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Five-Loop Static Contribution to the Gravitational Interaction Potential of Two Point Masses

J. Blümlein, A. Maier, and P. Marquard

Deutsches Elektronen–Synchrotron, DESY,
Platanenallee 6, D–15738 Zeuthen, Germany

Abstract

We compute the static contribution to the gravitational interaction potential of two point masses in the velocity-independent five-loop (and 5th post-Newtonian) approximation to the harmonic coordinates effective action in a direct calculation. The computation is performed using effective field methods based on Feynman diagrams in momentum-space in $d = 3 - 2\varepsilon$ space dimensions. We also reproduce the previous results including the 4th post-Newtonian order.

1 Introduction

The interpretation of the signals detected in gravitational wave interferometers like LIGO and VIRGO [1] requires a very accurate knowledge of the binary dynamics of large coalescing masses. This also applies to the planned projects like INDIGO, LISA Pathfinder and the Einstein telescope [2]. The increasing improvements of the detector sensitivity requires highly precise theoretical predictions.

Different approaches are used as the effective one-body formalism [3–8], numerical relativity [9–11], the self-force formalism [12, 13], the post-Newtonian (PN) [14–46] and post-Minkowskian approach [6, 7, 43, 47–57] and effective field theory methods [58]; for surveys see Refs. [59–64].

In this letter we calculate a first contribution to the fifth post-Newtonian approximation: the five-loop static gravitational interaction potential between two non-spinning point masses. The two-particle force receives a series of different higher order corrections, which can be parameterized by a small parameter ϵ, cf. Ref. [59],

$$F = F_N + \sum_{k=1}^{\infty} \epsilon^{2k} F_{kPN} + \epsilon^4 F_{SO} + \epsilon^4 F_{QO} + \epsilon^5 F_{RR} + \epsilon^6 F_{1PNSO} + \epsilon^6 F_{1PNQO} + \epsilon^6 F_{OO} + \epsilon^6 F_{SS} + \epsilon^6 F_{TO} + ... \tag{1}$$

Here F_N denotes Newton's force [65], F_{kPN} is the kth post-Newtonian force, F_{RR} the 2.5 PN radiation reaction force, F_{SO} and F_{1PNSO} etc. are the spin-orbit coupling force and their post-Newtonian corrections, F_{QO} and F_{1PNQO} etc. are the quadrupole-orbit coupling force and its post-Newtonian corrections. F_{OO}, F_{SS} and F_{TO} denote the octupole-orbit coupling force, the spin-spin coupling force, and tidal-orbit coupling force, respectively.[1]

We will concentrate here on the post-Newtonian corrections of the attraction of two spinless masses in the following, using non-relativistic gravitational fields obtained by a temporal Kaluza-Klein reduction [81] followed by a Weyl rescaling [82]. They are similar to the Arnowitt-Deser-Misner fields [83]. The corresponding action has been derived in Ref. [34]. In the representation time derivatives of arbitrary order occur, which also introduce higher derivatives of the accelerations $a_{1(2)}$. Since the Lagrange density of General Relativity is of second order, these terms shall be eliminated by adding suitable double (multiple)-zero terms [22]. One aims on the terms of the order

$$\sim \frac{G_N^k}{r^k} m_1^l m_2^{k+1-l}, \quad l \in [1, k], \tag{2}$$

where $(k-1)$ labels the kth post-Newtonian approximation. Here G_N denotes Newton's constant, $r = |\mathbf{r}|$ the distance of the two masses and $m_{1(2)}$ are the two point masses.

We calculate the contribution to the fifth post-Newtonian approximation in the static limit, i.e. leaving the velocity-dependent contributions for a later work. The virial theorem [84] relates $m_1 \mathbf{v}_1^2 + m_2 \mathbf{v}_2^2 \sim \frac{G_N}{r} m_1 m_2$ on temporal average, where $\mathbf{v}_{1(2)}$ denote the velocities of the two point masses. Therefore velocity terms have to be considered at this order in general. This also applies to higher derivatives of the velocities, which can finally be mapped to terms $\propto (G_N^k/r^k) v_i^l$ by applying the equation of motion.

We first outline the basic formalism and present then the details of the calculation. Finally, we compare to the results in the literature including the fourth post-Newtonian approximation.

[1] Radiation and spin effects are discussed in Refs. [21, 66–75] and Refs. [76–80], respectively.

2 Basic Formalism

The action of General Relativity for the present problem consists of the following three components,

$$S_{GR} = S_{\rm pp} + S_{\rm EH} + S_{\rm GF}\,, \tag{3}$$

where $S_{\rm pp}, S_{\rm EH}$ and $S_{\rm GF}$ are the point-particle-, the Einstein-Hilbert- , [85], and the gauge fixing contributions. Following [34] we parameterize the Riemann metric by

$$g_{\mu\nu} = e^{2\phi/\Lambda}\begin{pmatrix} -1 & A_j/\Lambda \\ A_i/\Lambda & e^{-c_d\phi/\Lambda}\gamma_{ij} - A_iA_j/\Lambda^2\end{pmatrix}\,. \tag{4}$$

Here, the scalar field ϕ, 3-vector field A_i and tensor field γ_{ij} are introduced which parameterize the ten components of the metric tensor and Λ and c_d are given by

$$\Lambda^{-1} = \sqrt{32\pi G_N}, \tag{5}$$

$$c_d = 2\frac{d-1}{d-2}, \tag{6}$$

where $d := 3-2\varepsilon$. In the Newtonian limit of flat space–time we adopt the 'mostly plus' convention $g_{\mu\nu} \to \eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$, with $\eta_{\mu\nu}$ the Minkowskian metric. The contribution to the spatial metric γ_{ij} is parameterized by

$$\gamma_{ij} = \delta_{ij} + \sigma_{ij}/\Lambda, \tag{7}$$

where δ_{ij} denotes Kronecker's symbol. The gravitational field amplitude is given by

$$h^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta} - \eta^{\alpha\beta} \tag{8}$$

and the harmonic coordinate condition reads

$$\partial_\mu h^{\alpha\mu} = 0. \tag{9}$$

The point-particle action

$$\begin{aligned} S_{\rm pp} &= -\sum_{k=1}^{2} m_k \int d\tau_k = -\sum_{k=1}^{2} m_k \int dt \sqrt{-g_{\mu\nu}\frac{dx_k^\mu}{dt}\frac{dx_k^\nu}{dt}} \\ &= -\sum_{k=1}^{2} m_k \int dt e^{\phi/\Lambda}\sqrt{(1-\mathbf{A}.\mathbf{v}_k)^2 - e^{-c_d\phi/\Lambda}\gamma_{ij}v_k^i v_k^j} \end{aligned} \tag{10}$$

describes the point masses themselves.

The dynamics of the metric in one temporal and d spatial dimensions are captured by the Einstein-Hilbert action, [34],

$$S_{\rm EH} = \frac{1}{16\pi G_N}\int d^dx\, dt\sqrt{-g}R[g] = S^{(0)}(\gamma,A,\phi) + S^{(1)}(\gamma,A,\phi) + S^{(2)}(\gamma,A,\phi), \tag{11}$$

sorting w.r.t. the occuring number of time-derivatives. For the static contribution calculated in the present paper only $S^{(0)}$ is relevant.

$$S^{(0)}(\gamma,A,\phi) = -\frac{1}{16\pi G_N}\int d^dx\, dt\sqrt{\gamma}\left[-R[\gamma] + \frac{1}{2}c_d\gamma^{ij}\partial_i\phi\partial_j\phi - \frac{1}{4}e^{c_d\phi}\gamma^{ik}\gamma^{jl}F_{ij}F_{kl}\right], \tag{12}$$

where $R[\gamma]$ denotes the Ricci scalar and $F_{ij} = \partial_i A_j - \partial_i A_j$. The determinant γ may be represented by

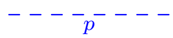
$$\begin{aligned}\gamma &= 1 + \sum_{k=1}^{3} \frac{\hat{\sigma}_k}{\Lambda^k} = 1 + \frac{1}{\Lambda}\mathrm{tr}(\sigma) + \frac{1}{2\Lambda^2}\left[\mathrm{tr}^2(\sigma) - \mathrm{tr}\left(\sigma^2\right)\right] + \frac{1}{6\Lambda^3}\left[\mathrm{tr}^3(\sigma) + 2\mathrm{tr}\left(\sigma^3\right)\right.\\ &\quad \left. -3\mathrm{tr}\left(\sigma^2\right)\mathrm{tr}(\sigma)\right], \qquad (13)\\ \sqrt{\gamma} &= 1 + \frac{\hat{\sigma}_1}{2\Lambda} - \frac{\hat{\sigma}_1^2 - 4\hat{\sigma}_2}{8\Lambda^2} + \frac{\hat{\sigma}_1^3 - 4\hat{\sigma}_1\hat{\sigma}_2 + 8\hat{\sigma}_3}{16\Lambda^3} - \frac{5\hat{\sigma}_1^4 - 24\hat{\sigma}_1^2\hat{\sigma}_2 + 32\hat{\sigma}_1\hat{\sigma}_3 + 16\hat{\sigma}_2^2}{128\Lambda^4} + O\left(\frac{1}{\Lambda^5}\right). \qquad (14)\end{aligned}$$

Finally, S_{GF} is a gauge fixing action. We use the harmonic gauge

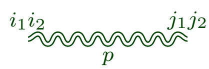
$$S_{\mathrm{GF}} = -\frac{1}{32\pi G_N}\int d^d x\ dt\sqrt{-g}\Gamma_\mu \Gamma^\nu, \qquad (15)$$

where $\Gamma^\mu = g^{\rho\sigma}\Gamma^\mu_{\rho\sigma}$ and $\Gamma^\mu_{\rho\sigma}$ denotes the Christoffel symbol.

The Feynman rules are derived from the path integral for the action (3) expanding in $1/\Lambda$ to the desired order and retaining the terms contributing in the static limit. Including the fifth post-Newtonian approximation the following relations are relevant. The Feynman rules for the propagators read

ϕ : 

$$= -\frac{i}{2c_d \mathbf{p}^2}, \qquad (16)$$

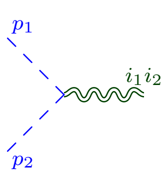
σ : 

$$= -\frac{i}{2\mathbf{p}^2}\left(\delta_{i_1 j_1}\delta_{i_2 j_2} + \delta_{i_1 j_2}\delta_{i_2 j_1} + (2 - c_d)\delta_{i_1 i_2}\delta_{j_1 j_2}\right). \qquad (17)$$

Note that the kinetic terms are not canonically normalized, hence the unusual form of the scalar propagator. The point-mass propagator in position space is $D(x, y) = \Theta(x^0 - y^0)\delta(\mathbf{x} - \mathbf{y})$. We can always arrange our calculation in such a way that only symmetric combinations $D(x, y) + D(y, x) = \delta(\mathbf{x} - \mathbf{y})$ appear. In the static case, the Fourier transform to momentum space and the resulting momentum space propagator are given by

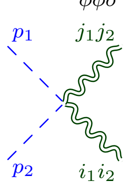
$$= 1\,. \qquad (18)$$

For the vertex Feynman rules, we choose all momenta as incoming. Choosing all momenta as outgoing or as left-to-right will give identical expressions. In contrast to [40] we follow the usual normalization for vertices involving multiple identical fields. For example, the coupling of a point mass to n scalars does not involve a factor of $1/n!$. The Feynman rules for the vertices do not agree with [40] when more than one tensor field is involved[2]



$$= i\frac{c_d}{2\Lambda}\left(V^{i_1 i_2}_{\phi\phi\sigma} + V^{i_2 i_1}_{\phi\phi\sigma}\right), \qquad (19)$$

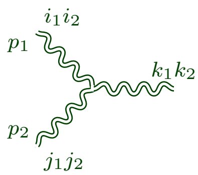
$$V^{i_1 i_2}_{\phi\phi\sigma} = \mathbf{p}_1 \cdot \mathbf{p}_2 \delta^{i_1 i_2} - 2p_1^{i_1} p_2^{i_2}\,, \qquad (20)$$



$$= i\frac{c_d}{16\Lambda^2}\left(V^{i_1 i_2, j_1 j_2}_{\phi\phi\sigma\sigma} + V^{i_2 i_1, j_1 j_2}_{\phi\phi\sigma\sigma} + V^{i_1 i_2, j_2 j_1}_{\phi\phi\sigma\sigma} + V^{i_2 i_1, j_2 j_1}_{\phi\phi\sigma\sigma}\right), \qquad (21)$$

[2]Yet we agree with their result at 4PN.

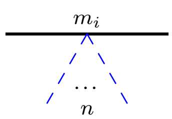
$$V_{\phi\phi\sigma\sigma}^{i_1i_2,j_1j_2} = \mathbf{p}_1\cdot\mathbf{p}_2(\delta^{i_1i_2}\delta^{j_1j_2} - 2\delta^{i_1j_1}\delta^{i_2j_2}) - 2(p_1^{i_1}p_2^{i_2}\delta^{j_1j_2} + p_1^{j_1}p_2^{j_2}\delta^{i_1i_2}) + 8\delta^{i_1j_1}p_1^{i_2}p_2^{j_2}\,, \tag{22}$$



$$= \frac{i}{32\Lambda}(\tilde{V}_{\sigma\sigma\sigma}^{i_1i_2,j_1j_2,k_1k_2} + \tilde{V}_{\sigma\sigma\sigma}^{i_2i_1,j_1j_2,k_1k_2})\,, \tag{23}$$

$$\tilde{V}_{\sigma\sigma\sigma}^{i_1i_2,j_1j_2,k_1k_2} = V_{\sigma\sigma\sigma}^{i_1i_2,j_1j_2,k_1k_2} + V_{\sigma\sigma\sigma}^{i_1i_2,j_2j_1,k_1k_2} + V_{\sigma\sigma\sigma}^{i_1i_2,j_1j_2,k_2k_1} + V_{\sigma\sigma\sigma}^{i_1i_2,j_2j_1,k_2k_1} \tag{24}$$

$$\begin{aligned} V_{\sigma\sigma\sigma}^{i_1i_2,j_1j_2,k_1k_2} = {} & (\mathbf{p}_1^2 + \mathbf{p}_1\cdot\mathbf{p}_2 + \mathbf{p}_2^2)\Big(-\delta^{j_1j_2}\big(2\delta^{i_1k_1}\delta^{i_2k_2} - \delta^{i_1i_2}\delta^{k_1k_2}\big) \\ & \quad + 2\big[\delta^{i_1j_1}\big(4\delta^{i_2k_1}\delta^{j_2k_2} - \delta^{i_2j_2}\delta^{k_1k_2}\big) - \delta^{i_1i_2}\delta^{j_1k_1}\delta^{j_2k_2}\big]\Big) \\ & + 2\Big\{4\big(p_1^{k_2}p_2^{i_2} - p_1^{i_2}p_2^{k_2}\big)\delta^{i_1j_1}\delta^{j_2k_1} \\ & \quad + 2\big[\big(p_1^{i_1} + p_2^{i_1}\big)p_2^{i_2}\delta^{j_1k_1}\delta^{j_2k_2} - p_1^{k_1}p_2^{k_2}\delta^{i_1j_1}\delta^{i_2j_2}\big] \\ & \quad + \delta^{j_1j_2}\big[p_1^{k_1}p_2^{k_2}\delta^{i_1i_2} + 2\big(p_1^{k_2}p_2^{i_2} - p_1^{i_2}p_2^{k_2}\big)\delta^{i_1k_1} - \big(p_1^{i_1} + p_2^{i_1}\big)p_2^{i_2}\delta^{k_1k_2}\big] \\ & \quad + p_2^{j_2}\Big(4p_1^{i_2}\delta^{i_1k_1}\delta^{j_1k_2} + p_1^{j_1}\big(2\delta^{i_1k_1}\delta^{i_2k_2} - \delta^{i_1i_2}\delta^{k_1k_2}\big) \\ & \qquad + 2\big[\delta^{i_1j_1}\big(p_1^{i_2}\delta^{k_1k_2} - 2p_1^{k_2}\delta^{i_2k_1}\big) - p_1^{k_2}\delta^{i_1i_2}\delta^{j_1k_1}\big]\Big) \\ & \quad + p_1^{j_2}\Big(p_1^{j_1}\big(2\delta^{i_1k_1}\delta^{i_2k_2} - \delta^{i_1i_2}\delta^{k_1k_2}\big) - 4p_2^{i_2}\delta^{i_1k_1}\delta^{j_1k_2} \\ & \qquad + 2\big[p_2^{k_2}\delta^{i_1i_2}\delta^{j_1k_1} + \delta^{i_1j_1}\big(2p_2^{k_2}\delta^{i_2k_1} - p_2^{i_2}\delta^{k_1k_2}\big)\big]\Big)\Big\}, \end{aligned} \tag{25}$$



$$= -i\frac{m_i}{\Lambda^n}\,. \tag{26}$$

Here $m_i,\ i = 1, 2$ denote the mass of the interacting world line.

3 Feynman diagrams contributing to the potential

We generate the Feynman diagrams using QGRAF [86]. To minimise the number of diagrams which will not contribute to the final result of the present calculation we do not generate self-energy diagrams, demand that the number of world line (WL) propagators are equal to the number of loops and eliminate diagrams in which two vertices are connected by more than one propagator. This results in the number of diagrams given in column 2 of Table 1. A series of sample diagrams is shown in Figure 1. Diagrams that factorise when cutting an arbitrary number of worldlines

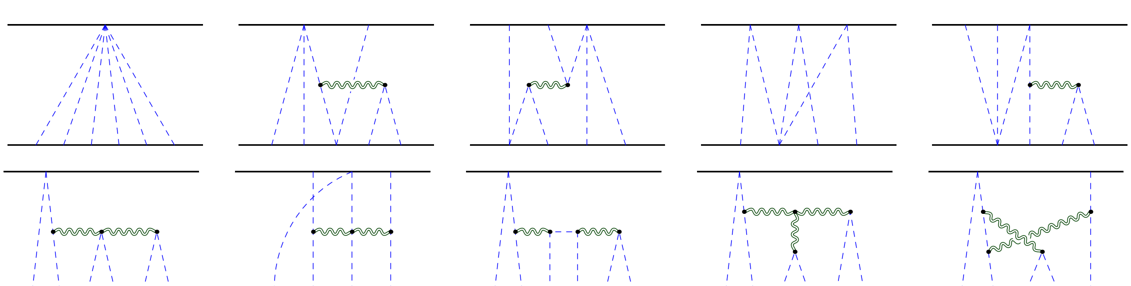


Figure 1: Sample diagrams for the $2 \to 2$ scattering process in the static limit at 5PN.

correspond to multiple potential interactions and therefore yield no additional information. We discard such diagrams (column 3). Furthermore, the contributing graphs should not have loops out of world lines (column 4) and not contain massless tadpoles in the static limit (column 5). At

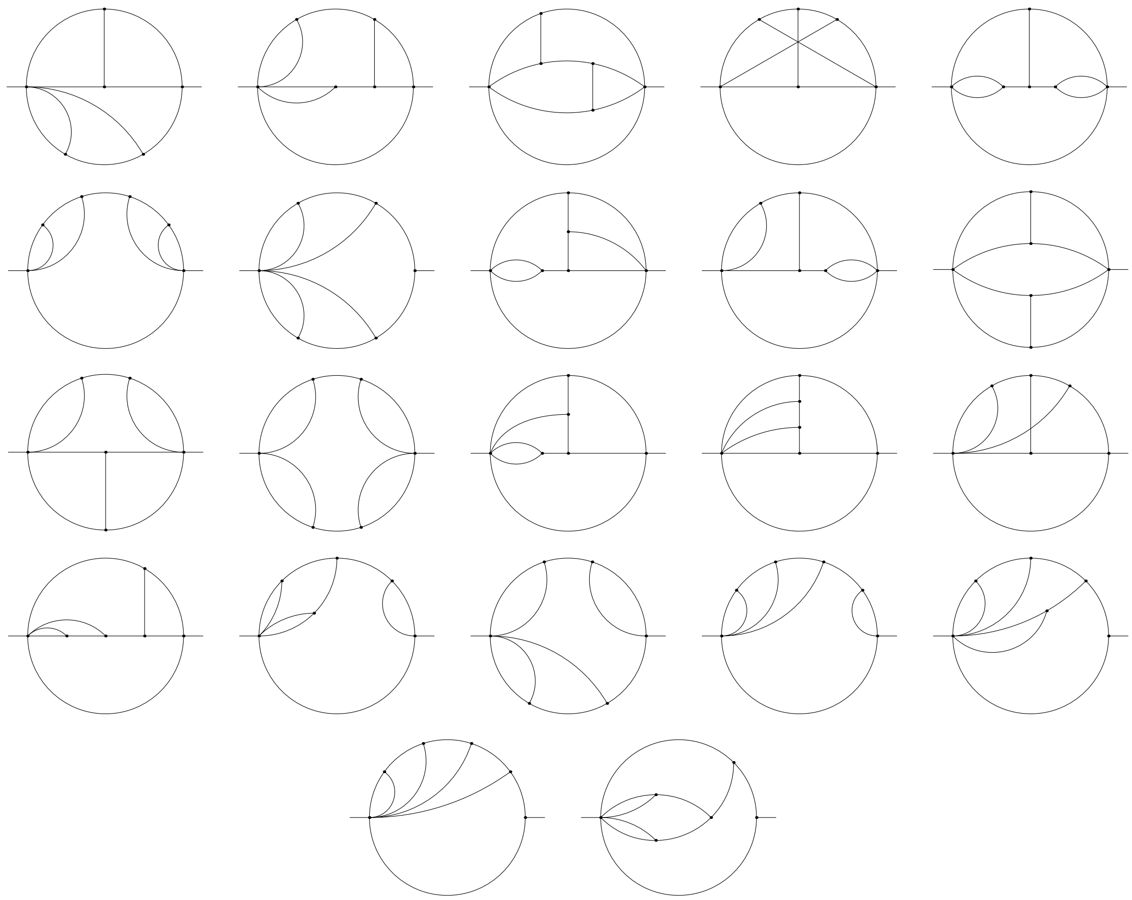


Figure 2: The topologies contributing to the potential in the static limit at 5PN.

5PN 27582 diagrams remain. In principle, these can be reduced to a smaller set of diagrams by using symmetry relations under the exchange of the world lines and all vertices on each world line, introducing additional symmetry factors. This procedure would lead to the number of diagrams shown in column 6. Up to 4PN order, we have checked that those diagrams agree with those listed in [36, 40]. However, we find that this last symmetrisation provides no benefit in our setup and do not apply it in our calculation.

In the static limit the diagrams for $2 \to 2$ scattering can be transformed into massless propagator-type diagrams representing the spacial potential between the two sources. At five-loop order we identify 22 top-level topologies, see Figure 2. We then insert the Feynman rules and perform algebraic simplifications using `FORM` [87, 88]. The remaining integrals are reduced using integration by parts [89] implemented in the package `Crusher` [90] leading to eight master integrals, out of which four contribute. In the same way we have calculated all the lower orders including 4PN. In the last column of Table 1 we list the number of master integrals, which turn out to be remarkably small. The present problem is by far simpler than the calculation of the five-loop β-function in QCD [91] and massive three-loop calculations in QCD, cf. [92].

	QGRAF	non fact.	no WL loops	no tadpoles	# Diag. [36]	# MI
N	1	1	1	1	1	-
1PN	2	2	2	2	1	1
2PN	19	19	19	15	5	1
3PN	360	276	258	122	8	1(1)
4PN	10081	5407	4685	1815	50	6(1)
5PN	332020	128080	101570	27582	154	4(4)

Table 1: Numbers of contributing diagrams at the different (post)-Newtonian levels and master integrals. The numbers in brackets denote the number of master integrals which occur during the reduction but do not contribute to the potential. In the next-to-last column the number of diagrams of equal value are given according to [36].

We have first calculated the static corrections up to 4PN in the way described above. The massless master integrals needed are known up to three loop order from Ref. [93] and at four loop order [40, 41, 94]. The master integrals depend on (multiple) zeta values [95], including ln(2). They have been compared to the numerical results given by `FIESTA` [96–99]. Here it is useful to use the Monte Carlo integrator `Divonne` [100] of the `CUBA` package [101] besides `VEGAS` [102].

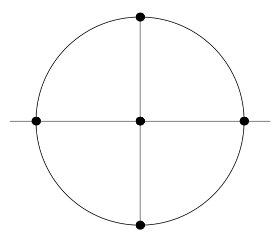
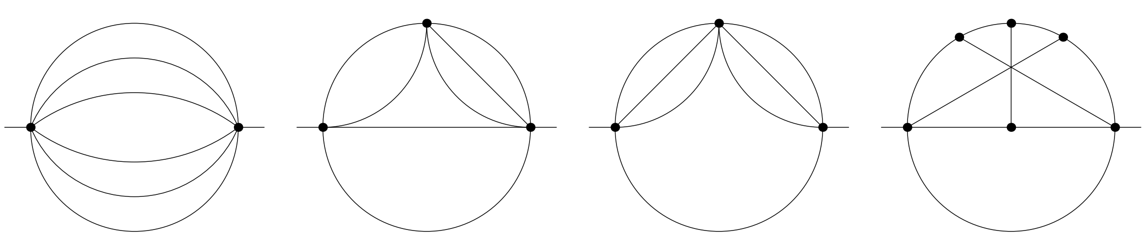


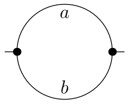
Figure 3: The master integral M_{36} contributing to the 4th post-Newtonian approximation.



M_{61} M_{72} M_{74} M_{91}

Figure 4: The five-loop master integrals contributing to the 5th post-Newtonian approximation in the static limit.

We use the $\overline{\mathsf{MS}}$-prescription in d dimensions. The one-loop two-point function is therefore defined as



$$= \int \frac{d^dk}{\pi^{d/2}} \frac{1}{((p-k)^2)^a} \frac{1}{(k^2)^b} = \frac{1}{(p^2)^{a+b-d/2}} \frac{\Gamma\left(\frac{d}{2}-a\right)\Gamma\left(\frac{d}{2}-b\right)\Gamma\left(a+b-\frac{d}{2}\right)}{\Gamma(a)\Gamma(b)\Gamma\left(d-a-b\right)}\,. \tag{27}$$

The contributing five loop master integrals can all be traced back to lower loop structures by identifying effective propagator insertions. This leads to relations of the form

$$M(p^2) = \int \frac{d^dk}{\pi^{d/2}} M_1\big(k^2\big) M_2\big((p-k)^2\big) \,, \tag{28}$$

where M_1, M_2 are suitably chosen propagator insertions and the argument of M, M_1, M_2 denotes the respective external momentum squared. As a shorthand notation we define $M := M(1)$.

The most complicated insertion is

$$M_{36} = 2\pi^2 \left[\frac{1}{\varepsilon^2} + \frac{2}{\varepsilon} - 2(16-\zeta_2) + 16\left[9 - 6\zeta_2 \left(\frac{13}{8} - \ln(2) \right) - \frac{77}{6}\zeta_3 \right] \right] \varepsilon + O(\varepsilon^2), \tag{29}$$

cf. [40, 41]. A closed form representation for general values of d of M_{36} is not known. Here $\zeta_k = \sum_{l=1}^{\infty}(1/l^k), \quad k \in \mathbb{N}, \quad k \geq 2$, denote Riemann's ζ-function at integers. With this, diagram M_{91} can be obtained as a bubble insertion of diagram M_{36} into the two-point function using

$$M_{91}(p^2) = \int \frac{d^dk}{\pi^{d/2}} \frac{1}{((p-k)^2)^a} \frac{M_{36}}{(k^2)^b} = \frac{1}{(p^2)^{a+b-d/2}} \frac{\Gamma\left(\frac{d}{2}-a\right)\Gamma\left(\frac{d}{2}-b\right)\Gamma\left(a+b-\frac{d}{2}\right)}{\Gamma(a)\Gamma(b)\Gamma\left(d-a-b\right)} M_{36}, \tag{30}$$

where $a = 1$.

The results for all master integrals contributing to the potential read

$$\begin{aligned} M_{61} &= e^{5\varepsilon\gamma_E} \frac{\Gamma\left(6-\frac{5d}{2}\right)\Gamma^6\left(-1+\frac{d}{2}\right)}{\Gamma(-6+3d)} \\ &= \pi^{7/2}\left[\frac{2}{3} + \left(\frac{134}{9} + \frac{4}{3}\ln(2) \right)\varepsilon + \left(\frac{5894}{27} + \frac{268}{9}\ln(2) + \frac{4}{3}\ln^2(2) + 19\zeta_2 \right)\varepsilon^2 \right] + O(\varepsilon^3) \,, \end{aligned} \tag{31}$$

$$\begin{aligned} M_{72} &= e^{5\varepsilon\gamma_E} \frac{\Gamma\left(7-\frac{5d}{2}\right)\Gamma\left(3-d\right)\Gamma\left(2-\frac{d}{2}\right)\Gamma^7\left(-1+\frac{d}{2}\right)\Gamma(5-2d)}{\Gamma\left(5-\frac{3}{2}d\right)\Gamma(-2+d)\Gamma\left(-3+\frac{3}{2}d\right)\Gamma\left(-7+3d\right)} \\ &= -\pi^{7/2}\Bigg[\frac{2}{\varepsilon} + 4\left(11+\ln(2)\right) + \left(656 + 88\ln(2) + 4\ln^2(2) - 25\zeta_2\right)\varepsilon \\ &\quad + \left(8288 + 1312\ln(2) + 88\ln^2(2) + 550\zeta_2 + \frac{8}{3}\ln^3(2) + 50\ln(2)\zeta_2 - \frac{2002}{3}\zeta_3 \right)\varepsilon^2 \Bigg] \\ &\quad + O(\varepsilon^3) \,, \end{aligned} \tag{32}$$

$$\begin{aligned} M_{74} &= e^{5\varepsilon\gamma_E} \frac{\Gamma\left(7-\frac{5d}{2}\right)\Gamma^2(3-d)\Gamma^7\left(-1+\frac{d}{2}\right)\Gamma\left(-6+\frac{5d}{2}\right)}{\Gamma\left(6-2d\right)\Gamma^2\left(-3+\frac{3d}{2}\right)\Gamma\left(-7+3d\right)} \\ &= -\pi^{7/2}\Bigg[\frac{4}{\varepsilon} + 72 + 8\ln(2) + \left(864 + 144\ln(2) + 8\ln^2(2) + 146\zeta_2\right)\varepsilon \\ &\quad + \left(8640 + 1728\ln(2) + 144\ln^2(2) + 2628\zeta_2 + \frac{16}{3}\ln^3(2) + 292\zeta_2\ln(2) - \frac{1988}{3}\zeta_3 \right)\varepsilon^2 \Bigg] \\ &\quad + O(\varepsilon^3) \,, \end{aligned} \tag{33}$$

$$M_{91} = 6\pi^{7/2}\Bigg[\frac{2}{\varepsilon} - 4(1-\ln(2)) - \left(48 + 8\ln(2) - 4\ln^2(2) - 105\zeta_2\right)\varepsilon + \Bigg(480 - 96\ln(2)$$

$$-8\ln^2(2)+\frac{8}{3}\ln^3(2)-530\zeta_2+402\ln(2)\zeta_2-\frac{1522}{3}\zeta_3\Big)\varepsilon^2\Big]+O(\varepsilon^3)\,. \tag{34}$$

4 Results

Inserting the master integrals into the expression for the static contribution to the Lagrangian it turns out that all pole contributions in the dimensional parameter ε cancel.

$$\mathcal{L}^S=\mathcal{L}^S_N+\mathcal{L}^S_{PN_1}+\mathcal{L}^S_{PN_2}+\mathcal{L}^S_{PN_3}+\mathcal{L}^S_{PN_4}+\mathcal{L}^S_{PN_5} \tag{35}$$

up to the fifth post-Newtonian order. One obtains

$$\mathcal{L}^S_N=\frac{G_N}{r}m_1m_2, \qquad [65] \tag{36}$$

$$\mathcal{L}^S_{PN_1}=-\frac{G_N^2}{2r^2}m_1m_2(m_1+m_2), \qquad [16] \tag{37}$$

$$\mathcal{L}^S_{PN_2}=\frac{G_N^3}{r^3}m_1m_2\left((m_1^2+m_2^2)+3m_1m_2\right), \qquad [32] \tag{38}$$

$$\mathcal{L}^S_{PN_3}=-\frac{G_N^4}{r^4}m_1m_2\left[\frac{3}{8}\left(m_1^3+m_2^3\right)+6m_1m_2(m_1+m_2)\right], \qquad [35] \tag{39}$$

$$\mathcal{L}^S_{PN_4}=\frac{G_N^5}{r^5}m_1m_2\left[\frac{3}{8}\left(m_1^4+m_2^4\right)+\frac{31}{3}m_1m_2\left(m_1^2+m_2^2\right)+\frac{141}{4}m_1^2m_2^2\right], \qquad [40] \tag{40}$$

$$\mathcal{L}^S_{PN_5}=-\frac{G_N^6}{r^6}m_1m_2\left[\frac{5}{16}(m_1^5+m_2^5)+\frac{91}{6}m_1m_2(m_1^3+m_2^3)+\frac{653}{6}m_1^2m_2^2(m_1+m_2)\right], \tag{41}$$

with $r=|\mathbf{r}_1-\mathbf{r}_2|$. We agree with the above results in the literature up to $\mathcal{L}^S_{PN_4}$. Moreover, including 5PN, all the contributions due to multiple zeta values also cancel and only rational coefficients remain.

To be explicit, the pole part of $\mathcal{L}^S_{PN_5}$ in terms of the master integrals is given by

$$\left[\mathcal{L}^S_{PN_5}\right]_{\varepsilon^{-1}}=-\frac{G_N^6}{r^6}(m_1m_2)^3(m_1+m_2)\pi^{-7/2}\frac{45}{32}\Big[2M_{72}-M_{74}\Big]_{\varepsilon^{-1}}=0\,, \tag{42}$$

where $[X]_{\varepsilon^n}$ denotes the coefficient of ε^n in X. With this in mind, the finite part can be written as

$$\begin{aligned}\left[\mathcal{L}^S_{PN_5}\right]_{\varepsilon^0}=&-\frac{G_N^6}{r^6}(m_1m_2)\pi^{-7/2}\Bigg\{\frac{15}{32}(m_1^5+m_2^5)\Big[M_{61}\Big]_{\varepsilon^0}+\frac{91}{4}m_1m_2(m_1^3+m_2^3)\Big[M_{61}\Big]_{\varepsilon^0}\\&+m_1^2m_2^2(m_1+m_2)\Bigg(\Big[\frac{293}{4}M_{61}-\frac{45}{16}M_{72}+\frac{45}{32}M_{74}\Big]_{\varepsilon^0}\\&+\Big[\frac{519}{16}M_{72}-\frac{627}{32}M_{74}+2M_{91}\Big]_{\varepsilon^{-1}}\Bigg)\Bigg\}\,.\end{aligned} \tag{43}$$

Let us first compare to the Lagrangian in harmonic coordinates, linear in the accelerations $a_{1(2)}$, [39],

$$\mathcal{L}^h_N=\frac{G_N}{r}m_1m_2, \tag{44}$$

$$
\begin{aligned}
\mathcal{L}^h_{PN_1} &= -\frac{G_N^2}{2r^2} m_1 m_2 (m_1+m_2), && (45)\\
\mathcal{L}^h_{PN_2} &= \frac{G_N^3}{r^3} m_1 m_2 \left(\frac{1}{2}(m_1^2+m_2^2) + \frac{19}{4} m_1 m_2\right), && (46)\\
\mathcal{L}^h_{PN_3} &= -\frac{G_N^4}{r^4} m_1 m_2 \left\{\frac{3}{8}\left(m_1^3+m_2^3\right) + \left(-\frac{9707}{420}(m_1+m_2)\right.\right.\\
&\quad \left.\left. +\frac{22}{3}\left[m_1 \ln\left(\frac{r}{r'_1}\right) + m_2 \ln\left(\frac{r}{r'_2}\right)\right]\right) m_1 m_2\right\}, && (47)\\
\mathcal{L}^h_{PN_4} &= \frac{G_N^5}{r^5} m_1 m_2 \left[\frac{3}{8}\left(m_1^4+m_2^4\right) + \left(\frac{1690841}{25200} + \frac{315}{16}\zeta_2\right) m_1 m_2 \left(m_1^2+m_2^2\right)\right.\\
&\quad - \left(\frac{242}{3}\ln\left(\frac{r}{r'_1}\right) + 16 \ln\left(\frac{r}{r'_2}\right)\right) m_1^3 m_2 - \left(\frac{242}{3}\ln\left(\frac{r}{r'_2}\right) + 16 \ln\left(\frac{r}{r'_1}\right)\right) m_1 m_2^3\\
&\quad \left. + \left(\frac{587963}{2800} - \frac{231}{8}\zeta_2 - \frac{110}{3}\ln\left(\frac{r^2}{r'_1 r'_2}\right)\right) m_1^2 m_2^2\right]. && (48)
\end{aligned}
$$

We see that the first terms of $\mathcal{L}^S_{PN_i}$ and $\mathcal{L}^h_{PN_i}$ agree. As outlined in the Section 1 the static contributions (35) are only one part of the Lagrangian. There are further the velocity-dependent contributions and terms due to their higher time derivatives, which have to be considered as well. Using harmonic coordinates one also obtains ε dependent and logarithmic terms $\sim \ln(r/r'_i), \;\; i = 1,2$ with constants r'_i, which have to be removed by redefining the coordinates, cf. [39].

The corresponding corrections are known up to the fourth post-Newtonian contribution.[3] For comparison we refer to the results up to $\mathcal{L}_{PN_4}$, setting the velocities $\mathbf{v}_{1,2} = 0$ in the final result of [39],

$$
\begin{aligned}
\mathcal{L}_N &= \frac{G_N}{r} m_1 m_2, && (49)\\
\mathcal{L}_{PN_1} &= -\frac{G_N^2}{2r^2} m_1 m_2 (m_1+m_2), && (50)\\
\mathcal{L}_{PN_2} &= \frac{G_N^3}{r^3} m_1 m_2 \left(\frac{m_1^2+m_2^2}{4} + \frac{5}{4} m_1 m_2\right) && (51)\\
\mathcal{L}_{PN_3} &= -\frac{G_N^4}{r^4} m_1 m_2 \left[\frac{1}{8}\left(m_1^3+m_2^3\right) + \frac{(454-189\zeta_2)}{48} m_1 m_2 (m_1+m_2)\right], && (52)\\
\mathcal{L}_{PN_4} &= \frac{G_N^5}{r^5} m_1 m_2 \left[\frac{1}{16}\left(m_1^4+m_2^4\right) + \left(\frac{3421459}{50400} - \frac{18711}{512}\zeta_2\right) m_1 m_2 \left(m_1^2+m_2^2\right)\right.\\
&\quad \left. + \left(\frac{4121669}{25200} - \frac{44825}{512}\zeta_2\right) m_1^2 m_2^2\right]. && (53)
\end{aligned}
$$

Here the coefficients of also the leading coefficients change for 3PN and higher. Starting with $\mathcal{L}_{PN_3}$ also zeta values contribute from the velocity-dependent sector.

5 Conclusions

We have calculated the five-loop correction to the gravitational interaction potential between two static point masses. This constitutes an important part of the effective gravitational Lagrangian

[3] See also Ref. [55].

at fifth post-Newtonian order. We also agree with the corresponding results in the lower post-Newtonian orders. The calculation of the velocity-dependent terms at this order is work in progress.

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Note added.
A few hours before the present submission the preprint [103] appeared, presenting an independent calculation of the same quantity. The final results, Eq. (41) in this work and Eq. (32) in [103], are in agreement.

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