\Pi(t)$ is given by

$$\frac{\Pi(t)}{\bar{\omega}_0(z(t))} = \begin{pmatrix} 1 \\ t \\ \frac{5t^2}{2} + \frac{11t}{2} - \frac{25}{12} \\ -\frac{5t^3}{6} - \frac{25t}{12} + \frac{25i\zeta(3)}{\pi^3} \\ \frac{t}{2} + \frac{1}{4} - 15\sqrt{q} \end{pmatrix} + \mathcal{O}(tq, q) \ , \tag{F.13}$$

where $q = e^{2\pi i t}$. These results are consistent with the discussion of [53].

G

Appendix G

From three-chains to Abel-Jacobi maps on the elliptic curve

In some of the examples considered in chapter 4, the domain wall tensions can be directly related to the Abel-Jacobi map on an elliptic curve in a certain limit in the moduli. This gives a check on the normalization obtained from the geometric surface periods. To this end, we consider the non-compact Calabi-Yau manifolds $X^\flat$ of ref. [186]

$$\begin{aligned} &\mathbb{P}^4_{3,2,1,1,-1}[6]: \quad P = y_1^2 + y_2^3 + y_3^6 + y_4^6 + \frac{1}{y_5^6} + \hat{\psi}\, y_1 y_2 y_3 y_4 y_5 \,, \quad z = \hat{\psi}^{-6} \,, \\ &\mathbb{P}^4_{2,1,1,1,-1}[4]: \quad P = y_1^2 + y_2^4 + y_3^4 + y_4^4 + \frac{1}{y_5^4} + \hat{\psi}\, y_1 y_2 y_3 y_4 y_5 \,, \quad z = \hat{\psi}^{-4} \,, \\ &\mathbb{P}^4_{1,1,1,1,-1}[3]: \quad P = y_1^3 + y_2^3 + y_3^3 + y_4^3 + \frac{1}{y_5^3} + \hat{\psi}\, y_1 y_2 y_3 y_4 y_5 \,, \quad z = \hat{\psi}^{-3} \,. \end{aligned} \tag{G.1}$$

The closed-string periods on the non-compact three-folds are solutions of the Picard-Fuchs operators

$$\mathcal{L}^{[n]} = \mathcal{L}_E^{[n]}(-z)\cdot\theta \,, \tag{G.2}$$

where $\mathcal{L}_E^{[n]}(z)$ denote the Picard-Fuchs operators for the representations of the elliptic curve E

$$\begin{aligned} &\mathbb{P}_{3,2,1}[6]: \quad &\mathcal{L}_E^{[6]}(z) &= \theta^2 - 12z(6\theta+5)(6\theta+1)\,, \\ &\mathbb{P}_{2,1,1}[4]: \quad &\mathcal{L}_E^{[4]}(z) &= \theta^2 - 4z(4\theta+3)(4\theta+1)\,, \\ &\mathbb{P}_{1,1,1}[3]: \quad &\mathcal{L}_E^{[3]}(z) &= \theta^2 - 3z(3\theta+1)(3\theta+2)\,, \end{aligned} \tag{G.3}$$

with $\theta = z\frac{d}{dz}$. The equation for the elliptic curve is given by the restriction to $(y_4y_5)^n = -1$ in (G.1).[1] Eq. (G.2) implies the relation $2\pi i\,\theta\Pi_\ell(z) = \pi_\ell(-z)$ between the periods $\Pi_\ell(z)$ of the non-compact threefold and the periods $\pi(z)$ on the elliptic curve.

A similar relation

$$2\pi i\,\theta T(z) = \tau(-z)\,, \tag{G.4}$$

holds for the chain integrals between the domain wall tension T of the non-compact threefold and the line integral τ of the associated elliptic curve E. They fulfill the inhomogeneous differential equation

$$\mathcal{L}^{[n]}T(z) = -\frac{c^{[n]}}{16\pi^2}\sqrt{z}\,, \qquad \mathcal{L}_E^{[n]}(z)\tau(z) = -\frac{c^{[n]}}{8\pi}\sqrt{z}\,, \tag{G.5}$$

[1] Keeping the convention eq. (2.82), the algebraic modulus of the Calabi-Yau manifold and the curve differ by a minus sign, as indicated in eq. (G.2) and below.

k	$n_k^{[6]}$	$n_k^{[4]}$	$n_k^{[3]}$
1	16	8	2
3	-432	-24	-2
5	45 440	320	10
7	-7 212 912	-6 776	-84
9	1 393 829 856	175 536	858
11	-302 514 737 008	-5 123 448	-9 878
13	70 891 369 116 256	161 777 200	123 110
15	-17 542 233 743 427 360	-5 401 143 120	-1 622 890
17	4 520 954 871 206 554 016	187 981 969 232	22 308 658
19	-1 202 427 473 254 100 406 128	-6 756 734 860 408	-316 775 410
21	327 947 495 234 600 477 004 048	249 179 670 525 576	4 616 037 426
23	-91 298 034 448 725 882 319 078 384	-9 384 048 140 182 200	-68 700 458 258

Table G.1.: Disc invariants for the on-shell superpotentials $W_{\text{inst}} = \frac{1}{2}T_{\text{inst}}$ for the non-compact hypersurfaces $X^\flat$ of degree $d = 6, 4, 3$.

in terms of the constants $c^{[n]}$

$$c^{[6]} = 16 \ , \qquad c^{[4]} = 8 \ , \qquad c^{[3]} = 2 \ , \tag{G.6}$$

which determine the normalization of the of the domain wall tension T. Then the domain wall tensions T, which are now solutions to the normalized inhomogeneous Picard-Fuchs equations (G.5), contains the quantum instanton contribution T_{inst}, which starts as

$$T_{\text{inst}}(z) \ = \ -\frac{1}{2\pi^2}\left(c^{[n]}\sqrt{z} + \ldots\right) \ ,$$

and yields for the three geometries (G.1) the normalized disc invariants in Tab. G.1.[2]

The normalization constants $c^{[n]}$ are determined by requiring integrality of the monodromy matrices with respect to the singularities of the moduli space of the extended period vector. The extended period vector consists of the bulk periods Π and the domain wall tension T. Alternatively, the constants $c^{[n]}$ can be determined by directly evaluating the line integral τ on the curve E and by exploiting its relation to the 3-chain integral T according to eq. (G.4). In the following we exemplify the two approaches for the non-compact sextic threefold (G.1) to determine the normalization constant $c^{[6]}$. The other two normalization constants $c^{[4]}$ and $c^{[3]}$ are obtained analogously.

The moduli space of the non-compact sextic threefold (G.1) exhibits three singularities $z = 0$, $z = -\frac{1}{432}$, and $z = \infty$, which correspond to a large radius, a conifold, and a orbifold point of the moduli space. In the vicinity of the large radius point $|z| < \frac{1}{432}$ a complete set

[2]Here we list the integral disc instanton numbers $n_k^{[n]}$. These invariants are related to the real invariants $n_{k,\text{real}}^{[n]}$ in ref. [187] by a factor 2, *i.e.* $n_k^{[n]} = 2 \cdot n_{k,\text{real}}^{[n]}$.

of solutions to the Picard-Fuchs operator $\mathcal{L}^{[6]}$ is given by

$$\begin{aligned}
\tilde{\Pi}_0(z) =&1 \ , \\
\tilde{\Pi}_1(z) =&\log z + \sum_{k=1}^{+\infty} \frac{(6k)!}{k!\,(2k)!\,(3k)!} \cdot \frac{(-z)^k}{k} \ , \\
\tilde{\Pi}_2(z) =&\frac{1}{2}(\log z)^2 + \sum_{k=1}^{+\infty} \frac{(6k)!}{k!\,(2k)!\,(3k)!} \cdot \frac{(-z)^k}{k} \cdot \Big(\log z - \frac{1}{k} + 6\Psi(6k+1) \\
&\qquad -\Psi(k+1) - 2\Psi(2k+1) - 3\Psi(3k+1)\Big) \ ,
\end{aligned} \tag{G.7}$$

in terms of the Polygamma function Ψ. Together with the solution $\tilde{T}$ to the inhomogeneous Picard-Fuchs equation $\mathcal{L}^{[6]}\tilde{T}(z) \sim \sqrt{z}$

$$\tilde{T}(z) \ = \ \frac{\pi}{32}\sqrt{z}\,\sum_{k=0}^{+\infty} \frac{\Gamma(6k+4)}{\Gamma(3k+\frac{5}{2})\Gamma(2k+2)\Gamma(k+\frac{3}{2})(k+\frac{1}{2})}(-z)^k \ , \tag{G.8}$$

they form the extended period vector $\tilde{\Pi} = \Big(\tilde{\Pi}_0, \tilde{\Pi}_1, \tilde{\Pi}_2, \tilde{T}\Big)$. For this vector we determine the large radius monodromy matrix $\tilde{M}_{\mathrm{LR}}$. Furthermore, by analytically continuation with the help of Barnes integrals to the other singular points in the moduli space we also infer the conifold and orbifold monodromy matrices $\tilde{M}_{\mathrm{con}}$ and $\tilde{M}_{\mathrm{orb}}$. Next we perform a change of basis to the integral extended period vector $\Pi = (\Pi_0, \Pi_1, \Pi_2, T)$ by demanding integrality of all the monodromy matrices. For the bulk sector these steps can be found in detail in ref. [186]. In addition to integrality of the monodromy matrices we require that the domain wall tension T vanishes at $z = \infty$. The latter condition arises because the domain wall tension T interpolates between two supersymmetric vacua that coincide at the orbifold point. After these steps we finally arrive at the integral periods

$$\begin{aligned}
\Pi_0(z) &= \tilde{\Pi}_0(z) \ = \ 1 \ , \\
\Pi_1(z) &= \frac{1}{2\pi i}\tilde{\Pi}_1(z) \ = \ t(z) \ , \\
\Pi_2(z) &= \frac{1}{(2\pi i)^2}\tilde{\Pi}_2(z) - \frac{1}{4\pi i}\tilde{\Pi}_1(z) - \frac{5}{12}\tilde{\Pi}_0(z) \ = \ \frac{1}{2}t(z)^2 - \frac{1}{2}t(z) - \frac{5}{12} + \Pi_{\mathrm{inst}}(z) \ , \\
T(z) &= \frac{32}{(2\pi i)^2}\tilde{T}(z) - \frac{1}{4\pi i}\tilde{\Pi}_1(z) + \frac{1}{4}\tilde{\Pi}_0(z) \ = \ -\frac{1}{2}t(z) + \frac{1}{4} + T_{\mathrm{inst}}(z) \ .
\end{aligned} \tag{G.9}$$

Here we also exhibit the classical terms in terms of the flat coordinate t and the instanton contributions Π_{inst} and T_{inst}. In particular the normalized domain wall tension yields the normalized instanton contribution

$$T_{\mathrm{inst}}(z) \ = \ -\frac{16}{2\pi^2}\left(\sqrt{z} - \frac{512}{9}z^{3/2} + \frac{229\,376}{25}z^{5/2} - \ldots\right) \ ,$$

and hence the normalization constant $c^{[6]} = 16$ in eq. (G.6). The integral monodromy matrices in the integral basis (G.9) are then given by

$$M_{\rm LR} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} , \quad M_{\rm con} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad M_{\rm orb} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} ,$$

with $M_{\rm con}M_{\rm orb} = M_{\rm LR}$.

As an independent calculation to determine normalized domain wall tensions we now directly reduce the three-chain integrals of the domain wall tensions between the curves $C^\flat_{\varepsilon,\pm}$ of eq. (4.48) to line integrals on the elliptic curve E. In order to evaluate the chain integrals we first change to the inhomogeneous coordinate α, y, z_1, z_2, which are suitable to evaluate the chain integrals [186]

$$y_1 = y - \frac{1}{2}\hat\psi\alpha z_1^3 z_2 \ , \quad y_2 = -\alpha z_1^2 \ , \quad y_3 = -iz_1 \ , \quad y_4 = -iz_2 \ , \quad y_5 = 1 \ .$$

In terms of these coordinates the hypersurface equation (G.1) of the non-compact sextic Calabi-Yau threefold $X^\flat$ becomes

$$X^\flat : \quad y^2 = z_1^6\left(1+\alpha^3+\frac{1}{4}(\hat\psi\alpha)^2 z_2^2\right) + (z_2^6-1) \ ,$$

while the holomorphic three form reads

$$\Omega(\hat\psi) = \frac{6}{(2\pi i)^3} \cdot \frac{\hat\psi\, z_1^2\, d\alpha\, dz_1\, dz_2}{2\sqrt{z_1^6\left(1+\alpha^3+\frac{1}{4}(\hat\psi\alpha)^2 z_2^2\right)+(z_2^6-1)}} \ .$$

We can think of this geometry as a complex surface given in terms of the coordinates z_1 and z_2 fibered over a $\mathbb{P}^1$ base parametrized by the affine coordinate α. Furthermore, in these coordinates the curves $C^\flat_{\varepsilon,\kappa}$ are given by[3]

$$C^\flat_{\varepsilon,\kappa} = \left\{ z_1 = i\, z_2 \, , \ \alpha z_2 = -i\kappa\sqrt{\varepsilon\, i\, \hat\psi} \, , \ \varepsilon = \frac{i}{2}\hat\psi\alpha z_1^4 + y \right\} \ , \quad \varepsilon = \pm i \quad \kappa = \pm i \, .$$

The goal is now to evaluate the domain wall tensions

$$T_\ell(\hat\psi) = \int_{\Gamma_\ell} \Omega(\hat\psi) \ ,$$

where we consider the two three chains Γ_1 and Γ_2 bounded by

$$\partial\Gamma_1 = C_{+i,+i} - C_{-i,-i} \ , \qquad \partial\Gamma_2 = C_{-i,+i} - C_{-i,-i} \ .$$

As we will see in the calculation the domain wall tensions for the remaining combinations of curves do not yield independent results. The steps to reduce the three dimensional integral to

[3]For ease of notation we have chosen here the explicit root $\eta = i$ for $\eta^6 = -1$ in eq. (4.48).

a line integral over the $\mathbb{P}^1$ base are worked out and explained in detail in ref. [186]. Therefore for completeness we merely sketch the necessary steps here.

Instead of calculating the domain wall tension, it is easier to derive the line integral τ of eq. (G.4). With $z = \hat{\psi}^{-6}$ we get

$$\begin{aligned}\tau_\ell(-z) = 2\pi i\,\theta\, T_\ell(\hat{\psi}(z)) &= -\frac{2\pi i}{6}\,\hat{\psi}\,\frac{dT_\ell(\hat{\psi})}{d\hat{\psi}} \\ &= -\frac{\hat{\psi}}{(2\pi i)^2}\int_{\Gamma_\ell} d\alpha\, dz_2\, dz_1\,\frac{d}{dz_2}\frac{1}{2\sqrt{z_1^6\left(1+\alpha^3+\frac{1}{4}(\hat{\psi}\alpha)^2 z_2^2\right)+(z_2^6-1)}}\ .\end{aligned}$$

The simplification occurs because for the integrand the derivative with respect to $\hat{\psi}$ is equivalent to the derivative with respect to z_2. Then the integral over z_2 becomes trivial.[4] We now evaluate the integral over the coordinate z_1 along a closed contour encircling the six branch points of the square root. Next we integrate the coordinate z_2 along the interval from $z_2 = 1$ to $z_2 = -1$ to arrive at [186]

$$\int \Omega = -\frac{1}{2\pi i}\left(\int \frac{\hat{\psi}\, z_2\, d\alpha}{2\sqrt{1+\alpha^3+\frac{1}{4}(\hat{\psi}z_2)^2\alpha^2}}\Bigg|_{z_2=1} - \int \frac{\hat{\psi}\, z_2\, d\alpha}{2\sqrt{1+\alpha^3+\frac{1}{4}(\hat{\psi}z_2)^2\alpha^2}}\Bigg|_{z_2=-1}\right)\ . \tag{G.10}$$

Note that the performed integration is equivalent to the integration over a homology 2-sphere, as the contour in the z_1 coordinate can be shrunk to a point at the endpoints $z_2 = \pm 1$ of the interval.

If we now carry out the remaining integral (G.10) over α along a closed contour encircling the two branch points with leading behavior $\sim \hat{\psi}^{-1}$ for large $\hat{\psi}$, we integrate over a one cycle of the elliptic curve E and obtain the fundamental period of the elliptic curve

$$\pi_0(-z) \;=\; {}_2F_1(\tfrac{1}{6},\tfrac{5}{6};1;-432z) \;=\; 1 - 60z + 13\,860z^2 - \ldots\ .$$

If, instead, we reduce the three chains Γ_1 and Γ_2 to line integrals over α in eq. (G.10), we need to evaluate the integrals

$$\begin{aligned}\tau_1(-z) &= -\frac{1}{2\pi i}\left(\int_{i\sqrt{\hat{\psi}}}^{-\sqrt{\hat{\psi}}}\frac{\hat{\psi}\, d\alpha}{2\sqrt{1+\alpha^3+\frac{1}{4}\hat{\psi}^2\alpha^2}} + \int_{-i\sqrt{\hat{\psi}}}^{\sqrt{\hat{\psi}}}\frac{\hat{\psi}\, d\alpha}{2\sqrt{1+\alpha^3+\frac{1}{4}\hat{\psi}^2\alpha^2}}\right)\ ,\\ \tau_2(-z) &= -\frac{1}{2\pi i}\left(\int_{\sqrt{\hat{\psi}}}^{-\sqrt{\hat{\psi}}}\frac{\hat{\psi}\, d\alpha}{2\sqrt{1+\alpha^3+\frac{1}{4}\hat{\psi}^2\alpha^2}} + \int_{-\sqrt{\hat{\psi}}}^{\sqrt{\hat{\psi}}}\frac{\hat{\psi}\, d\alpha}{2\sqrt{1+\alpha^3+\frac{1}{4}\hat{\psi}^2\alpha^2}}\right)\ .\end{aligned} \tag{G.11}$$

Here the integration boundaries for α are determined by requiring that the coordinates $(z_2 = \pm 1, \alpha)$ associated to the endpoints of the line integral correspond to a point on the appropriate curve $C^{\flat}_{\varepsilon,\kappa}$.

[4] Similarly as for the examples discussed in ref. [188], there is no contribution from the derivative $\frac{d}{d\hat{\psi}}$ acting on the three chain Γ_ℓ.

While the line integral (G.11) trivially vanishes for Γ_2, namely $\tau_2(-z) = 0$, we evaluate the integral over Γ_1 and arrive at

$$\tau(-z) \equiv \tau_1(-z) = \frac{16\sqrt{z}}{2\pi i}\ {}_3F_2(\tfrac{2}{3},\tfrac{4}{3},1;\tfrac{3}{2},\tfrac{3}{2};-432z) - \frac{1}{2}\pi_0(-z)\ . \tag{G.12}$$

The resulting domain wall tension $\tau(-z) = 2\pi i\,\theta T(z)$ is in agreement with the result in eq. (G.9) and in eq. (G.5) together with the normalization $c^{[6]} = 16$ of eq. (G.6).

H

Appendix H

(Mock) modularity

Contents

In this section we want to review the definition and some basic properties of mock modular forms. In addition we give the definitions of modular forms appearing in the main body text. For more details on mock modular forms the reader is referred to the mathematics' literature [61, 60, 225].

H.1. Mock modular forms

Following [60], we denote the space of mock modular forms of weight k by $\mathbb{M}_k$ and the space of modular forms by M_k. Mock modular forms are holomorphic functions of τ, which is an element of the upper half plane $\mathcal{H}$, but do not transform in a modular covariant way. However, to every mock modular form h of weight k there exists a shadow $g \in M_{2-k}$ such that the function $\hat{h}$, given by

$$\hat{h}(\tau) = h(\tau) + g^*(\tau) \tag{H.1}$$

transforms as of weight k. Denoting by $g^c(z) = \overline{g(-\bar{z})}$, the completion $g^*(\tau)$ is defined by

$$g^*(\tau) = -(2i)^k \int_{-\bar{\tau}}^{\infty} (z+\tau)^{-k} g^c(z)\, dz \ . \tag{H.2}$$

Thus, $\hat{h}$ is modular but has a non-holomorphic dependence. The corresponding space containing forms of type (H.1) is denoted by $\widehat{\mathbb{M}}_k$. Given g as the expansion $g(\tau) = \sum_{n\geq 0} b_n q^n$, the completion $g^*(\tau)$ can also be written as

$$g^*(\tau) = \sum_{n\geq 0} n^{k-1} \bar{b}_n\, \beta_k(4n\tau_2)\, q^{-n} \ , \tag{H.3}$$

with $\tau_2 = \mathrm{Im}\,(\tau)$ and β_k defined by

$$\beta_k(t) = \int_t^{\infty} u^{-k} e^{-\pi u} du \ . \tag{H.4}$$

Conversely, given $\hat{h}$, one determines the shadow g by taking the derivative of $\hat{h}$ with respect to $\bar{\tau}$. One easily sees that

$$\frac{\partial \hat{h}}{\partial \bar{\tau}} = \frac{\partial g^*}{\partial \bar{\tau}} = \tau_2^{-k}\overline{g(\tau)} \ . \tag{H.5}$$

This viewpoint opens another characterization of $\widehat{\mathbb{M}}_k$ as the set of real-analytic functions F that fulfill a certain differential equation. To be precise, let us define the space $\mathfrak{M}_k$ as the space of real-analytic functions F in the upper half-plane $\mathcal{H}$ transforming as a modular form under $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$, i.e.

$$F(\gamma\tau) = \rho(\gamma)(c\tau+d)^k F(\tau) \ ,$$

where $\rho(\gamma)$ denotes some character of Γ and we demand exponential growth at the cusps. Hence, the space of completed mock modular forms $\widehat{\mathbb{M}}_k$ can now be characterized by

$$\widehat{\mathbb{M}}_k = \left\{ F \in \mathfrak{M}_k \,\Big|\, \frac{\partial}{\partial \tau}\left(\tau_2^k \frac{\partial F}{\partial \bar{\tau}}\right) = 0 \right\}. \tag{H.6}$$

This definition induces the following maps[1]

$$\mathfrak{M}_k = \mathfrak{M}_{k,0} \overset{\tau_2^k \partial_{\bar{\tau}}}{\longrightarrow} \mathfrak{M}_{0,2-k} \overset{\tau_2^{2-k} \partial_{\tau}}{\longrightarrow} \mathfrak{M}_{k,0} = \mathfrak{M}_k \ , \tag{H.7}$$

so that the composition can be converted to the Laplace operator in weight k. Hence, mock modular forms in $\widehat{\mathbb{M}}_k$ have the special eigenvalue $\frac{k}{2}\left(1-\frac{k}{2}\right)$ and are sometimes also called harmonic weak Maass forms.

Zwegers showed in [61] that mock modular forms can be realized in three different ways, namely either as Appell-Lerch sums, indefinite theta-series or as Fourier coefficients of meromorphic Jacobi forms. Further, there is a notion of mixed mock modular forms, which are functions that transform in the tensor space of mock modular forms and modular forms. However, we will call them simply mock modular forms as well.

In the following a simple example of a mock modular form is presented.

Example: E_2 as a mock modular form

The modular completion of the holomorphic Eisenstein series E_2 (see below for a definition) has the form

$$\widehat{E_2}(\tau) = E_2(\tau) - \frac{3}{\pi\tau_2} \ .$$

From $\partial_{\bar{\tau}}\widehat{E}_2 = \tau_2^{-2}\frac{3i}{2\pi}$ we get $\overline{g} = \frac{3i}{2\pi}$, a constant shadow. Doing the integral indeed yields

$$g^*(\tau) = -\,(2i)^2 \int_{-\bar{\tau}}^{\infty} (z+\tau)^{-2}\frac{3i}{2\pi}dz = -\frac{6i}{\pi}\left[\frac{-1}{z+\tau}\right]_{-\bar{\tau}}^{\infty} = -\frac{3}{\pi\tau_2} \ . \tag{H.8}$$

[1] A function $f \in \mathfrak{M}_{k,l}$ transforms under modular transformations $\gamma \in \Gamma$ with bi-weight (k,l) and character ρ, i.e. $f(\gamma\tau) = \rho(\gamma)(c\tau+d)^k(c\bar{\tau}+d)^l f(\tau)$.

H.2. Modular forms

Let us collect the definitions of various modular forms appearing in the main body text. We denote the following standard theta-functions by

$$\begin{aligned}\vartheta_1(\tau,\nu) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} (-1)^n q^{\frac{1}{2}n^2} e^{2\pi i n\nu},\\ \vartheta_2(\tau,\nu) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} q^{\frac{1}{2}n^2} e^{2\pi i n\nu},\\ \vartheta_3(\tau,\nu) &= \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi i n\nu},\\ \vartheta_4(\tau,\nu) &= \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2} e^{2\pi i n\nu}.\end{aligned} \tag{H.9}$$

In the case that $\nu = 0$ we simply denote $\vartheta_i(\tau) = \vartheta_i(\tau, 0)$ (notice that $\vartheta_1(\tau) = 0$). Under modular transformations the theta functions $\vartheta_i(\tau)$ behave as vector-valued modular forms of weight $\frac{1}{2}$. They transform as

$$\vartheta_2(-1/\tau) = \sqrt{\frac{\tau}{i}}\vartheta_4(\tau), \qquad \vartheta_2(\tau+1) = e^{\frac{i\pi}{4}}\vartheta_2(\tau), \tag{H.10}$$

$$\vartheta_3(-1/\tau) = \sqrt{\frac{\tau}{i}}\vartheta_3(\tau), \qquad \vartheta_3(\tau+1) = \vartheta_4(\tau), \tag{H.11}$$

$$\vartheta_4(-1/\tau) = \sqrt{\frac{\tau}{i}}\vartheta_2(\tau), \qquad \vartheta_4(\tau+1) = \vartheta_3(\tau) \ . \tag{H.12}$$

Further, the eta-function is defined by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) \ , \tag{H.13}$$

and transforms according to

$$\eta(\tau+1) = e^{\frac{i\pi}{12}}\eta(\tau), \qquad \eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\,\eta(\tau) \ . \tag{H.14}$$

The Eisenstein series are defined by

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\frac{n^{k-1}q^n}{1-q^n} \ , \tag{H.15}$$

where B_k denotes the k-th Bernoulli number. E_k is a modular form of weight k for $k > 2$ and even.

H.3. Modular properties of the elliptic genus

We denote by $Z_P^{(r)}(\tau, z)$ the elliptic genus of r M5-branes wrapping P as defined previously in sect. 5.2.2. The elliptic genus should transform like a Jacobi form of bi-weight $(-\frac{3}{2}, \frac{1}{2})$ and bi-index $(\frac{r}{2}(d_{AB}-\frac{J_AJ_B}{J^2}),\frac{r}{2}\frac{J_AJ_B}{J^2})$ under the full modular group. In particular, we impose

$$\begin{aligned} Z_P^{(r)}(\tau+1,z) &= \varepsilon(T)\, Z_P^{(r)}(\tau,z), \\ Z_P^{(r)}(-\frac{1}{\tau},\frac{z_-}{\tau}+\frac{z_+}{\bar\tau}) &= \varepsilon(S)\,\tau^{-\frac{3}{2}}\bar\tau^{\frac{1}{2}}e^{\pi i r(\frac{z_-^2}{\tau}+\frac{z_+^2}{\bar\tau})}\, Z_P^{(r)}(\tau,z)\ , \end{aligned} \tag{H.16}$$

where ε are certain phases [310].

Siegel-Narain theta-function and its properties

Let us start by recalling the definition of the Siegel-Narain theta-function of eq. (5.13)

$$\theta_{\mu,J}^{(r)}(\tau,z)=\sum_{\xi\,\in\,\Lambda+\frac{[P]}{2}}(-)^{r(\xi+\mu)\cdot[P]}\bar q^{-\frac{r}{2}(\xi+\mu)_+^2}q^{\frac{r}{2}(\xi+\mu)_-^2}e^{2\pi i r(\xi+\mu)\cdot z}\ , \tag{H.17}$$

where we define

$$\xi_+^2=\frac{(\xi\cdot J)^2}{J\cdot J},\quad \xi_-^2=\xi^2-\xi_+^2\ . \tag{H.18}$$

Note, that $\xi_+^2<0$ if J lies in the Kähler cone.

If we denote by $\mathcal{D}_k=\partial_{\bar\tau}+\frac{i}{4\pi k}\partial_{z_+}^2$, the theta-function fulfills the heat equation

$$\mathcal{D}_r\,\theta_{\mu,J}^{(r)}(\tau,z)=0\ . \tag{H.19}$$

Further, we denote by Λ^* the dual lattice to Λ w.r.t. the metric rd_{AB}. For $\mu\in\Lambda^*/\Lambda$, we can deduce the following set of transformation rules

$$\begin{aligned} \theta_{\mu,J}^{(r)}(\tau+1,z) &= (-1)^{r(\mu+\frac{[P]}{2})^2}\theta_{\mu,J}^{(r)}(\tau,z), \\ \theta_{\mu,J}^{(r)}(-\frac{1}{\tau},\frac{z_+}{\bar\tau}+\frac{z_-}{\tau}) &= \frac{(-1)^{r\frac{[P]^2}{2}}}{\sqrt{|\Lambda^*/\Lambda|}}(-i\tau)^{\frac{r(\Lambda)-1}{2}}(i\bar\tau)^{\frac{1}{2}}e^{\pi i r(\frac{z_-^2}{\tau}+\frac{z_+^2}{\bar\tau})}\sum_{\delta\,\in\,\Lambda^*/\Lambda}e^{-2\pi i r\mu\cdot\delta}\theta_{\delta,J}^{(r)}(\tau,z)\ . \end{aligned} \tag{H.20}$$

Rank one

At rank one we have the universal answer

$$f_{\mu,J}^{(1)}(\tau)=\frac{\vartheta_{\Lambda^\perp}(\tau)}{\eta(\tau)^\chi}\ . \tag{H.21}$$

The transformation rules are simply given by (H.14) for the eta-function and for $\vartheta_{\Lambda^\perp}$ we obtain (assuming $\Lambda^\perp$ even and self-dual)

$$\begin{aligned} \vartheta_{\Lambda^\perp}(\tau+1) &= \vartheta_{\Lambda^\perp}(\tau)\ , \\ \vartheta_{\Lambda^\perp}(-\frac{1}{\tau}) &= \left(\frac{\tau}{i}\right)^{\frac{r(\Lambda^\perp)}{2}}\vartheta_{\Lambda^\perp}(\tau)\ . \end{aligned} \tag{H.22}$$

Rank two

Using Zwegers' theta-function with characteristics $\vartheta_{a,b}^{c,c'}(\tau)$ given in def. 2.1 of his thesis [61], we can write

$$\widehat\Theta_{\Lambda,\mu}^{c,c'}(\tau,x)=q^{-\frac{1}{2}\langle a,a\rangle}e^{-2\pi i\langle a,b\rangle}\vartheta_{a+\mu,b}^{c,c'}(\tau)\ , \tag{H.23}$$

where $x = a\tau + b$, i.e.

$$a = \frac{\mathrm{Im}(x)}{\mathrm{Im}(\tau)}, \quad b = \frac{\mathrm{Im}(\bar{x}\tau)}{\mathrm{Im}(\tau)} \ . \tag{H.24}$$

Following Corollary 2.9 of Zwegers [61], we can deduce the following set of transformations

$$\begin{aligned} \widehat{\Theta}^{c,c'}_{\Lambda,\mu}(\tau+1,x) &= (-1)^{\langle\mu,\mu\rangle}\widehat{\Theta}^{c,c'}_{\Lambda,\mu}(\tau,x) \ , \\ \widehat{\Theta}^{c,c'}_{\Lambda,\mu}(-\frac{1}{\tau},\frac{x}{\tau}) &= \frac{i(-i\tau)^{r(\Lambda)/2}}{\sqrt{|\Lambda^*/\Lambda|}} e^{\pi i \frac{\langle x,x\rangle}{\tau}} \sum_{\delta\,\in\,\Lambda^*/\Lambda} e^{-2\pi i\langle\delta,\mu\rangle}\,\widehat{\Theta}^{c,c'}_{\Lambda,\delta}(\tau,x) \ . \end{aligned} \tag{H.25}$$

This input enables us to write down the transformation rules for $\hat{f}^{(2)}_{\mu,J}$. They read

$$\begin{aligned} \hat{f}^{(2)}_{\mu,J}(\tau+1) &= (-1)^{\frac{\chi}{6}+2\mu^2}\hat{f}^{(2)}_{\mu,J}(\tau) \ , \\ \hat{f}^{(2)}_{\mu,J}(-\frac{1}{\tau}) &= -\frac{(-i\tau)^{-\frac{r(\Lambda)+2}{2}}}{\sqrt{|\Lambda^*/\Lambda|}} \sum_{\delta\,\in\,\Lambda^*/\Lambda} e^{4\pi i\delta\cdot\mu}\hat{f}^{(2)}_{\delta,J}(\tau) \ . \end{aligned} \tag{H.26}$$

This gives the conjectured transformation properties (H.16).

The blow-up factor

For completeness we elaborate on the transformation properties of the blow-up factor. We define

$$B_{r,k}(\tau) = \eta(\tau)^{-r} \sum_{a_i\,\in\,\mathbb{Z}+\frac{k}{r}} q^{\sum_{i\le j\le r-1} a_i a_j} \ . \tag{H.27}$$

We can deduce the following set of transformation rules

$$\begin{aligned} B_{r,k}(\tau+1) &= (-1)^{\frac{r}{12}+\frac{k^2(r-1)}{r}} B_{r,k}(\tau) \ , \\ B_{r,k}(-\frac{1}{\tau}) &= \frac{1}{\sqrt{r}}\left(\frac{\tau}{i}\right)^{-\frac{1}{2}} \sum_{0\le l\le r-1} (-1)^{\frac{2kl(r-1)}{r}} B_{r,l}(\tau) \ . \end{aligned} \tag{H.28}$$

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