Stochastic Simulation of Backward Runaway Electrons

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Abstract

In this paper a stochastic approach to solve the probability of backward runaway. A stochastic differential equation for electrons in fully ionized plasma is derived using Cadjan-Ivanov method. An energy-preserving algorithm is presented. Using this method, the backward runaway probability is calculated statically.

Introduction

In fully ionized plasma, when an electron is moving along external electric field, it suffers from a competition between electric field and the velocity drag $-\boldsymbol{v}/v^3$ due to collisions. If the velocity is small enough that the velocity drag dominant, the electron will be stopped by collisions. But if the velocity is high and the electric field is larger, it will be accelerated to extreme high velocity until is hit the boundary. This phenomenon is called the forward runaway. The Dreicer velocity[1] indicates which electrons have high enough velocity to run away.

However, if the initial velocity of an electrons is in the opposite direction of the external electric field, it is not certain to run away or be stopped. If an electrons gains a high perpendicular velocity large enough before its parallel velocity reduces to zero, it could transformed into a forward runaway electron, otherwise, it will be stopped. Since the collision effect is a random factor, an electrons has a certain probability to run away[2].

This paper is organized as following, In Section. II, the stochastic differential equation is derived for electrons in fully ionized plasma. In Section. III, an energy-preserving algorithm is present to numerically solve the SDEs. And the simulation results are presented in Section IV.

Stochastic Differential Equation for Lorentz Plasma

The distribution function of runaway electrons obeys Boltzmann equation with collisional operator

$$\frac{\partial f}{\partial t}+v\cdot\frac{\partial f}{\partial x}+\frac{q}{m}(E+v\times B)\cdot\frac{\partial f}{\partial v}=\left(\frac{\partial f}{\partial t}\right)_c \quad (1)$$

For electron-ion collision with Z_i being the charge carried by ion, Lorentz collision term is included

$$\left(\frac{\partial f}{\partial t}\right)_{e-i}=\frac{Z_i\Gamma}{2}\frac{\partial}{\partial v}\cdot\left(\frac{v^2I-vv}{v^3}\cdot\frac{\partial f}{\partial v}\right)$$

For electron-electron collision, it should be noted that a velocity drag damping is introduced

$$\left(\frac{\partial f}{\partial t}\right)_{e-e}=\Gamma\frac{\vec{v}}{v^3}\cdot\frac{\partial f}{\partial \vec{v}}+\frac{\Gamma}{2}\frac{\partial}{\partial \vec{v}}\cdot\left(\frac{v^2I-vv}{v^3}\cdot\frac{\partial f}{\partial \vec{v}}\right)$$

To sum up, the final Boltzmann equation reads[1, 2]

$$\frac{\partial f}{\partial t}+\left(\frac{qE}{m}-\Gamma\frac{v}{v^3}\right)\cdot\frac{\partial f}{\partial v}=\frac{\Gamma(1+Z_i)}{2}\frac{\partial}{\partial v}\cdot\left(\frac{v^2I-vv}{v^3}\cdot\frac{\partial f}{\partial v}\right) \quad (2)$$

Boltzmann equation for spatial homogeneous Lorentz plasma can be rewritten in form

$\frac{\partial f}{\partial t}+v\cdot\frac{\partial f}{\partial x}+\frac{e}{m}(E+v\times B)\cdot\frac{\partial f}{\partial v}=\left(\frac{\partial f}{\partial t}\right)_c$, where

$$\left(\frac{\partial f}{\partial t}\right)_c=\frac{\partial}{\partial v_i}\left(-\left\{-Z_i\Gamma\frac{v_i}{v^3}\right\}f+\frac{1}{2}\frac{\partial}{\partial v_k}\left\{Z_i\Gamma\frac{v^2\delta_{ik}-v_iv_k}{v^3}\right\}f\right).$$

And Γ is a constant defined $\Gamma=\frac{ne^4\ln\Lambda}{4\pi\epsilon_0^2m_e^2}$.

Tt can also be solved by statistics over solutions of a Stratonovich stochastic equation[3, 4],

$$dv=\frac{e}{m}(E+v\times B)+\sqrt{\frac{Z_i\Gamma}{v^5}}v\times v\times{}^\circ d\vec{W} \quad (3)$$

One can easily verify the square root by multiplication.

The final Boltzmann equation with collisions of electrons and ions included is

$$\frac{\partial f}{\partial t}+\left(\frac{qE}{m}-\Gamma\frac{v}{v^3}\right)\cdot\frac{\partial f}{\partial v}=\frac{\Gamma(1+Z_i)}{2}\frac{\partial}{\partial \vec{v}}\cdot\left(\frac{v^2I-vv}{v^3}\cdot\frac{\partial f}{\partial v}\right)$$

The corresponding stochastic differential equation is

$$d\vec{v}=\left(\frac{qE}{m}-\Gamma\frac{v}{v^3}\right)dt-\sqrt{\frac{(1+Z_i)\Gamma}{v^5}}v\times v\times{}^\circ dW,$$

Which is renormalized parameters by $\tilde{v}=v/v_{Dreicer}$, $\tilde{t}=t/\tau_{Dreicer}$, where $\tau_{Dreicer}=v_{Dreicer}^3/\Gamma$, $\tilde{W}=W(\tau_{Drecier}\tilde{t})/\sqrt{\tau_{Drecier}}$, $\tilde{E}=\frac{qE}{m}/F_0$, where $F_0=m\Gamma/v_{Dreicer}$

The dimensionless equation is

$$d\tilde{v}=\left(\tilde{E}-\frac{\tilde{v}}{\tilde{v}^3}\right)dt-\sqrt{1+Z_i}\tilde{v}^{-\frac{5}{2}}\tilde{v}\times\tilde{v}\times{}^\circ d\tilde{W} \quad (4)$$

Energy-Preserving Algorithms

For $dX(t)=\mu(X,t)dt+\sigma(X,t){}^\circ dW_t$ in Stratonovich sense.

The midpoint implicit method[5, 6] is

$$X_0^n=X_0$$

$$X_i^n - X_{i-1}^n = \mu\left(\frac{1}{2}(X_i^n + X_{i-1}^n), \frac{1}{2}(t_i + t_{i-1})\right)\Delta_i + \sigma\left(\frac{1}{2}(X_i^n + X_{i-1}^n), \frac{1}{2}(t_i + t_{i-1})\right)\Delta_i\widetilde{W} \quad (5)$$

Where

$$\Delta_i\widetilde{W} = \begin{cases} -A_h, & \Delta_i W < A_h \\ \Delta_i W, & |\Delta_i W| \le A_h \\ A_h, & \Delta_i W > A_h \end{cases}$$

$A_h = \sqrt{2k|\ln h|}$, $h = \min(\Delta_i)$ and $k \ge 1$. And $\widetilde{W}$ is the standard Winer process. It can easily to proof that this algorithm preserves that energy of an electron.

Simulation Results

A sample solution path of a test electron without velocity drag is plotted in Fig. 1. Because of the energy is conserved, it looks like a random walk on energy spherical surface. The energy of using energy-preserving method is compared with commonly used Euler-Maruyama[7] method in Fig. 2. It can be seen that the energy is numerically conserved.

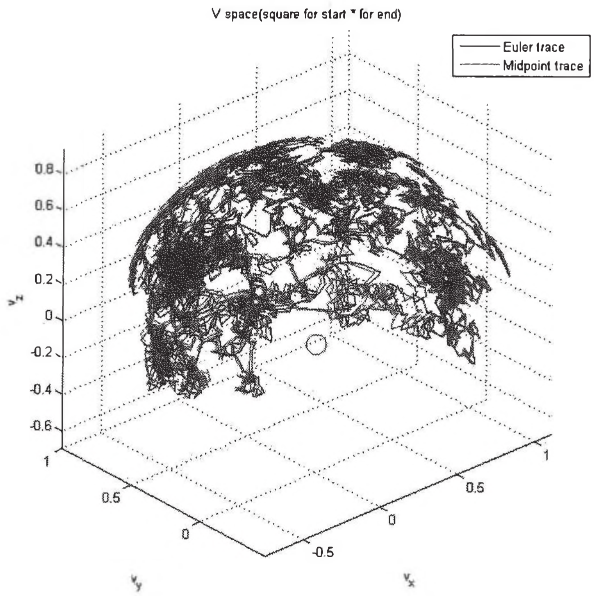

Fig. 1 A sample solution process of electron

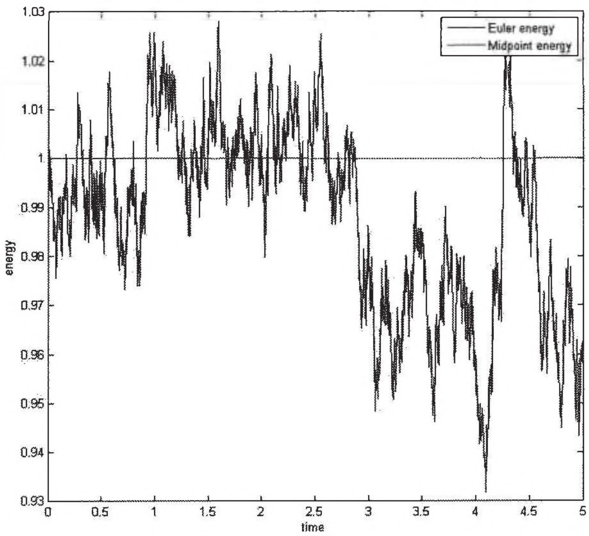

Fig. 2 Energy-time of the sample process. Midpoint method preserves the energy exactly.

In Fig. 3, the left side shows the typical backward runaway and the stopped trajectories. Sample electrons are all initialized at velocity $v_0 = (-5,0,0)v_D$, external electric field is set to $qE = (1,0,0)F_D$, time step for simulation is $\Delta\tau = \tau_D/500$ and simulation ends at $t_{end} = 10\tau_D$. Charge carried by ion is $Z_i = 1$. When the magnitude of velocity is less than $0.1v_D$, the particle is assigned to be stopped.

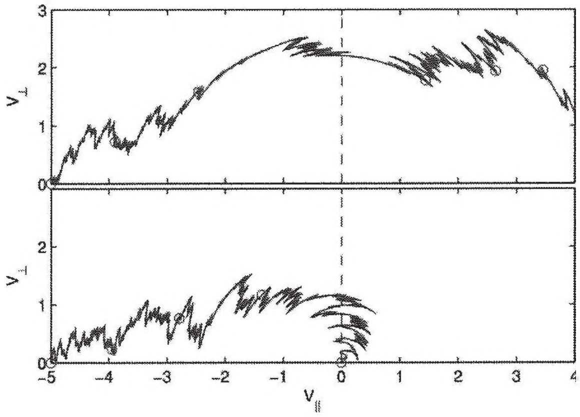

Fig. 3. Typical backward runaway and stopped particle trajectories.

Runaway probability defined by unstopped/total number of particles at time $t = 10\tau_D$ on $v_\parallel$-$v_\perp$ space. It is coincident with [1, 2] obtained by adjoint method. The contour is plotted in Fig. 4. In addition, the contour of averaged perpendicular velocity when partial passing through runaway surface defined to be $v_\parallel = 3$. Still it is coincident with previous work. It is shown in Fig. 5.

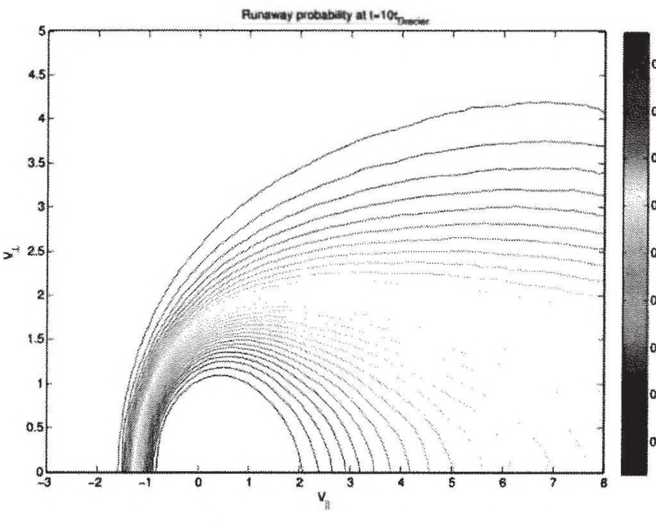

Fig. 4 Contour of runaway probability

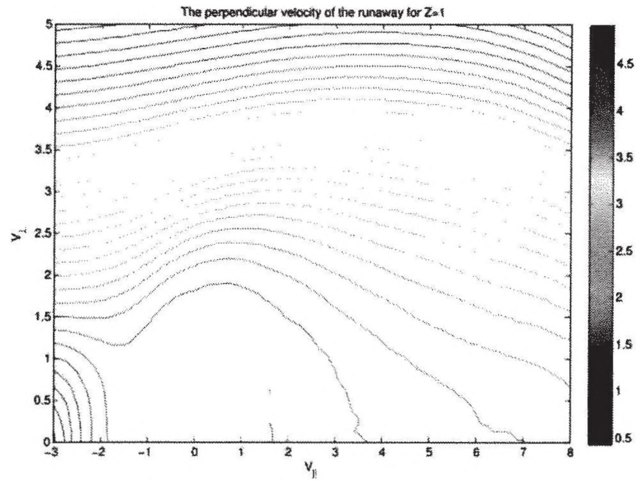

Fig. 5 Contour of averaged perpendicular velocity when partial passing through runaway surface

Summary

There is a connection between partial differential equation and stochastic differential equation. Collision effects can be described by a stochastic differential equation. Energy-preserving algorithm is applied to the stochastic differential equation.

In the following works, we are going to investigate the runaway electrons in Tokamak geometry. And figure out the characteristics of backward runaway electrons in order to find out the way to reduce the backward runaway electrons in real tokamak.

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