The invariant polarisation–tensor field for deuterons in storage rings and the Bloch equation for the polarisation-tensor density

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Abstract

I extend and update earlier work, summarised in [1], whereby the invariant polarisation–tensor field (ITF) for deuterons in storage rings was introduced to complement the invariant spin field (ISF). Taken together, the ITF and the ISF provide a definition of the equilibrium spin density–matrix field which, in turn, offers a clean framework for describing equilibrium spin-1 ensembles in storage rings. I show how to construct the ITF by stroboscopic averaging, I give examples, I discuss adiabatic invariance and I introduce a formalism for describing the effect of noise and damping.

Contents

1 Introduction

This paper explores the concept of the invariant polarisation–tensor field for charged spin-1 particles such as deuterons in storage rings.

Spin motion for charged particles moving in electric and magnetic fields is governed by the T–BMT equation [2]. This describes the rate of precession of the rest–frame, pure–state, spin expectation value $\vec{S}$ ("the spin") of a particle and in general, the independent variable is the time. In circular particle accelerators and storage rings the electric and magnetic guide fields are fixed in space so that it is more convenient to take the distance s around the ring as the independent variable [3]. The motion of a particle is governed by the Lorentz force [2] and I describe particle motion in the 6–dimensional phase space in terms of the position u relative to the 6–dimensional closed (periodic) orbit. Then, at the position s and the point u in phase space, I write the T–BMT equation as $d\vec{S}/ds = \vec{\Omega}(u(s);s) \times \vec{S}$ where the vector $\vec{\Omega}(u(s);s)$ describing the precession axis and the rate of precession, depends on the electric and magnetic fields in the laboratory, and on the reference energy of the ring. Thus both the motion of the particle and the motion of the spin expectation value can be treated classically. Nevertheless, as we shall see, for some aspects of spin motion we still need to look at the quantum mechanics. For this I exploit the spin density matrix.

Earlier works have emphasised the utility of the invariant spin field (ISF) for describing equilibrium spin distributions for beams of spin-1/2 particles and have shown how the amplitude dependent spin tune (ADST) can be exploited [4, 5, 6, 7, 8, 9, 10]. In this paper I embrace spin-1 particles too by introducing the concept of the invariant field of the Cartesian polarisation tensor, which I call the "invariant tensor field" (ITF). Then I show how to define equilibrium spin density–matrix fields (EDMF) [1] in terms of the ISF and ITF and explain how an EDMF can be diagonalised by rotations of the coordinate system when a certain ansatz for the ITF is valid. I continue the discussion by giving examples of ensembles with and without EDMFs. Following this, I address the matter of adiabatic invariants and then mention other representations for the spin-1 density matrix. The paper is rounded off by extending earlier work on the influence of noise and damping on the evolution of the so–called vector–polarisation density to include their effect on the tensor–polarisation density.

Of course, the mathematics surrounding the density matrix for spin-1 particles was established decades ago [11, 12, 13] but this work shows how to embed that mathematics within modern structures for systematising spin motion in storage rings.

I begin by reviewing the concept of the ISF for spin-1/2 particles.

2 Equilibrium spin distributions for spin-1/2 particles

The properties of a mixed spin state for spin-1/2 particles are completely defined by the 2×2 density matrix ρ [11]. The density matrix is hermitian and its trace is constrained to a definite value (in fact unity) and it is therefore defined by just three real parameters which

[1] In [1] the longer acronym, ESDeMF, was used.

can be chosen as the three components of the vector polarisation $\vec{P}$. Then I write

$$\rho = \frac{1}{2}\{I_{2\times 2} + \vec{P}\cdot\vec{\sigma}\}\ , \tag{2.1}$$

where the $\vec{\sigma}$ is the matrix–valued 3–vector formed from the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \ \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \ \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \tag{2.2}$$

representing the normalised spin operators $\hat{s}_i$ $(i = 1, 2, 3)$ and I is the unit matrix. The labels 1, 2 and 3 are chosen to be consistent with those in [8, 14] and refer to the axes of a right–handed Frenet–Serret coordinate system attached to the design orbit whereby in the arcs of a flat ring, axis 2 is vertical, axis 1 is radial and axis 3 is longitudinal. For spin-1/2 particles we normalise the length of $\vec{S}$ to unity. The polarisation vector is the mixed–state expectation value of the normalised spin operator and it can be written as $\vec{P} = \mathrm{Tr}(\rho\vec{\sigma})$. Its components are $P_i = \langle\hat{s}_i\rangle$ where the brackets $\langle\rangle$ signal computing the expectation value. I now follow the argumentation in [15, Section 8.6.1] whereby in accelerators the statistical properties of a spin-orbit system can be expressed in terms of a spin-orbit Wigner function [16] and I then write a density matrix as a function of classical variables, namely the position u in phase space and the distance s around the ring. In Sections 2, 3 and 4, the classical orbital motion is defined by a Hamiltonian so that the density in phase space is conserved along a particle trajectory. This allows us to focus our attention on the spin density matrix which I write in terms of the local vector polarisation $\vec{P}_{\mathrm{loc}}(u;s)$ [16, 17, 9, 10] as $\rho = \frac{1}{2}\{I + \vec{P}_{\mathrm{loc}}(u;s)\cdot\vec{\sigma}\}$. The degree of polarisation at a point in phase space, $P_{\mathrm{loc}} = |\vec{P}_{\mathrm{loc}}|$, is at most unity. Of course, since the density of particles in phase space is finite, the notion of a mixed spin state at each position $(u;s)$ is an idealisation, but this and other idealisations in this paper will not detract from the value of the basic concepts presented. In Section 6 I show how to proceed when the phase space density is not preserved along a trajectory.

2.1 The invariant spin field

Since the T–BMT equation is linear in $\vec{S}$ and since the particles at $(u;s)$ all see the same $\vec{\Omega}(u;s)$, $\vec{P}_{\mathrm{loc}}(u(s);s)$ obeys the T–BMT equation along trajectories. Of course, the same result emerges by exploiting the equation of motion for ρ [18, Section 1–8c], [16]. Furthermore, P_{loc} is constant along a trajectory. For a storage ring at fixed energy, $\vec{\Omega}$ is 1–turn periodic in s at a fixed position in phase space u so that $\vec{\Omega}(u;s) = \vec{\Omega}(u;s+C)$ where C is the circumference. This opens the possibility of a configuration for the polarisation that is the same from turn to turn in the sense that $\vec{P}_{\mathrm{loc}}(u;s)$ is 1–turn periodic in s for fixed u, i.e., $\vec{P}_{\mathrm{loc}}(u;s+C) = \vec{P}_{\mathrm{loc}}(u;s)$. For reasons that will become clear I denote such a $\vec{P}_{\mathrm{loc}}$ by $\vec{P}_{\mathrm{eq}}$. Since $\vec{P}_{\mathrm{eq}}$ also obeys the T–BMT equation we then have

$$\vec{P}_{\mathrm{eq}}(M(u;s+C,s);s+C) = \vec{P}_{\mathrm{eq}}(M(u;s+C,s);s) = R(u;s+C,s)\vec{P}_{\mathrm{eq}}(u;s)\ , \tag{2.3}$$

where $M(u;s+C,s)$ is the new position in phase space after one turn starting at u and s, $R(u;s+C,s)$ is the corresponding 3×3 spin transfer matrix representing the solution to

the T–BMT equation for one turn from s to $s + C$ and where here, $\vec{P}_{\rm eq}$ is represented by a column vector of its components. By writing the T–BMT equation in matrix form we have

$$\frac{d}{ds}R(u(s); s, s_0) = \tilde{\Omega}(u(s); s)R(u(s); s, s_0) \ , \tag{2.4}$$

where

$$\tilde{\Omega} = \begin{pmatrix} 0 & -\vec{\Omega}_3 & \vec{\Omega}_2 \\ \vec{\Omega}_3 & 0 & -\vec{\Omega}_1 \\ -\vec{\Omega}_2 & \vec{\Omega}_1 & 0 \end{pmatrix} \ . \tag{2.5}$$

Of course, since it represents a pure rotation of a 3-vector, R is $SO(3)$-valued, i.e., $R^{\rm T}R = I_{3\times 3}$ and $\det(R) = 1$. Also, note that the fact that $R \in SO(3)$ follows easily from the fact that $\tilde{\Omega}$ is antisymmetric [19].

The relations (2.3) motivate the introduction of a vector *field* $\hat{n}(u; s)$ of fixed, and in particular, unit length, obeying similar constraints, namely

$$\hat{n}(M(u; s + C, s); s + C) = \hat{n}(M(u; s + C, s); s) = R(u; s + C, s)\hat{n}(u; s) \ , \tag{2.6}$$

where $\hat{n}$ is represented by a column vector of its components. Where appropriate, a similar convention will be used in the rest of this paper. When it exists, the field $\hat{n}$ is simply a 3–vector function of u and s obeying the T–BMT equation and the periodicity conditions in (2.6). No reference to real particles and their spin states is required. Since the vector field $\hat{n}(u; s)$ is invariant from turn to turn and independent of the real state of a beam it is called the invariant spin field (ISF). The ISF can be used to define the amplitude dependent spin tune (ADST) and together they provide a most elegant way to systematise spin motion in storage rings and circular accelerators [4, 5, 6, 7, 8, 9, 10, 20]. Note that if all parameters of the system, such as the energy, are fixed, the scalar product $I_{\rm sn} = \vec{S} \cdot \hat{n}$ is invariant along a particle trajectory, since both vectors obey the T–BMT equation. Thus the motion of $\vec{S}$ is simply a precession around the local $\hat{n}$.

On the closed orbit, $u = 0$ and I denote the vector $\hat{n}$ on the closed orbit by $\hat{n}_0(s) := \hat{n}(0; s)$. The vector $\hat{n}_0(s)$ is 1–turn periodic and is given by the real, unit–length eigenvector of $R(0; s + C, s)$.

For the remainder of the paper I assume that the orbital motion is integrable to a good approximation so that u can be parametrised in terms of three pairs of action–angle variables $(J_i, \phi_i, \ i = 1, 2, 3)$ which I abbreviate by (J, ϕ). I will use the symbols u and (J, ϕ) interchangeably. The actions J are constants of the motion. Thus the orbital phase space is partitioned into disjoint tori, each of which is characterised by a unique set J. I also assume that the orbital motion is nonresonant so that, in time, a trajectory covers its torus. In [15] this is referred to as topological transitivity. Then from (2.3) the norm, $P_{\rm eq}$, of $\vec{P}_{\rm eq}$ must be the same at all points ϕ on a torus but it can depend on the J. Furthermore I will consider only ISFs which are continuous in ϕ and I avoid spin–orbit resonances [9, 10]. Then apart from a global sign, $\hat{n}(J, \phi; s)$ is a unique function of ϕ and s, 2π–periodic in ϕ [9, 10]. So I may write $\vec{P}_{\rm eq}(J, \phi; s) = P_{\rm eq}(J)\,\hat{n}(J, \phi; s)$, bearing in mind that $|\hat{n}| = 1$. The requirement

that $\hat{n}$ be continuous in ϕ is chosen to reflect the expectation that the polarisation $\vec{P}_{\rm eq}$ in a real ring varies continuously in ϕ, and it corresponds to the requirement in [15] [2].

The key aspects of the ISF and the ADST can be summarised as follows.

1. For a turn–to–turn invariant particle distribution in phase space, a distribution of spins in which each is initially aligned parallel to the ISF at its position in phase space, remains invariant (i.e., in equilibrium) from turn to turn, and the ISF gives the direction of the equilibrium polarisation $\vec{P}_{\rm eq}$ at each $(u; s)$. Of course, the polarisation for the whole torus, defined as the average $P_{\rm eq}(J)/(2\pi)^3 \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \hat{n}(J, \phi; s) d\phi$, is invariant from turn to turn. An example in which the polarisation for the torus is not invariant from turn to turn is presented in [5, 7].

2. For integrable orbital motion and away from orbital resonances and spin–orbit resonances [9, 10] the ISF determines both the maximum attainable time-averaged polarisation and the maximum equilibrium polarisation, $P_{\rm lim} = |\langle \hat{n}(J, \phi; s) \rangle_\phi|$, on a phase space torus at each s, where the brackets $\langle \rangle_\phi$ denote the average over the orbital phases.

3. Away from orbital resonances and spin-orbit resonances, $I_{\rm sn}$ is an adiabatic invariant along a particle trajectory, i.e., it does not change as a parameter such as the reference energy is slowly varied. See [22] for details. So, if a $\vec{S}$ at some $(u; s)$ is set parallel to the $\hat{n}(u; s)$ and then propagated forwards with R as the parameters are slowly varied, the $\vec{S}$, which is changing *dynamically* finds itself parallel to the *pre-calculated* $\hat{n}(u'; s')$ at each new position $(u'; s')$ along the trajectory.

4. The ISF provides the main axis for orthonormal coordinate systems constructed at each point in phase space for defining the ADST. This, in turn, is used to define the concept of spin–orbit resonance. Away from orbital resonance and spin–orbit resonance, $\hat{n}(u; s)$ is unique up to a global sign [9, 10]. These coordinate systems comprise the so-called invariant frame field (IFF) mentioned later.

5. On the closed orbit, the ADST reduces to the number of precessions of a spin, per turn, around $\hat{n}_0$. I denote this spin tune by ν_0. Its fractional part can be extracted from the complex eigenvalues $e^{\pm 2\pi i \nu_0}$ of $R(\vec{0}; s + C, s)$. For a perfectly aligned flat ring with no solenoids, $\nu_0 = a\gamma_0$ where a is the gyromagnetic anomaly and γ_0 is the Lorentz factor for the beam energy.

These and other matters are explained and illustrated in great detail in the sources cited above. In order to limit this paper to a reasonable length I will assume that the reader is familiar with that material.

[2] Note that, for most rational orbital tunes an $\hat{n}$ can be extracted trivially as the normalised real eigenvector of a multi–turn spin map [14]. However, for the subset of rational tunes corresponding to so-called "'snake resonance" the vector obtained in this way is discontinuous in ϕ and it then does not match our requirements. Moreover, an infinite number of totally discontinuous invariant fields satisfying (2.6) can easily be envisaged through the so–called "filling–up method" [21].

The most general, *model–independent* way, to construct the ISF is by so–called stroboscopic averaging [5, 7, 8, 21]. This just requires a spin–orbit tracking code such as those listed in [8] and [23] which deliver the matrices R along particle orbits. As explained in [5, 7, 8, 21], with stroboscopic averaging the ISF, $\hat{n}(u_0; s_0)$, at the starting positions u_0 and $s = s_0$ can be found in terms of multi–turn spin transfer matrices by taking the average

$$\vec{f}_N(u_0; s_0) := \frac{1}{N+1} \sum_{k=0}^{N} R(u(s_0 - kC); s_0, s_0 - kC))\hat{n}_0(s_0) \; , \tag{2.7}$$

for very large N and normalising this to unity: $\hat{n}(u_0; s_0) = \vec{f}_N(u_0; s_0)/|\vec{f}_N(u_0; s_0)|$. In this expression I have used notation similar to that in [5, eq. 22] and chosen $\hat{n}_0$ as the "seed" spin field, although more general choices could be used [21]. If the orbital motion is integrable and nonresonant, the vector $\hat{n}$ need only be calculated by this means at one position (u_0, s_0). After that, $\hat{n}$ can be found all over the corresponding torus by propagating this initial $\hat{n}$ along the trajectory. Thus for integrable, nonresonant orbital motion, there is usually *no* need to execute stroboscopic averages on a grid of pre–chosen positions ϕ on a torus since the approach just suggested suffices for obtaining the significant information, namely averages.

Typical plots of $P_{\lim}$, the ADST or the components of the ISF, can be found in [5, 7, 8, 14, 24]. The figures in [25] show how to confirm that the $\hat{n}$ obtained by forward propagation is a single valued function of the position in phase space.

For detailed discussions on convergence of stroboscopic averages, see [21, 5, 7]. See [15] for a framework for addressing the question of the existence of the ISF for topologically transitive systems.

2.2 The equilibrium spin density–matrix field for spin-1/2 particles

The existence of a 1–turn periodic $\vec{P}_{\rm eq}$ and the corresponding ISF implies the existence of an equilibrium spin density–matrix field (EDMF), $\rho^{\rm eq}$, obeying the periodicity condition $\rho^{\rm eq}(u; s + C) = \rho^{\rm eq}(u; s)$ and which we can write in the form:

$$\rho^{\rm eq}_{P_{\rm eq}(J)}(J, \phi; s) = \frac{1}{2}\{I + P_{\rm eq}(J)\, \hat{n}(J, \phi; s) \cdot \vec{\sigma}\} \; , \tag{2.8}$$

in an obvious notation. Different ensembles of spins at a $(u; s)$ with the same $\vec{P}_{\rm eq}$ cannot be distinguished by measurements from a mixture of spins in eigenstates of the operator $\hat{n} \cdot \vec{\sigma}$ with that $\vec{P}_{\rm eq}$. The corresponding density matrix at each $(u; s)$ is diagonal in a coordinate system in which the components of $\hat{n}$ are $(0,\ 1,\ 0)$. The invariant frame field (IFF) [9] provides such coordinate systems [3] .

[3] Note that in contrast to our treatment, $\hat{n}$ has components $(0,\ 0,\ 1)$ in the IFF of [9].

3 Equilibrium spin distributions for spin-1 particles

Spin–1 particles such as deuterons have three eigenvalues, namely $+1, 0, -1$, for the projection of the normalised spin operator $\hat{s}$ onto a chosen quantisation axis, and a 3×3 density matrix. Since it is hermitian and its trace is constrained to a definite value, namely to unity, this density matrix can be completely specified in terms of eight real parameters. Three of these can be the components of a vector polarisation $\vec{P}$ analogous to those for spin-1/2 particles and the other five are the so–called tensor polarisations. Various representations of the latter are in use. For us, a particularly useful parametrisation for the density matrix ρ is that given in [11, Section 3.1.12] in terms of a rank–2, 3×3, real, symmetric, traceless, Cartesian tensor T as:

$$\rho = \frac{1}{3}\left\{ I_{3\times 3} + \frac{3}{2}\vec{P}\cdot\vec{\mathfrak{J}} + \sqrt{\frac{3}{2}}\sum_{i,j} T_{ij}\ (\mathfrak{J}_i\mathfrak{J}_j + \mathfrak{J}_j\mathfrak{J}_i) \right\}\ , \tag{3.1}$$

where the three matrices $\mathfrak{J}$

$$\mathfrak{J}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix},\ \mathfrak{J}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},\ \mathfrak{J}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\ , \tag{3.2}$$

representing the normalised spin operators $\hat{s}_i\ \ (i = 1, 2, 3)$, are the analogues for spin-1 of the Pauli matrices and where $\vec{\mathfrak{J}}$ is the corresponding matrix–valued 3–vector. The tensor T has just five independent components and is irreducible in that it contains no non–zero tensors of lower rank [26, 27]. Since $\mathrm{Tr}(\mathfrak{J}_j\mathfrak{J}_i) = 2\delta_{ij}$ where δ_{ij} is the Kronecker delta, the tracelessness of the tensor T ensures that $\mathrm{Tr}(\rho) = 1$, as required. The matrices $\mathfrak{J}$ are cyclically permuted with respect to the corresponding matrices in [11] because of the labeling of the axes explained in Section 2. See the Appendix for an informal way to derive (3.1).

The vector polarisation, with its components $P_i = \langle \hat{s}_i \rangle$, is now [4] $\vec{P} = \mathrm{Tr}(\rho\vec{\mathfrak{J}})$ and again, $P = |\vec{P}|$ is at most unity. However, the components of T which are also needed for the spin-1 density matrix, depend on quadratic combinations of spin operators. In particular [11],

$$T_{ij} = \frac{1}{2}\sqrt{\frac{3}{2}}\left\{ \langle \hat{s}_i\hat{s}_j + \hat{s}_j\hat{s}_i \rangle - \frac{4}{3}\delta_{ij} \right\}\ . \tag{3.3}$$

That this is consistent with (3.1) can be seen by computing the $\langle \hat{s}_i\hat{s}_j \rangle$ using $\mathrm{Tr}(\rho\mathfrak{J}_i\mathfrak{J}_j)$. The degree of tensor polarisation is defined as $\mathfrak{T} := \sqrt{\sum_{i,j} T_{ij}^2} = \sqrt{\mathrm{Tr}(TT^{\mathrm{T}})} = \sqrt{\mathrm{Tr}(T^2)}$ and it is at most unity [11]. If $\mathfrak{T}^2 = 0$, $T = 0_{3\times 3}$. Another convention for defining the degree of tensor polarization is described in Section 4.

In contrast to the case of spin-1/2 particles, spin-1 particles can exist in pure states for which $|\vec{S}|$ is zero. In that case $|\vec{S}|$ cannot be normalised to unity [28]. If $|\vec{S}|$ is zero I_{sn} vanishes and it is automatically invariant. For the other pure states, $|\vec{S}|$ can be normalised to unity. The overall degree of polarisation for spin-1 particles is $\frac{3}{4}P^2 + \mathfrak{T}^2$ [11, eqs. 3.1.55, 3.1.65]. So an ensemble is unpolarised only when *both* P and $\mathfrak{T}$ are zero. In that case $\rho = \frac{1}{3}I$ so that the probabilities for the three substates are equal at 1/3.

[4] Note that $\mathrm{Tr}(\mathfrak{J}_i(\mathfrak{J}_j\mathfrak{J}_k + \mathfrak{J}_k\mathfrak{J}_j)) = 0\ \forall\ i, j, k$.

3.1 The invariant tensor field

Given the existence of ISFs in most cases of interest, it is natural to ask whether invariant tensor *fields* (ITF) can exist. I now examine this possibility by defining the ITF and then suggesting how to construct it by stroboscopic averaging and then in terms of the ISF.

For this I use the analogue for T of solutions of the T–BMT equation for spin. In particular, if along a trajectory, an initial spin, $\vec{S}^{\rm i}$, is transformed into a final spin, $\vec{S}^{\rm f}$, by a spin transfer matrix R according to the relation

$$\vec{S}^{\rm f} = R\ \vec{S}^{\rm i}\ , \tag{3.4}$$

then the components of T are transformed according to the rule [26]

$$T^{\rm f} = R\ T^{\rm i} R^{\rm T}\ . \tag{3.5}$$

Of course, $\vec{P}^{\rm f} = R\,\vec{P}^{\rm i}$. The similarity transform, (3.5) conserves the trace and the symmetry of the tensor as well as $\mathfrak{T}$. Moreover,

$$\frac{dT}{ds}\ =\ [\,\tilde{\Omega},\ T] := \tilde{\Omega}\,T - T\,\tilde{\Omega} \tag{3.6}$$

while

$$\frac{d\vec{P}}{ds}\ =\ \tilde{\Omega}\,\vec{P}\ .$$

I emphasise the fact, obvious from (3.4) and (3.5), that as soon as the matrix R for rotating spins is known, the transformation for T follows trivially. Of course, T remains traceless and symmetric even if a parameter such as the reference energy is varied along a trajectory since the $\tilde{\Omega}$ in (3.6) is still antisymmetric while the R in (3.5) therefore remains orthogonal [19].

I now define the ITF $T^{\rm I}$, in analogy with the definition for the ISF, by the periodicity condition

$$T^{\rm I}(M(u;s+C,s);s+C) = T^{\rm I}(M(u;s+C,s);s) = R(u;s+C,s)\ T^{\rm I}(u;s) R^{\rm T}(u;s+C,s) \tag{3.7}$$

with $\sqrt{{\rm Tr}(T^{\rm I}T^{\rm I})} = 1$. This normalisation is preserved along a trajectory since $(T^{\rm I})^2$ is transported as $R\ (T^{\rm I})R^{\rm T}R\ (T^{\rm I})R^{\rm T} = R\ (T^{\rm I})^2 R^{\rm T}$ and a similarity transformation preserves a trace. Just as with the ISF, when it exists, the ITF is simply a single valued tensor function of u and s obeying (3.5) and the periodicity conditions as in (3.7). No reference to real particles and their spin states is required. Intuition suggests that the ITF is unique up to a global sign away from orbital resonances and spin–orbit resonances. Stroboscopic averaging to calculate the ISF is grounded in the linearity of the T-BMT equation and the fact that $\vec{\Omega}(u;s)$ is 1–turn periodic in s. The equation of motion (3.6) for T is also linear. So I now postulate that the ITF can also be constructed using stroboscopic averaging:

$$g_N(u_0;s_0) := \frac{1}{N+1}\sum_{k=0}^{N} R(u(s_0-kC);s_0,s_0-kC)\ T(s_0)\ R^{\rm T}(u(s_0-kC);s_0,s_0-kC) \tag{3.8}$$

where N is very large and $T(s_0)$ is a fixed 3×3 symmetric matrix with zero trace. The symmetry and tracelessness ensure that the 3×3 matrix $g_N(u_0; s_0)$ is traceless and symmetric. The ITF should be obtained as $T^{\rm I}(u_0; s_0) = g_N/\sqrt{{\rm Tr}(g_N^2)}$.

In fact numerical tests away from orbital resonances and spin-orbit resonances in a typical ring confirm that stroboscopic averaging as in (3.8) does indeed deliver a tensor T with the required properties, namely those of a $T^{\rm I}$. In particular when one propagates this $T^{\rm I}(u_0; s_0)$ forward turn-by-turn to obtain T^I at s_0 all over the corresponding torus, one finds a single valued tensor function of the position in phase space u, as required. Moreover, if the stroboscopic average in (3.8) is calculated for many different, non–zero, traceless, symmetric matrices $T(s_0)$, then, away from orbital resonances and spin–orbit resonances, the same normalised stroboscopic average is obtained (up to a sign). This suggests, but does not prove, that the ITF is unique up to a global sign.

Given the simplicity of the rule (3.5) it is also tempting to try to express the ITF in terms of the ISF, and the form:

$$T^{\rm I} = \pm\sqrt{\frac{3}{2}}\left\{\hat{n}\hat{n}^{\rm T} - \frac{1}{3}I_{3\times3}\right\} \ , \tag{3.9}$$

suggests itself. The term with the 3×3 unit matrix ensures that the tensor is traceless and the factor $\sqrt{\frac{3}{2}}$ ensures the chosen normalisation. In fact one finds in numerical tests that the ansatz (3.9) agrees perfectly with the normalised stroboscopic average of (3.8).

Since a spin transfer matrix R is orthogonal, it is trivial that this ansatz satisfies (3.5). Note that the required periodicity conditions (3.7) are fulfilled owing to the analogous periodicity of $\hat{n}$. The ansatz (3.9) clearly satisfies all requirements for $T^{\rm I}$.

Later we shall see that the ansatz (3.9) is supported by another consideration based on requirements for pure states. In fact, as argued in [29], the ansatz (3.9) is unique if the ISF is unique and it then takes this form. The same conclusion is reached in a broader context and in a more powerful manner in [15, Section 8] whereby invariant fields are associated with symmetry groups. I therefore accept the ansatz of (3.9) from here on. Moreover it will suffice to choose the + sign. Then, for example, $T^{\rm I}_{2,2}(J, \phi; s) = \sqrt{\frac{2}{3}}(\frac{3}{2}\cos^2\theta - \frac{1}{2})$ where $\theta = \cos^{-1}(\hat{n}_2(J, \phi; s))$.

Since $T^{\rm I}$ is unique, it can be obtained using (3.9) as soon as the ISF is known, without further stroboscopic averaging. Then, for example, in a parameter regime where the so–called single resonance model [14, 25, 30] provides an approximation to the spin dynamics, one not only has an approximate analytical formula for the ISF, but also for the ITF.

3.2 The EDMF for spin-1 particles

The kind of argument that leads to the concept of the equilibrium polarisation $\vec{P}_{\rm eq}(J, \phi; s) = P_{\rm eq}\,\hat{n}(J, \phi; s)$ on a torus, also leads to the concept of the equilibrium polarisation tensor $T_{\rm eq}(J, \phi; s) := \mathfrak{T}_{\rm eq}(J)\,T^{\rm I}(J, \phi; s)$ where $\mathfrak{T}_{\rm eq}(J)$ is the degree of equilibrium tensor polarisation. If the particles are distributed uniformly in ϕ on the torus J and $T(J, \phi; s) = T_{\rm eq}(J, \phi; s)$ at

some s, the polarisation tensor for the torus, defined as $\mathfrak{T}_{\rm eq}(J)/(2\pi)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} T^{\rm I}(J,\phi;s)d\phi$, is invariant from turn to turn.

With the ITF we can now construct an EDMF for spin-1 particles:

$$\rho^{\rm eq}_{[P_{\rm eq},\mathfrak{T}_{\rm eq}](J)}(J,\phi;s) = \frac{1}{3}\left\{ I + \frac{3}{2}P_{\rm eq}(J)\hat{n}\cdot\vec{\mathfrak{J}} + \sqrt{\frac{3}{2}}\mathfrak{T}_{\rm eq}(J)\sum_{i,j} T^{\rm I}_{ij}\ (\mathfrak{J}_i\mathfrak{J}_j + \mathfrak{J}_j\mathfrak{J}_i)\right\}\ . \quad (3.10)$$

So on a torus, the EDMF is defined by two free parameters, $P_{\rm eq}(J)$ and $\mathfrak{T}_{\rm eq}(J)$. As required, $\mathrm{Tr}(\rho^{\rm eq}) = 1$ for any pure or mixed state. However, for a pure state we require that $\mathrm{Tr}(\rho^2) = 1$. Then in a pure state, from (3.10) and (3.9) and after some matrix algebra, I find the constraint

$$1 = \frac{1}{9}\left\{3 + \frac{9}{2}P^2_{\rm eq} + 6\mathfrak{T}^2_{\rm eq}\right\} \quad \Rightarrow \quad 1 = \frac{3}{4}P^2_{\rm eq} + \mathfrak{T}^2_{\rm eq}\,. \quad (3.11)$$

which is clearly independent of the precise values of the components of $\hat{n}$ and $T^{\rm I}$. In this context it should be emphasised that the ISF and the ITF just encode the relative sizes of their components and that to specify actual equilibrium spin distributions on tori we need $P_{\rm eq}(J)$ and $\mathfrak{T}_{\rm eq}(J)$ too. Relation (3.11) corresponds to the fact that pure states are fully polarised according to the definition in [11, eqs. 3.1.55, 3.1.65].

As we are reminded in [11], although the density matrix for spin-1/2 particles can always be diagonalised by a rotation of the coordinate system, this is not always the case for spin-1 particles. However, it is always possible for the EDMF of spin-1 particles when the ansatz in (3.9) is valid since the ITF, and thus the EDMF, will be diagonal if $\hat{n}$ has components (0, 1, 0) and this can always be achieved at each $(u;s)$ by rotating the coordinate frame to make $\hat{n}$ vertical in that frame, i.e., by viewing the system in an IFF. According to the definition in [11], the ISF therefore defines the quantisation axes for the EDMF on a torus for spin-1 particles too. In particular, different ensembles of spins at a $(u;s)$ with the same $\vec{P}_{\rm eq}$ and $T_{\rm eq}$ cannot be distinguished by measurements from a mixture of spins in eigenstates of the operator $\hat{n}\cdot\vec{\mathfrak{J}}$ with those $\vec{P}_{\rm eq}$ and $T_{\rm eq}$. Of course, since all spin transformations here are rotations according to the matrices R, it is clear that if $\rho^{\rm eq}$ can be diagonalised at one point on a trajectory, it can be diagonalised by a rotation everywhere else along the trajectory.

A way to ensure that a tensor field is invariant at high energy is described in Section 4.

I now illustrate these concepts with simple examples in which I choose some special configurations of spins and then check to see if they can be described by EDMFs.

3.3 Examples

Example 1

Consider the case where all particles at some arbitrary $(J,\phi;s)$ are in the eigenstate of $\hat{n}\cdot\mathfrak{J}$ whose eigenvalue is $+1$ so that $P_{\rm loc} = 1$. I write this eigenstate as $|n^+\rangle$. The corresponding

normalised 3-spinor is

$$e^{i\chi}\begin{pmatrix} \frac{1+n_2}{2} \\ \frac{n_1+in_3}{\sqrt{2}} \\ \frac{1-n_2}{2}\frac{n_1+in_3}{n_1-in_3} \end{pmatrix} = e^{i(\chi+\psi)}\begin{pmatrix} \frac{1+n_2}{2}e^{-i\psi} \\ \frac{r}{\sqrt{2}} \\ \frac{1-n_2}{2}e^{+i\psi} \end{pmatrix} , \tag{3.12}$$

where χ is an arbitrary phase, $r=\sqrt{1-n_2^2}$, $n_1 = r\cos\psi$ and $n_3 = r\sin\psi$. Then, by evaluating the expectation values $\langle n^+|\ \mathfrak{J}_i\mathfrak{J}_j\ |n^+\rangle$ in (3.3) for $|n^+\rangle$, I find for the local polarisation tensor:

$$T_{\rm loc} = \frac{1}{2}\sqrt{\frac{3}{2}}\left\{\hat{n}\hat{n}^{\rm T} - \frac{1}{3}I\right\} . \tag{3.13}$$

So $T_{\rm loc}(J,\phi;s)$ is proportional to the $T^{\rm I}(J,\phi;s)$ of (3.9). The proportionality factor is $1/2$ and the same value will be obtained at any other $(J,\phi;s)$. Also, it is preserved as the ensemble moves over the torus. So there is an EDMF. In particular I have $T_{\rm eq}(J,\phi;s) = T_{\rm loc}(J,\phi;s) = T^{\rm I}(J,\phi;s)/2$, i.e., $T_{\rm eq}(J,\phi;s) := \mathfrak{T}_{\rm eq}(J)\,T^{\rm I}(J,\phi;s)$ where $\mathfrak{T}_{\rm eq}(J) = 1/2$. With $P_{\rm loc} = P_{\rm eq} = 1$, this value of $\mathfrak{T}_{\rm eq}(J)$ satisfies (3.11), as expected for a pure state. I can now write

$$\begin{aligned} \rho^{\rm eq} &= \left\{\frac{I}{3} + \frac{1}{2}P_{\rm eq}\hat{n}\cdot\vec{\mathfrak{J}} + \mathfrak{T}_{\rm eq}\sum_{i,j}(\hat{n}_i\hat{n}_j - \frac{1}{3}\delta_{ij})\frac{(\mathfrak{J}_i\mathfrak{J}_j + \mathfrak{J}_j\mathfrak{J}_i)}{2}\right\} \\ &= \left\{\frac{I}{3} + \frac{1}{2}\hat{n}\cdot\vec{\mathfrak{J}} + \frac{1}{2}\sum_{i,j}(\hat{n}_i\hat{n}_j - \frac{1}{3}\delta_{ij})\frac{(\mathfrak{J}_i\mathfrak{J}_j + \mathfrak{J}_j\mathfrak{J}_i)}{2}\right\} , \end{aligned} \tag{3.14}$$

where I have suppressed the dependences on J,ϕ and s for convenience. It is now clear that the ansatz (3.9) is natural if (3.10) is to be able to encompass these pure states on a torus. It is easily checked that this density matrix is idempotent as required for pure states.

Note that these states are the so–called coherent states appearing in the discussion of the matter of defining the classicality of spin states in [31]. There, the density operator for the pure state with a spin expectation value $\hat{\mathfrak{n}}$ of unit magnitude, is given by the projector

$$|\hat{\mathfrak{n}}\rangle\langle\hat{\mathfrak{n}}| = \left\{\frac{I}{3} + \frac{1}{2}\hat{\mathfrak{n}}\cdot\vec{\mathfrak{J}} + \frac{1}{2}\sum_{i,j}(\hat{\mathfrak{n}}_i\hat{\mathfrak{n}}_j - \frac{1}{3}\delta_{ij})\frac{(\mathfrak{J}_i\mathfrak{J}_j + \mathfrak{J}_j\mathfrak{J}_i)}{2}\right\} . \tag{3.15}$$

With the explicit representation (3.12), the equality of the 3×3 matrices on each side of (3.15) is readily confirmed.

Of course, if all particles are in the eigenstate $|n^-\rangle$ of $\hat{n}\cdot\mathfrak{J}$ whose eigenvalue is -1, $P_{\rm eq} = -1$. However, $\mathfrak{T}_{\rm eq}$ is again $1/2$. The 3-spinor for $|n^-\rangle$ can be obtained by simply reversing the signs of the components of $\hat{n}$ in (3.12).

Example 2

Consider the case where all particles at some $(J,\phi;s)$ are in the eigenstate $|n^0\rangle$ of $\hat{n}\cdot\mathfrak{J}$ whose eigenvalue is 0 so that $P_{\rm loc} = 0$. The corresponding normalised 3-spinor is

$$e^{i\chi}\sqrt{\frac{1-n_2^2}{2}}\begin{pmatrix} 1 \\ -\frac{\sqrt{2}n_2}{n_1-in_3} \\ -\frac{n_1+in_3}{n_1-in_3} \end{pmatrix} = e^{i(\chi+\psi)}\frac{r}{\sqrt{2}}\begin{pmatrix} e^{-i\psi} \\ -\frac{\sqrt{2}n_2}{r} \\ -e^{+i\psi} \end{pmatrix} , \tag{3.16}$$

where χ is an arbitrary phase. Then, by evaluating the expectation values in (3.3) for $|n^0\rangle$, I find for the local polarisation tensor:

$$T_{\rm loc} = -\sqrt{\frac{3}{2}}\left\{\hat{n}\hat{n}^{\rm T} - \frac{1}{3}I\right\} \ . \tag{3.17}$$

So the ansatz (3.9) for $T^{\rm I}$ appears again. Thus there is again an EDMF and in this case $\mathfrak{T}_{\rm eq}(J) = -1$. Since $P_{\rm eq} = 0$, (3.11) is satisfied as expected for a pure state.

Example 3

Although ensembles in the pure states $|n^+\rangle$ or $|n^-\rangle$ can be described by EDMFs, this is not necessarily true for ensembles in coherent superpositions of these states. For example, consider the case where all particles at some $(J, \phi; s)$ are in the pure state $|N^+\rangle := \{|n^+\rangle + |n^-\rangle\}/\sqrt{2}$ or where all particles are in the pure state $|N^-\rangle := \{|n^+\rangle - |n^-\rangle\}/\sqrt{2}$. In both cases $P_{\rm loc} = 0$ and $\mathfrak{T} = 1$ so that the relation $1 = \frac{3}{4}P^2 + \mathfrak{T}^2$ for pure states is fulfilled. However, neither of the local polarisation tensors, $T^+_{\rm loc}$ for $|N^+\rangle$ and $T^-_{\rm loc}$ for $|N^-\rangle$, is proportional to the $T^{\rm I}$ calculated using (3.9). So ensembles in one or the other of these pure states are not described by an EDMF.

Alternatively we may say that the states $|N^\pm\rangle$ cannot be represented by a projector of the form (3.15). This is in contrast to the case of spin-1/2 particles. There, all pure states can be represented in the corresponding form namely, $|\mathfrak{n}\rangle\langle\mathfrak{n}| = \frac{1}{2}\{I + \hat{\mathfrak{n}}\cdot\vec{\sigma}\}$, since every pure state is an eigenstate of $\hat{\mathfrak{n}}\cdot\vec{\sigma}$ for some unit vector $\hat{\mathfrak{n}}$.

Example 4

I now turn to mixed states. Consider the mixed state at some arbitrary $(J, \phi; s)$ comprised of equal proportions of the pure states $|N^+\rangle$ and $|N^-\rangle$. The density matrix is the average of the density matrices for the two pure states. Then $T_{\rm loc} = \frac{1}{2}T^+_{\rm loc} + \frac{1}{2}T^-_{\rm loc}$ and we find

$$T_{\rm loc} = \frac{1}{2}\sqrt{\frac{3}{2}}\left\{\hat{n}\hat{n}^{\rm T} - \frac{1}{3}I\right\} \ , \tag{3.18}$$

so that there is an EDMF and $\mathfrak{T}_{\rm eq}(J) = 1/2$. Since $P_{\rm eq} = 0$, $\frac{3}{4}P^2_{\rm eq} + \mathfrak{T}^2_{\rm eq} < 1$ as expected in a mixed state.

This example illustrates nicely how an EDMF can exist for a mixture of states which individually do not have EDMFs. There is an unlimited number of such examples.

Example 5

Consider now the mixed state at some arbitrary $(J, \phi; s)$ in which the fraction p_+ is in the state $|n^+\rangle$ and the remainder is in the state $|n^-\rangle$. Then

$$T_{\rm loc} = \frac{1}{2}\sqrt{\frac{3}{2}}\left\{\hat{n}\hat{n}^{\rm T} - \frac{1}{3}I\right\} \ , \tag{3.19}$$

and there is an EDMF with $P_{\rm eq} = 2p_+ - 1$ and $\mathfrak{T}_{\rm eq}(J) = 1/2$.

With $p_+ = 1/2$, the EDMF is the same as that in Example 4 so that the two ensembles cannot be distinguished by measurements.

Example 6

As a final example, consider the mixed state at some arbitrary $(J, \phi; s)$ comprising equal proportions of the pure states $|n^+\rangle$, $|n^-\rangle$ and $|n^0\rangle$. As we have seen, each of these pure states has a $T_{\rm loc}(J, \phi; s)$ proportional to the $T^{\rm I}(J, \phi; s)$ of (3.9) and thus an EDMF. The $\mathfrak{T}_{\rm eq}(J)$ is the average of those for the three pure states and is zero. Since $P_{\rm eq} = 0$ as well, this ensemble is fully unpolarised.

4 Adiabatic invariants based on the ISF and ITF

As mentioned earlier, for a pure state with spin expectation value $\vec{S}$, and under appropriate conditions [22], the scalar product $I_{\rm sn} = \vec{S} \cdot \hat{n}$ is an adiabatic invariant along a trajectory, i.e., it does not change as a parameter such as the reference energy is slowly varied. So if, for example, spins start in the eigenstates considered in Examples 1 and 5 in Section 3.3, they stay in those eigenstates. This is reminiscent of the Adiabatic Theorem of Quantum Mechanics [32]. For these initial states the initial $T_{\rm loc}$ is proportional to the ITF. Then since with adiabatic variation a spin stays in its eigenstate, the $T_{\rm loc}$ remains proportional to the ITF evaluated at the new parameters, with the same $\mathfrak{T}_{\rm eq}$, namely 1/2 in this case. On the other hand if parameters are changed quickly enough, $I_{\rm sn}$ changes along trajectories and I expect that $T_{\rm loc}$ ceases to be proportional to the ITF. The variation of $I_{\rm sn}$ provides a measure of the deviation from adiabaticity and of course the scalar product $I_{\rm sn}$ is invariant under a rotation of the coordinate system.

I now put this picture on a quantitative basis by defining a second adiabatic invariant, namely a "scalar product" $I_{\rm tt} := {\rm Tr}(T_{\rm loc}\, T^{\rm I})$ between the ITF and $T_{\rm loc}$ at each $(u; s)$ [33]. For $I_{\rm sn}$ the vector $\hat{n}$ acts as the reference. For $I_{\rm tt}$ the tensor $T^{\rm I}$ provides the reference. Since a trace is invariant under a similarity transformation, $I_{\rm tt}$ like $I_{\rm sn}$, is constant along trajectories when the parameters do not change. Then with $T_{\rm loc} = \mathfrak{T}_{\rm eq} T^{\rm I}$ and the initial states, say, of Examples 1 and 5, $I_{\rm tt} = \mathfrak{T}_{\rm eq}$ remains at 1/2. So the deviation of $I_{\rm tt}$ from 1/2 also provides a measure of the deviation from adiabaticity.

The calculation of the variation of $I_{\rm tt}$ is straightforward. Consider a beam with $T_{\rm loc} = \mathfrak{T}_{\rm eq} T^{\rm I}$ so that $I_{\rm tt} = \mathfrak{T}_{\rm eq}$ and so that on a trajectory starting at $(u^{\rm i}; s^{\rm i})$, $T_{\rm loc}(u^{\rm i}; s^{\rm i}) = \mathfrak{T}_{\rm eq} T^{\rm I}(u^{\rm i}; s^{\rm i})$. After a parameter p has been varied from the starting value $p^{\rm i}$ to a final value $p^{\rm f}$ the particle is at $(u^{\rm f}; s^{\rm f})$ and $T_{\rm loc}(u^{\rm i}; s^{\rm i})$ has evolved to $R(u^{\rm i}; s^{\rm f}, s^{\rm i}) T_{\rm loc}(u^{\rm i}; s^{\rm i}) R^{\rm T}(u^{\rm i}; s^{\rm f}, s^{\rm i})$. For brevity in the following, I will use the superscripts i and f to indicate that objects are to be evaluated at $(u^{\rm i}; s^{\rm i})$ and $(u^{\rm f}; s^{\rm f})$ respectively and I define $\hat{m} := r\, \hat{n}^{\rm i}$ [5] with $r := R(u^{\rm i}; s^{\rm f}, s^{\rm i})$.

[5] This vector, $\hat{m}$, has no connection with the vector, $\hat{m}$, appearing in the SLIM formalism [41].

Then

$$\begin{aligned} I^{\rm f}_{\rm tt} &= {\rm Tr}(r\ T^{\rm i}_{\rm loc}\ r^{\rm T}\ (T^{\rm I})^{\rm f}) = \mathfrak{T}_{\rm eq}{\rm Tr}(r\ (T^{\rm I})^{\rm i}\ r^{\rm T}\ (T^{\rm I})^{\rm f}) \\ &= \sqrt{\frac{3}{2}}\mathfrak{T}_{\rm eq}{\rm Tr}(r\ \hat{n}^{\rm i}(\hat{n}^{\rm i})^{\rm T}\ r^{\rm T}\ (T^{\rm I})^{\rm f}) = \sqrt{\frac{3}{2}}\mathfrak{T}_{\rm eq}{\rm Tr}(\hat{m}\hat{m}^{\rm T}(T^{\rm I})^{\rm f}) \\ &= \sqrt{\frac{3}{2}}\mathfrak{T}_{\rm eq}\hat{m}^{\rm T}(T^{\rm I})^{\rm f}\hat{m} \\ &= \frac{3}{2}\mathfrak{T}_{\rm eq}\left(\hat{m}^{\rm T}\hat{n}^{\rm f}(\hat{n}^{\rm f})^{\rm T}\hat{m} - \frac{1}{3}\right) \\ &= \frac{3}{2}\mathfrak{T}_{\rm eq}\left((\hat{m}\cdot\hat{n}^{\rm f})^2 - \frac{1}{3}\right) = \mathfrak{T}_{\rm eq}\left(\frac{3}{2}{\rm cos}^2\theta_{\rm mn} - \frac{1}{2}\right) \\ &= I^{\rm i}_{\rm tt}\left(\frac{3}{2}{\rm cos}^2\theta_{\rm mn} - \frac{1}{2}\right)\ , \end{aligned} \tag{4.1}$$

where $\theta_{\rm mn}$ is the angle between $\hat{m}$ and $\hat{n}^{\rm f}$.

Thus $I^{\rm f}_{\rm tt}/I^{\rm i}_{\rm tt} \leq 1$ and the change in $I_{\rm tt}$ depends on just one angle irrespective of the mixture of pure states leading to the $\mathfrak{T}_{\rm eq}$.

Like $I_{\rm sn}$, $I_{\rm tt}$ is also invariant under a rotation of the coordinate system. This can be exploited for calculating $I_{\rm tt}$ in another way, for example in the field of coordinate systems formed by the principal axes [29]. In the following the superscript "P" will indicate that objects are evaluated w.r.t principal axes. Then $\hat{n}^{\rm P}$ has the components (0, 1, 0) and $(T^{\rm I})^{\rm P}$ is diagonal with diagonal elements (eigenvalues) $(T^{\rm I}_{11})^{\rm P} = (T^{\rm I}_{33})^{\rm P} = -(T^{\rm I}_{22})^{\rm P}/2 = -1/\sqrt{6}$. So since $T_{\rm loc}$ is traceless, $I^{\rm f}_{\rm tt} = \sqrt{3/2}\ T^{\rm f,P}_{\rm loc,22}$. Thus we have an interpretation of $I_{\rm tt}$ — it simply describes $T^{\rm P}_{\rm loc,22}$.

With (4.1),

$$T^{\rm f,P}_{\rm loc,22} = T^{\rm i,P}_{\rm loc,22}\left(\frac{3}{2}{\rm cos}^2\theta_{\rm mn} - \frac{1}{2}\right)\ . \tag{4.2}$$

I now note that

$$\cos\,\theta_{\rm mn} = \hat{m}\cdot\hat{n}^{\rm f} = \hat{m}^{\rm P}\cdot\hat{n}^{\rm f,P} \text{ with } \hat{m}^{\rm P} = r^{\rm P}\hat{n}^{\rm i,P}\ .$$

Then if $\cos\,\theta_{\rm mn} = 1$, $\hat{m}^{\rm P} = \hat{n}^{\rm f,P}$ so that $r^{\rm P}$ can only represent a rotation around $\hat{n}^{\rm i,P}$. Therefore with an initially diagonal $T^{\rm i,P}_{\rm loc} = \mathfrak{T}_{\rm eq}(T^{\rm I})^{\rm i,P}$, we find $T^{\rm f,P}_{\rm loc} = r^{\rm P}T^{\rm i,P}_{\rm loc}(r^{\rm P})^{\rm T} = T^{\rm i,P}_{\rm loc} = \mathfrak{T}_{\rm eq}(T^{\rm I})^{\rm i,P} = \mathfrak{T}_{\rm eq}(T^{\rm I})^{\rm f,P}$. Moreover, if $\cos\,\theta_{\rm mn} \neq 1$, $T^{\rm f,P}_{\rm loc} = r^{\rm P}T^{\rm i,P}_{\rm loc}(r^{\rm P})^{\rm T}$ will not be diagonal so that $T^{\rm f}_{\rm loc}$ will not be proportional to $(T^{\rm I})^{\rm f}$.

The vector $\hat{m} = r\ \hat{n}^{\rm i}$ behaves just like a notional normalised spin starting parallel to $\hat{n}$ so that these calculations vindicate the predictions from the heuristic discussion about the invariance of $T_{\rm loc}$ in the first paragraph of this section but without choosing special initial spin states. It is also now clear that $I^{\rm f}_{\rm tt}$ does not add to the information already contained in the $I^{\rm f}_{\rm sn}$.

If, after the variation to $p^{\rm f}$, p then remains fixed while the particle continues along its trajectory, a $T_{\rm loc}$ which is not proportional to $T^{\rm I}$ remains non-invariant but $I_{\rm tt}$ does not

change. In other words, the fact that $I_{\rm tt}$ is constant along trajectories does not mean that $T_{\rm loc}$ is proportional to the ITF. If an initial field $T_{\rm loc}(u;s)$ is proportional to the ITF and the parameters are not varied, $\cos\theta_{\rm mn}$ will remain at the value 1 in agreement with the fact that $T_{\rm loc}(u;s)$ should remain proportional to the ITF. Obviously, if $\theta_{\rm mn}$ is the same on all trajectories $I_{\rm tt}$ is the same on all trajectories.

At low beam energy in a simple flat storage ring with no solenoids and standard transverse emittances, where ν_0 is far from an integer and where the ADST is far from a spin-orbit resonance, $\hat{n}_0(s)$ will be vertical and $\hat{n}(u;s)$ will be essentially parallel to $\hat{n}_0(s)$ at each s. So if a beam is injected with polarisation parallel to $\hat{n}_0$ at the injection point with $\vec{P}(u;s) = P_{\rm inj}\ \hat{n}_0$, $T_{\rm loc}$ will be proportional to $T^{\rm I}$ and the same at all $(u;s)$. Then if the beam is accelerated in a way that ensures that the spins stay parallel to their ISF vectors $\hat{n}$, which might spread out away from $\hat{n}_0$, the final equilibrium polarisation distribution $\vec{P}_{\rm loc}(u;s) = P_{\rm inj}\hat{n}(u;s)$ will be established. Moreover the final $T_{\rm loc}$ will still be proportional to $T^{\rm I}$ with the same $\mathfrak{T}_{\rm eq}$. This is how an EDMF can be established at high energy in this case.

In another convention, useful when the vector polarisation is vertical, the degree of tensor polarization is defined to be the so-called alignment and it is equal to $\sqrt{6}\,T_{22}$. See [34] and [11, pages 52-53]. Together with P_2, knowledge of the alignment is frequently needed for the analysis of the results of scattering experiments at low energies involving deuterons. See, for example, the discussion surrounding [11, eq. 3.172]. There, only P_2 and T_{22} appear in the density matrix and it is diagonal. Thus other components of the tensor are not needed. See [12, 35] too.

The component T_{22} of the tensor provides a useful window into the behaviour of the tensor part of the spin dynamics of deuterons in storage rings at low energy. Consider a simple flat storage ring at low beam energy with standard transverse emittances and no solenoids where ν_0 is far from an integer so that $\hat{n}_0(s)$ is vertical. If the ADST is far from a spin-orbit resonance then $\hat{n}(u;s)$ will be essentially vertical too at each $(u;s)$. If the vector polarisation is $\vec{P}(u;s) = P_{\rm inj}\ \hat{n}_0$ then the EDMF is diagonal and $\sqrt{6}T_{\rm loc,22}$ is simply the alignment [11, pages 52-53] as mentioned above. Next, if, for example, the vertical betatron tune or the frequency of a radio-frequency dipole is varied non-adiabatically through, say, a state of spin-orbit resonance to a state far from resonance, $\hat{n}$ will again be very close to the vertical at all $(u;s)$ at the end of the variation. However, $\cos\theta_{\rm mn}$ for each particle will in general differ from its initial values of 1. Then, as we have seen, the final $T_{\rm loc}$ is not proportional to the $T^{\rm I}$ and at each $(u;s)$ $T^{\rm f}_{\rm loc,22} = T^{\rm i}_{\rm loc,22}(\frac{3}{2}\cos^2\theta_{\rm mn} - \frac{1}{2})$. If the Froissart-Stora formalism applies [42], $\theta_{\rm mn}$ will be the same for all trajectories. For related experimental work with a radio-frequency dipole see [34, 36]. In particular, if the parameter variation is, in fact, adiabatic, $\cos\theta_{\rm mn} = 1$ so that $T^{\rm f}_{\rm loc,22} = T^{\rm i}_{\rm loc,22}$.

However, at high energy and/or close to a spin-orbit resonance, $\hat{n}(u;s)$ can be tilted away from $\hat{n}_0(s)$. Then although $T_{\rm loc}$ can be proportional to $T^{\rm I}$, the $T^{\rm I}$ is not diagonal and $T_{\rm loc,22}$ is not, on its own, sufficient for analysing scattering experiments. Nevertheless, insights are available via numerical simulation. For example the system can be viewed in an IFF or w.r.t. principal axes. Then if the vector polarisation at $(u;s)$ is parallel to $\hat{n}(u;s)$, the density matrix is diagonal and depends just on $|P_{\rm loc}(u;s)|$ and on the $T_{\rm loc,22}$ viewed in the

IFF. With this, the analysis of the effect of a variation of a parameter can be carried through in the IFF (or w.r.t. the principal axes) in the same way as in the discussion leading to (4.2). An example of working in the IFF for protons is given in [25]. See [24] too.

5 Other parametrisations

Once the Cartesian version of the ITF has been established, the five components of the corresponding invariant spherical polarisation–tensor field follow trivially using the relations in [11, eqs. 3.1.66 - 67]. See [37, 12, 35, 13] too. Thus it is simple to find the invariant spherical polarisation–tensor field for, for example, the single resonance model. In analogy with the expansion for spin-1/2 particles in (2.1), the density matrix for spin-1 particles can also be expanded in terms of the generators of the group SU(3) and the so–called coherence (or Bloch) vector [38, 39]. Other matrix bases and the corresponding Bloch vectors can be used too: see [40] and the Appendix too. However, the expansion in terms of the polarisation and the Cartesian tensor seems to be the most convenient for calculating and discussing the EDMF. Moreover, although not all Bloch vectors inside their so–called Bloch hyper–spheres lead to admissible density matrices [40], this matter does not arise for density matrices based on the ITFs of (3.9).

6 The Bloch equations

In the previous sections, the orbital motion is deterministic and governed just by a Hamiltonian so that the density in phase space is conserved along a trajectory and one can work with spin density matrices instead of spin–orbit density matrices. However, if the particles are subject to noise and damping, the evolution of the density of particles in phase space is more complicated and spin–orbit density matrices and their Wigner functions come into play [16, 43]. Alternatively, we can work with the classical phase space density and the so–called polarisation density as in [17, 44]. The effect of noise and damping on the polarisation tensor can be studied in an analogous way. So I round off this paper on the polarisation tensor by showing how to do this. In order to avoid undue repetition I assume that the reader is familiar with the concepts in [17, 44]. The final result of this section is perhaps no more than a curiosity but it would be remiss not to include it here while the ITF is being presented in such detail.

Charged particles suffering deflection in magnetic fields emit synchrotron radiation and, as is well known, this has important consequences for the properties of electron (positron) beams in storage rings. For example, the stochastic nature of photon emission together with damping mechanisms causes the phase space density $W_{\rm orb}(u;s)$ to reach equilibrium, i.e., to become 1–turn periodic: $W_{\rm orb}(u;s+C) = W_{\rm orb}(u;s)$ [45]. It often suffices to approximate the effects of the photon emission as a Gaussian white noise process so that if interparticle forces can be ignored, the evolution of $W_{\rm orb}$ can be described in terms of the Fokker–Planck equation [45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55]. For this account it is sufficient to write

the Fokker–Planck equation somewhat symbolically as in [17] in the form

$$\frac{\partial W_{\rm orb}}{\partial s} = \mathcal{L}_{\rm FP,orb}\, W_{\rm orb}\ , \tag{6.1}$$

where the orbital Fokker–Planck operator can be decomposed into the form:

$$\mathcal{L}_{\rm FP,orb} = \mathcal{L}_{\rm ham} + \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2\ , \tag{6.2}$$

whereby $\mathcal{L}_0,\ \mathcal{L}_1,\ \mathcal{L}_2$ are terms due to damping and noise containing respectively zeroth, first and second order derivatives w.r.t. the components of u. The term $\mathcal{L}_{\rm ham}$ is the Poisson bracket $\{H_{\rm orb}, W_{\rm orb}\}$ of $W_{\rm orb}$ with the orbital Hamiltonian. Detailed forms for $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$ can be found in [50, 51, 52] but are not important for the argument that follows. We normalise $W_{\rm orb}$ to unity: $\int d^6u\ W_{\rm orb}(u;s) = 1$.

I have introduced the Fokker–Planck equation by mentioning synchrotron radiation but the Fokker–Planck equation can be applied when the particles are subject to other sources of noise and damping by adopting appropriate forms for $\mathcal{L}_0,\ \mathcal{L}_1$ and $\mathcal{L}_2$. Of course, except at very large energies, synchrotron radiation is irrelevant for particles other than electrons (positrons). But as in the case of synchrotron radiation, noise can lead to irreversible loss of vector polarisation [16], and even when $W_{\rm orb}$ has reached periodicity. The chief mechanism is simple: the noise causes random perturbations to the trajectories and thereby causes random perturbations in $\vec{\Omega}$ in the non–uniform fields of the quadrupoles.

Since the local vector polarisation $\vec{P}_{\rm loc}(u;s)$ is not a density, its evolution cannot be described by a Fokker–Planck equation. However, as explained in [17, 44], if the noise has no *direct* effect on spins, the vector–polarisation density $\vec{\mathcal{P}}(u;s) := W_{\rm orb}(u;s)\vec{P}_{\rm loc}(u;s)$ evolves according to the equation

$$\frac{\partial \vec{\mathcal{P}}}{\partial s}\ =\ \mathcal{L}_{\rm FP,orb}\vec{\mathcal{P}} + \vec{\Omega}\times\vec{\mathcal{P}}\ . \tag{6.3}$$

where the Poisson bracket $\mathcal{L}_{\rm ham}$ is now $\{H_{\rm orb}, \vec{\mathcal{P}}\}$. Thus, if $\mathcal{L}_{\rm FP,orb}$ is known, we have an immediate, succinct, classical encapsulation of the way in which $\vec{P}_{\rm loc}$ is modified both by precession and by the effect of noise and damping on the mixture of spin states at each $(u;s)$. We call (6.3) the Bloch equation for the vector–polarisation density to reflect the analogy with the equations for magnetisation in magnetised solids [56][6]. Although (6.3) was derived with spin-1/2 particles in mind, it applies to spin-1 particles too. The vector–polarisation density is proportional to the density in phase space of spin angular momentum. To obtain $\vec{P}_{\rm loc}(u;s)$, (6.1) and (6.3) should be solved in parallel and then $\vec{\mathcal{P}}(u;s)/W_{\rm orb}(u;s)$ should be calculated.

If $\mathcal{L}_0,\ \mathcal{L}_1$ and $\mathcal{L}_2$ vanish, leaving just the Poisson bracket $\mathcal{L}_{\rm ham}$, then (6.3) reduces to the T–BMT equation for $\vec{\mathcal{P}}$ along a trajectory, just as expected when we recall that in this case $W_{\rm orb}$ is preserved along the trajectory. Then, if $\vec{P}_{\rm loc}(u;s)$ is parallel to $\hat{n}(u;s)$ at each point in phase space with $\vec{P}_{\rm loc}(J,\phi;s) = P_{\rm eq}(J)\,\hat{n}(J,\phi;s)$, and $W_{\rm orb}$ is in equilibrium, the

[6] This equation has nothing directly to do with the Bloch vectors mentioned earlier.

polarisation of the beam will be independent of s. On the other hand if $\vec{P}_{\rm loc}(u;s)$ is not parallel to $\hat{n}(u;s)$, the polarisation of the beam will oscillate as illustrated in [7, p.72]. If noise and damping are included, the polarisation of the beam will die away in the long term [16, 50, 57].

In order to study the evolution of the Cartesian polarisation tensor I introduce the polarisation-tensor density $\mathcal{T}(u;s) := W_{\rm orb}(u;s)T_{\rm loc}(u;s)$. Then it can be shown that the Bloch–like equation for $\mathcal{T}$ is

$$\frac{\partial \mathcal{T}}{\partial s} \;=\; \mathcal{L}_{\rm FP,orb}\mathcal{T} + [\,\tilde{\Omega},\,\mathcal{T}\,]\;, \tag{6.4}$$

where the commutator derives from (4.1) and where the Poisson bracket $\mathcal{L}_{\rm ham}$ is $\{H_{\rm orb},\mathcal{T}\}$. Given the previous discussion, (6.4) is already expected on purely heuristic grounds. In fact (6.4) is valid for an arbitrary tensor obeying (4.1) but, in particular, for the traceless symmetric tensor T_{loc}. To obtain $T_{\rm loc}(u;s)$, (6.1) and (6.4) should be solved in parallel and then $\mathcal{T}(u;s)/W_{\rm orb}(u;s)$ should be calculated.

Since, by definition, $\mathcal{T}$ is symmetric and traceless at all $(u;s)$, the right-hand side of (6.4) is traceless and symmetric too. Therefore I expect that (6.4) preserves the symmetry and tracelessness of $T_{\rm loc}$. In any case, this should happen since the form (3.1) must be preserved in the presence of noise and damping. In the absence of a polarising mechanism, $P_{\rm loc}$ falls to zero in beams of spin-1/2 particles subject to noise and damping [16, 50, 57, 41]. This will also happen for spin-1 particles and I expect that $\mathfrak{T}_{\rm loc}$ will fall to zero as the spin states become fully mixed. Moreover, with noise and damping $P_{\rm loc}(u;s)$ aligns itself almost parallel to $\hat{n}(u;s)$ [57] during the depolarisation. Then I expect that $T_{\rm loc}(u;s)$ becomes almost proportional to $T^I(u;s)$ at the same time.

Note that the relationships between (6.1) and (6.3) and between (6.1) and (6.4) survive if $\mathcal{L}_{\rm FP,orb}$ is replaced by any physically admissible transport operator $\mathcal{K}_{\rm orb}$ — which could even contain derivatives beyond second order.

7 Summary

I have proposed a definition of an invariant rank–2 Cartesian polarisation–tensor field (ITF) for spin-1 particles, and a procedure for calculating it numerically by stroboscopic averaging or analytically once the ISF is known. The ISF and ITF provide "chassis" on which to "hang" equilibrium vector and tensor spin distributions for spin-1 particles on a phase–space torus and I have adopted the two fields to construct equilibrium spin density–matrix fields which depend on just two parameters, the degree of equilibrium vector polarisation $P_{\rm eq}(J)$ and the degree of equilibrium tensor polarisation $\mathfrak{T}_{\rm eq}(J)$. I have also pointed out that the EDMF for spin-1 particles can always be diagonalised by a rotation of the coordinate system when the ansatz (3.9) for the ITF is valid and I have shown with examples how that ansatz can accommodate a typical spin-1 tensor in a natural way. Moreover, I have identified adiabatic invariants associated with the ITF. Of course this work is mainly relevant for deuterons but from the arguments in this paper, it is clear that an EDMF and an accompanying invariant

tensor field could be defined for spin-3/2 particles too. Of course, such objects would have no practical use owing to the small lifetimes of those particles.

Finally, I have extended earlier work to include the effects of noise and damping on the polarisation tensor and I have provided an evolution equation for the polarisation-tensor density.

A follow-up paper [29] will discuss the uniqueness of the ansatz in (3.9).

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Appendix

In this appendix I give a heuristic and pedagogical demonstration of how to arrive at the representation (3.1) for the spin-1 density matrix.

We need to write ρ as a linear combination of 3×3 matrices. There is a variety of possibilities [38, 39, 40] but we wish to use the matrices $\mathfrak{J}$. Since eight real parameters are needed, an expansion of the form $\rho = \sum_{i=1-3} U_i \, \mathfrak{J}_i$ with complex U_i does not suffice. So I try an expansion of the form

$$\rho = \sum_{i,j} U_{ij} \, \mathfrak{J}_j \mathfrak{J}_i \ , \tag{A.1.1}$$

with complex U_{ij}. Then we have space for the required eight independent real parameters but their identity is not immediately evident among the eighteen real parameters defining the U_{ij}.

To come further I recall that $\mathrm{Tr}(\mathfrak{J}_i \mathfrak{J}_j) = 2\delta_{ij}$. Then $\mathrm{Tr}(U) = 1/2$ since $\mathrm{Tr}(\rho) = 1$. Moreover, with the rule $\langle \hat{s}_i \, \hat{s}_j \rangle = \mathrm{Tr}(\rho \mathfrak{J}_i \mathfrak{J}_j)$ I then find

$$U_{ij} = \langle \hat{s}_i \hat{s}_j \rangle - \frac{1}{2} \delta_{ij} \ , \tag{A.1.2}$$

which I can write in matrix form as

$$U = \langle \hat{s} \, \hat{s}^{\mathrm{T}} \rangle - \frac{1}{2} I \ , \tag{A.1.3}$$

in an obvious notation. The matrix $\langle \hat{s} \, \hat{s}^{\mathrm{T}} \rangle$ transforms as a Cartesian tensor. Therefore U transforms as a Cartesian tensor too. Thus $U^{\mathrm{f}} = R \, U^{\mathrm{i}} R^{\mathrm{T}}$ in analogy with (3.5) so that $(U^{\mathrm{T}})^{\mathrm{f}} = R \, (U^{\mathrm{T}})^{\mathrm{i}} R^{\mathrm{T}}$.

For the next step I write U as the sum of its symmetric and antisymmetric parts: $U = t + a$ with $t = (U + U^{\mathrm{T}})/2$ and $a = (U - U^{\mathrm{T}})/2$. Both t and a are Cartesian tensors. Furthermore, using the hermiticity of ρ it can be shown that t is real and a is pure imaginary.

Equation (A.1.1) can now be written as

$$\rho = \sum_{i,j} \frac{t_{ij}}{2}(\mathfrak{J}_i\mathfrak{J}_j + \mathfrak{J}_j\mathfrak{J}_i) + a_{12}(\mathfrak{J}_1\mathfrak{J}_2 - \mathfrak{J}_2\mathfrak{J}_1) + a_{23}(\mathfrak{J}_2\mathfrak{J}_3 - \mathfrak{J}_3\mathfrak{J}_2) + a_{13}(\mathfrak{J}_1\mathfrak{J}_3 - \mathfrak{J}_3\mathfrak{J}_1) \ , \quad \text{(A.1.4)}$$

and from the commutation relations among the matrices $\mathfrak{J}$ we obtain

$$\rho = \sum_{i,j} \frac{t_{ij}}{2}(\mathfrak{J}_i\mathfrak{J}_j + \mathfrak{J}_j\mathfrak{J}_i) + ia_{12}\mathfrak{J}_3 + ia_{23}\mathfrak{J}_1 - ia_{13}\mathfrak{J}_2 \ . \quad \text{(A.1.5)}$$

I now define a real 3–component object $\vec{v}$ with elements $\vec{v}_1 := i\,a_{23}, \vec{v}_2 := -i\,a_{13}$ and $\vec{v}_3 := i\,a_{12}$. Then, using the fact that a is a Cartesian tensor, it is easily shown that $\vec{v}$ transforms as an axial 3–vector. Thus (A.1.5) takes the form

$$\rho = \sum_{i,j} \frac{t_{ij}}{2}(\mathfrak{J}_i\mathfrak{J}_j + \mathfrak{J}_j\mathfrak{J}_i) + \vec{v}\cdot\vec{\mathfrak{J}} \ . \quad \text{(A.1.6)}$$

With the definition $C = t/2$ so that $\mathrm{Tr}(C) = 1/4$, I rewrite (A.1.6) as

$$\rho = \sum_{i,j} C_{ij}(\mathfrak{J}_i\mathfrak{J}_j + \mathfrak{J}_j\mathfrak{J}_i) + \vec{v}\cdot\vec{\mathfrak{J}} \ . \quad \text{(A.1.7)}$$

This, in turn, can be rearranged in terms of the traceless tensor $C - \frac{1}{12}I$ to give

$$\rho = \frac{1}{3}I + \sum_{i,j}(C_{ij} - \frac{1}{12}\delta_{ij})(\mathfrak{J}_i\mathfrak{J}_j + \mathfrak{J}_j\mathfrak{J}_i) + \vec{v}\cdot\vec{\mathfrak{J}} \ . \quad \text{(A.1.8)}$$

Then by making the identifications $\vec{P} = \mathrm{Tr}(\rho\vec{\mathfrak{J}}) = 2\vec{v}$ and $\frac{1}{\sqrt{6}}T = C - \frac{1}{12}I$ I arrive at (3.1).

The passage from (A.1.1) to (A.1.8) is a demonstration of how to usefully decompose the tensor U into its scalar, vector and irreducible tensor parts [26]. Moreover, by explicitly isolating the scalar term, $\frac{1}{3}I$, in (A.1.8) we obtain a transparent form for ρ when the three spin substates are equally populated: $\vec{P}$ and T must vanish.

The parametrisation (A.1.8) contains nine real parameters, but of course, only eight of them are independent owing to the condition $\mathrm{Tr}(C) = 1/4$. However, we can also begin with a parametrisation based on the matrices $\mathfrak{J}$ but with just eight parameters. For this we recall that the density matrix can be written in the form:

$$\rho = \frac{1}{3}I + \sum_{k=1-8} \lambda_k\mathcal{O}_k \ , \quad \text{(A.1.9)}$$

where the 3×3 matrices $\mathcal{O}$ are traceless and hermitian and $\mathrm{Tr}(\mathcal{O}_k\mathcal{O}_l) = \alpha_k\delta_{kl}$ so that together with $\frac{1}{3}I$ they comprise an orthogonal basis. The coefficients α are real and the components, $\lambda_k = \mathrm{Tr}(\rho\,\mathcal{O}_k)/\alpha_k$, of the Bloch vector are real and mutually independent.

Then we can, for example, choose the eight assignments:

$$\begin{aligned}
\mathcal{O}_1 &= \mathfrak{J}_1 & \mathcal{O}_2 &= \mathfrak{J}_2 & \mathcal{O}_3 &= \mathfrak{J}_3 \\
\mathcal{O}_4 &= \mathfrak{J}_1\mathfrak{J}_2 + \mathfrak{J}_2\mathfrak{J}_1 & \mathcal{O}_5 &= \mathfrak{J}_2\mathfrak{J}_3 + \mathfrak{J}_3\mathfrak{J}_2 & \mathcal{O}_6 &= \mathfrak{J}_1\mathfrak{J}_3 + \mathfrak{J}_3\mathfrak{J}_1 \\
\mathcal{O}_7 &= \mathfrak{J}_1\mathfrak{J}_1 - \mathfrak{J}_3\mathfrak{J}_3 & \mathcal{O}_8 &= \mathfrak{J}_2\mathfrak{J}_2 - \frac{2}{3}I\ . & & \qquad (A.1.10)
\end{aligned}$$

It is simple to show that, as required, $\langle \hat{s}_1\,\hat{s}_1 + \hat{s}_2\,\hat{s}_2 + \hat{s}_3\,\hat{s}_3 \rangle = 2$ independently of the coefficients λ, i.e, for any mixed state. Of course, $\vec{P}$ and T are simply related to the coefficients λ. Moreover, the parametrisation (A.1.10) is convenient for using the equation of motion of the density matrix as in [18, Section 1–8c], but with the commutation relations for the matrices $\mathfrak{J}$, to demonstrate that $\vec{P}$ for spin–1 particles obeys the T-BMT equation. That approach also leads to the equation of motion (4.1) for T.

References

[1] D.P. Barber and M. Vogt, *The Invariant Polarisation-Tensor Field for Spin-1 Particles in Storage Rings*, Proc.18th Int. Spin Physics Symposium, Charlottesville, Virginia, USA, October 2008, AIP Conference Proceedings 1149 (2008).

[2] J.D. Jackson, *"Classical Electrodynamics"*, 3rd edition, Wiley (1998).

[3] D.P. Barber, K. Heinemann and G. Ripken, Z. f. Physik **C64**, p.117 (1994).

[4] Ya.S. Derbenev and A.M. Kondratenko, Sov. Phys. JETP **37**, p.968 (1973).

[5] K. Heinemann and G.H. Hoffstaetter, Phys.Rev. **E54**(4), 4240 (1996) .

[6] G.H. Hoffstaetter, M. Vogt and D.P. Barber, Phys. Rev. ST Accel. Beams **11**(2), 114001 (1999).

[7] G.H. Hoffstaetter, *"High Energy Polarised Proton Beams: a Modern View"*, Springer Tract in Modern Physics, Vol 218 (2006).

[8] M. Vogt, PhD Thesis, University of Hamburg, Germany, DESY-THESIS-2000-054 (2000).

[9] D.P.Barber, J.A. Ellison and K. Heinemann, Phys. Rev. ST Accel. Beams **7**, 124002 (2004).

[10] D.P.Barber, J.A. Ellison and K. Heinemann, Phys. Rev. ST Accel. Beams **8**, 089002 (2005).

[11] E. Leader, *"Spin in particle physics"*, Cambridge Univ. Press (2001).

[12] H. Paetz gen. Scheick, *"Nuclear Physics with Polarized Particles"*, Springer Lecture Notes in Physics, Vol. 842 (2011).

[13] Proc. 3rd Int. Symp. on Polarization Phenomena in Nuclear Reactions, Madison, USA, 1970, Eds. H.H. Barschall and W. Haeberli, University of Wisconsin Press (1971).

[14] D.P. Barber and M. Vogt, New J. Phys. **8**, p.296 (2006).

[15] K. Heinemann, D.P.Barber, J.A. Ellison and M. Vogt, arXiv:1501.02747 (2015). Submitted for publication.

[16] D.P. Barber et al., five articles in proceedings of ICFA workshop "Quantum Aspects of Beam Physics", Monterey, U.S.A., 1998, edited by P. Chen, World Scientific (1999). Also in extended form as DESY Technical Report 98-096 (1998) and arXiv:physics/9901038, 9901041, 9901042, 9901043, 9901044.

[17] K. Heinemann and D.P. Barber, Nucl. Instr. Meth. **A463**, p.62 and **A469**, p.294 (2001).

[18] P. Roman, *"Advanced quantum theory"*, Addison Wesley (1965).

[19] S. Blanes, F. Casas, J.A. Oteo and J. Ros, Phys. Rep. **470**, p.151 (2009).

[20] K. Yokoya, DESY Report 86-57 (1986).

[21] J.A. Ellison and K. Heinemann, Physica **D234**(2), p.131 (2007).
[22] G. H. Hoffstaetter, H.S. Dumas and J.A. Ellison, Phys. Rev. ST Accel. Beams **9**, 014001 (2006).
[23] D. T. Abell, D. Meiser, V. H. Ranjbar and D.P. Barber, Phys. Rev. ST Accel. Beams **18**, 024001 (2015).
[24] G.H. Hoffstaetter and M. Vogt, Phys. Rev. **E70**, 056501 (2004).
[25] D.P. Barber, *Pre-existing betatron motion and spin flipping with RF fields in storage rings*, Proc.19th Int. Spin Physics Symposium, Juelich, Germany, October 2010.
[26] J.S. Bell, *"Polarised particles for accelerator physicists"*, CERN Report 75-11 (1975).
[27] M.E. Rose, *"Elementary Theory of Angular Momentum"*, Dover Publications (1995).
[28] I thank S.R. Mane for reminding me that $|\vec{S}|$ can vanish for spin-1 particles.
[29] D.P. Barber, A. Kling and M. Vogt, arXiv:1510.07963 (2015).
[30] S.R. Mane, Fermilab technical report TM-1515 (1988).
[31] O. Giraud, P. Braun and D. Braun, arXiv:0805.2592 (2008).
[32] A. Messiah, *"Quantum Mechanics"*, John Wiley & Sons, New York (1958).
[33] I thank M. Vogt for suggesting that there might be adiabatic invariants associated with the ITF and that traces could be a way of quantifying them.
[34] V. Morozov et al., Phys. Rev. Letts **91**, 214801 (2003).
[35] H.E. Conzett, *"Interactions of polarized beams with particles and nuclei'"*, Rep. Prog. Phys., p.1 (1994).
[36] V. Morozov et al., Phys. Rev. ST Accel. Beams **8**, 089002 (2005).
Note that contrary to the claim in the title of this paper, the results of the measurements were entirely expected once correct estimates of the expected resonance strengths were recognised.
[37] J. Werle, *"Relativistic Theory of Reactions"*, North Holland, (1966).
[38] M.S. Byrd and N. Khaneja, Phys. Rev. **A68**, 062322 (2003).
[39] G. Kimura, Phys. Lett. **A314**, p.339 (2003).
[40] R.A. Bertlmann and P. Krammer, J. Phys. **A41**, 235303 (2008).
[41] D.P. Barber and G. Ripken, *"Handbook of Accelerator Physics and Engineering"*, Eds: A.W. Chao and M. Tigner, 1st edition, 3rd printing, World Scientific (2006) and 2nd edition, Eds. A.W. Chao, K.H. Mess, M. Tigner and F. Zimmermann (2013).
See also D.P. Barber and G. Ripken, arXiv:physics/9907034.
[42] M. Froissart and R. Stora, Nucl. Instr. Meth. **7**, p.297 (1960).
[43] Ya.S. Derbenev and A. M. Kondratenko, Sov. Phys. Dokl. **19**, p.438 (1975).
[44] D.P. Barber and K. Heinemann, *"Red, blue and green electrons"*, in the proceedings of the workshop "Polarized Protons at High Energies – Accelerator Challenges and Physics Opportunities", DESY, May 1999, Eds. A. De Roeck et al.
DESY report DESY–PROC–1999–03 (1999).
[45] H. Wiedemann, *"Particle Accelerator Physics"*, 3rd edition, Springer, Berlin (2007).
[46] H. Risken, *"The Fokker-Planck Equation: Methods of Solution and Applications"*, Springer Series in Synergetics, Springer (1996).
[47] C.W. Gardiner, *"Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences"*, Springer Series in Synergetics, Springer (2004).
[48] N.G. van Kampen, *"Stochastic Processes in Physics and Chemistry"*, Elsevier (1981).
[49] H. Mais and G. Ripken, DESY Report 83-62 (1983).
[50] D.P. Barber, K. Heinemann, H. Mais and G. Ripken, DESY Report 91-146 (1991).

[51] J.M. Jowett, *"Introductory Statistical Mechanics for Electron Storage Rings"*, AIP Conference Proceedings 153 (1987).
[52] F. Ruggiero et al., Annals of Physics **197**(2), p.396 (1990).
[53] K. Heinemann, DESY Report 86-129 (1986) and arXiv:physics/9709025.
[54] J.A. Ellison, H. Mais and G. Ripken, *"Handbook of Accelerator Physics and Engineering"*, Eds: A.W. Chao and M. Tigner, 1st edition, 3rd printing, World Scientific (2006).
[55] S. Tzenov, *"Contemporary Accelerator Physics"*, World Scientific, (2004).
[56] F. Bloch, Phys. Rev. **70**, p.460 (1946).
[57] D.P. Barber and K. Heinemann, DESY Report 15-142 and arXiv:1508.05318 (2015).