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Asymptotics and closed form of a generalized incomplete gamma function

M.A. Chaudhry[a], N.M. Temme[b,*], E.J.M. Veling[c]

[a] *Department of Mathematical Sciences, King Fahd University, Dhahran, Saudi Arabia*
[b] *CWI, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands*
[c] *RIVM, P.O. Box 1, 3720 BA Bilthoven, The Netherlands*

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Abstract

We consider the asymptotic behavior of the function

$$\Gamma(\alpha,x;b)=\int_x^\infty t^{\alpha-1}\mathrm{e}^{-t-b/t}\,\mathrm{d}t,\quad x>0,\ \alpha>0,\ b\geqslant 0,$$

as x tends to infinity. We give several expansions for the case that α and b are fixed and we give a uniform expansion in which α and b may range through unbounded intervals. We give a closed form for half-integer values of α.

Keywords: Generalized incomplete gamma functions; Asymptotic expansion; Error function

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1. Introduction

The function

$$\Gamma(\alpha,x;b)=\int_x^\infty t^{\alpha-1}\mathrm{e}^{-t-b/t}\,\mathrm{d}t,\quad x>0,\ \alpha>0,\ b\geqslant 0,\tag{1.1}$$

has been studied in detail in the recent paper [4]; see also the problem in [3]. The function $\Gamma(\alpha,x;b)$ can be used in closed form solutions to several problems in heat conduction with time-dependent boundary conditions. Moreover, the function in (1.1) plays a role in cumulative probability functions. For details we refer to the above mentioned paper.

* Corresponding author.

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We concentrate on the parameter domains indicated in (1.1), but all results to be derived in this paper can be extended to large complex domains of the parameters. In fact, when $x \neq 0$, the function $\Gamma(\alpha, x; b)$ is an entire function of both variables α and b. The function $\Gamma(\alpha, x; b)$ can be viewed as a generalization of the well-known incomplete gamma function: when we take $b = 0$ it reduces to $\Gamma(\alpha, x)$. When we take $x = 0$ it becomes a modified Bessel function:

$$\Gamma(\alpha, 0; b) = \int_0^\infty t^{\alpha-1} e^{-t-b/t} dt = 2b^{\alpha/2} K_\alpha(2\sqrt{b}), \quad b > 0. \tag{1.2}$$

In this paper we give several expansions of $\Gamma(\alpha, x; b)$ that are valid when the parameter x is large. When α and b are fixed, we give expansions in terms of incomplete gamma functions, Laguerre polynomials, and confluent hypergeometric functions. Although several of these expansions turn out to be convergent, we concentrate on the asymptotic character of the expansions. We also consider a result in which all three parameters α, x, b may range through unbounded domains. In that case, which is of special interest from the viewpoint of cumulative probability functions, an error function is used as basic approximant in a uniform asymptotic expansion. In the final section we give a closed form expression in terms of modified Bessel functions and error functions for $\alpha = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$

2. An expansion in terms of incomplete gamma functions

Using the series expansion of $\exp(-b/t)$ in (1.1), we obtain the expansion

$$\Gamma(\alpha, x; b) = \sum_{n=0}^\infty \frac{(-b)^n}{n!} \Gamma(\alpha - n, x), \tag{2.1}$$

which has already been given in [4]. This expansion is the Maclaurin expansion of $\Gamma(\alpha, x; b)$; recall that this function is an entire function of the variable b. For fixed values of n and α we have

$$\Gamma(\alpha - n, x) \sim x^{\alpha-n-1} e^{-x}, \quad x \to \infty.$$

It follows that (2.1) can be viewed as an asymptotic expansion as $x \to \infty$, when b and α are restricted to bounded intervals of the domains indicated in (1.1). The incomplete gamma functions in (2.1) can be obtained by recursion:

$$(\alpha - n)\Gamma(\alpha - n, x) = \Gamma(\alpha - n + 1, x) - x^{\alpha-n} e^{-x},$$

which is not numerically stable (see [5]). The expansion in (2.1) looses its asymptotic character when α and/or b are $\mathcal{O}(x)$ or larger.

3. An expansion in terms of Laguerre polynomials

To obtain (2.1) the expansion of $\exp(-b/t)$ was used, which can be viewed as an expansion at $t = \infty$. But the main contributions to the integral in (1.1) come from an immediate neighborhood at the right of $t = x$. Hence, let us expand

$$t^{\alpha-1} e^{-b/t} = \sum_{n=0}^\infty c_n(x)(t - x)^n, \tag{3.1}$$

where the coefficients will be given below. Substituting this into (1.1), we obtain

$$\Gamma(\alpha, x; b) \sim \mathrm{e}^{-x} \sum_{n=0}^{\infty} n!\, c_n(x). \tag{3.2}$$

This expansion has an asymptotic character as $x \to \infty$, with α, b fixed, and reduces to the asymptotic expansion of $\Gamma(\alpha, x)$ as $b \to 0$.

The coefficients $c_n(x)$ can be written in terms of Laguerre polynomials. To show this we use the generating function for the Laguerre polynomials:

$$(1-t)^{-\mu-1} \mathrm{e}^{-tx/(1-t)} = \sum_{n=0}^{\infty} L_n^{\mu}(x) t^n, \quad \mu, x \in \mathbb{C}, \ |t| < 1.$$

From this result it easily follows that

$$c_n(x) = (-1)^n x^{\alpha-1-n} \mathrm{e}^{-b/x} L_n^{-\alpha}(b/x).$$

The coefficients can be computed by using the recurrence relation of the Laguerre polynomials:

$$(n+1) L_{n+1}^{\mu}(x) = (2n + \mu + 1 - x) L_n^{\mu}(x) - (n+\mu) L_{n-1}^{\mu}(x), \quad n = 1, 2, 3, \ldots,$$

with $L_0^{\mu}(x) = 1$, $L_1^{\mu}(x) = 1 + \mu - x$. This relation is numerically stable.

4. An expansion in terms of confluent hypergeometric functions

Substituting in (1.1) the expansion

$$\mathrm{e}^{-b/t} = \mathrm{e}^{-b/x} \mathrm{e}^{b(t-x)/(xt)} = \mathrm{e}^{-b/x} \sum_{n=0}^{\infty} \frac{b^n}{n!} \frac{(t-x)^n}{(xt)^n},$$

we obtain

$$\Gamma(\alpha, x; b) = x^{\alpha} \mathrm{e}^{-x} \mathrm{e}^{-b/x} \sum_{n=0}^{\infty} \frac{b^n}{x^n} d_n, \tag{4.1}$$

where

$$d_n = \frac{1}{n!} \int_0^{\infty} u^n (u+1)^{\alpha-n-1} \mathrm{e}^{-xu} \,\mathrm{d}u = U(n+1, \alpha+1, x), \tag{4.2}$$

where U denotes the confluent hypergeometric function. Since $U(a, c, x) \sim x^{-\alpha}$ as $x \to \infty$, we see that the asymptotic behavior of the terms in the series is $\mathcal{O}(b^n / x^{2n+1})$, which is much better than in the previous cases. Moreover, the series is convergent and has positive terms. The convergence follows from the fact that $d_n = \mathcal{O}(1/n!)$ as $n \to \infty$ (we can use in the integral in (4.2) the estimation $|u/(1+u)| \leqslant 1$, $u \geqslant 0$).

The coefficients d_n can be computed by using a recursion relation for the U-functions:

$$U(a-1, c, x) + (c - 2a - x) U(a, c, x) + a(1 + a - c) U(a+1, c, x) = 0,$$

which is numerically unstable in forward direction. For the computational aspects of the U-function, in particular for the problem of computing a sequence of values $\{U(a+n, c, x)\}$, we refer to [9].

5. A uniform expansion in terms of the error function

In this section we consider the parameter α as the large parameter, and x and b as free parameters, which may range through the interval $(0, \infty)$. This will not be a restriction on the parameter x, which was the large parameter in the previous sections. For instance, in the expansion of this section we can take α fixed and let x tend to infinity. The expansion has a double (or perhaps triple) asymptotic property. First we scale the parameters in (1.1) with respect to α:

$$t = \alpha u, \qquad b = \tfrac{1}{4}\alpha^2\beta^2, \qquad x = \alpha\xi,$$

with $\beta > 0$, and obtain:

$$\Gamma(\alpha, x; b) = \alpha^\alpha \int_\xi^\infty \mathrm{e}^{-\alpha\phi(u)} \frac{\mathrm{d}u}{u},$$

with

$$\phi(u) = u + \tfrac{1}{4}\beta^2/u - \ln u.$$

The function ϕ assumes its minimal value on $(0, \infty)$ at

$$u_0 = \tfrac{1}{2}(1 + \sqrt{1+\beta^2}),$$

which point will be outside the interval of integration $[\xi, \infty)$ when x and/or b are small with respect to α. When x and/or b grow, the point u_0 will pass the point $u = \xi$, and the asymptotic behavior of the function $\Gamma(\alpha, x; b)$ will change considerably. In this case an error function (a normal distribution function) is needed to describe the asymptotic behavior.

We write:

$$\Gamma(\alpha, x; b) = \alpha^\alpha \mathrm{e}^{-\alpha\phi(u_0)} \int_\xi^\infty \mathrm{e}^{-\alpha[\phi(u)-\phi(u_0)]} \frac{\mathrm{d}u}{u},$$

and substitute

$$\tfrac{1}{2}v^2 = \phi(u) - \phi(u_0), \qquad \mathrm{sign}(v) = \mathrm{sign}(u - u_0),$$

which gives

$$\Gamma(\alpha, x; b) = \alpha^\alpha \mathrm{e}^{-\alpha\phi(u_0)} \int_\eta^\infty \mathrm{e}^{-\alpha v^2/2} f(v)\,\mathrm{d}v, \tag{5.1}$$

where

$$\tfrac{1}{2}\eta^2 = \phi(\xi) - \phi(u_0), \qquad \mathrm{sign}(\eta) = \mathrm{sign}(\xi - u_0), \qquad f(v) = \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}v}.$$

We observe that η is positive when α is large with respect to x and/or b (that is, $u_0 < \xi$), and that η will change sign when x and/or b become large (that is, $u_0 > \xi$).

We use the error function erfc z defined by

$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2}\, dt.$$

It follows that

$$\sqrt{\frac{\alpha}{2\pi}} \int_\eta^{\infty} e^{-\alpha v^2/2} f(v)\, dv = \tfrac{1}{2} \operatorname{erfc}(\eta\sqrt{\alpha/2}) f(0) - \sqrt{\frac{1}{2\pi\alpha}} \int_\eta^{\infty} \frac{f(v) - f(0)}{v}\, d e^{-\alpha v^2/2}$$

$$= \tfrac{1}{2} \operatorname{erfc}(\eta\sqrt{\alpha/2}) f(0) + \sqrt{\frac{1}{2\pi\alpha}} \frac{f(\eta) - f(0)}{\eta} e^{-\alpha\eta^2/2}$$

$$+ \frac{1}{\alpha} \sqrt{\frac{\alpha}{2\pi}} \int_\eta^{\infty} e^{-\alpha v^2/2} f_1(v)\, dv,$$

where

$$f_1(v) = \frac{d}{dv} \frac{f(v) - f(0)}{v}.$$

Repeating this process, we obtain the expansion

$$\sqrt{\frac{\alpha}{2\pi}} \int_\eta^{\infty} e^{-\alpha v^2/2} f(v)\, dv \sim \tfrac{1}{2} \operatorname{erfc}(\eta\sqrt{\alpha/2}) \sum_{n=0}^{\infty} \frac{f_n(0)}{\alpha^n} + \frac{e^{-\alpha\eta^2/2}}{\sqrt{2\pi\alpha}} \sum_{n=0}^{\infty} \frac{f_n(\eta) - f_n(0)}{\eta\alpha^n}, \tag{5.2}$$

where the sequence of functions $\{f_n\}$ is defined by $f_0 = f$ and

$$f_n(v) = \frac{d}{dv} \frac{f_{n-1}(v) - f_{n-1}(0)}{v}, \quad n = 1, 2, 3, \ldots$$

We observe that the first series in (5.2) does not depend on x. In fact we can relate the first series with a known expansion. First we introduce two functions $A(\alpha, \beta)$, $B(\alpha, \beta, \eta)$ by writing

$$\sqrt{\frac{\alpha}{2\pi}} \int_\eta^{\infty} e^{-\alpha v^2/2} f(v)\, dv = \tfrac{1}{2} \operatorname{erfc}(\eta\sqrt{\alpha/2})\, A(\alpha, \beta) + \frac{e^{-\alpha\eta^2/2}}{\sqrt{2\pi\alpha}} B(\alpha, \beta, \eta). \tag{5.3}$$

Using (5.1) and (5.3) we obtain

$$\sqrt{\frac{\alpha}{2\pi}} \alpha^{-\alpha} e^{-\alpha\phi(u_0)}\, \Gamma(\alpha, x; b) = \tfrac{1}{2} \operatorname{erfc}(\eta\sqrt{\alpha/2})\, A(\alpha, \beta) + \frac{e^{-\alpha\eta^2/2}}{\sqrt{2\pi\alpha}} B(\alpha, \beta, \eta), \tag{5.4}$$

which we consider as an exact representation. When we let $x \to 0$, we observe that from definition of η it follows that $\eta \to -\infty$, and that the error function in (5.4) tends to the limi Using (1.2), we infer that

$$A(\alpha, \beta) = \sqrt{\frac{2\alpha}{\pi}}\, e^{\alpha\phi(u_0)} (\beta/2)^\alpha K_\alpha(\alpha\beta), \qquad ($$

and that the first series in (5.2) corresponds to the uniform expansion of the modified Be function $K_\alpha(\alpha\beta)$; see [1, p. 378, 7], [p. 378].

The function $B(\alpha,\beta,\eta)$ is rather difficult to compute; that is, the terms in the second series of (are complicated functions of β and η. By differentiating (5.3) with respect to η, we obtain a rat simple differential equation for B:

$$\frac{1}{\alpha}\frac{d}{d\eta} B(\alpha,\beta,\eta) - \eta B(\alpha,\beta,\eta) = A(\alpha,\beta) - f(\eta), \qquad ($$

which may be a starting point for numerical algorithms. Substituting the expansions

$$A(\alpha,\beta) \sim \sum_{n=0}^{\infty} \frac{A_n(\beta)}{\alpha^n}, \qquad \beta(\alpha,\beta,\eta) \sim \sum_{n=0}^{\infty} \frac{B_n(\beta,\eta)}{\alpha^n},$$

in which the coefficients A_n are known from the modified Bessel function expansion, we obtain recursion relation

$$\eta B_{n+1}(\beta,\eta) = \frac{d}{d\eta} B_n(\beta,\eta) - A_{n+1}(\beta), \quad n = 0,1,2,\ldots,$$

with

$$B_0(\beta,\eta) = \frac{f(\eta) - A_0(\beta)}{\eta} = \frac{f(\eta) - f(0)}{\eta}.$$

The result of the final part of this section is based on a method introduced in [8], where method is applied to the incomplete gamma functions.

6. A closed form for $\alpha = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$

In this section, we derive an explicit expression for $\Gamma(\alpha, x; b)$ for half-integer values of α. Alre in [2] the possibility is mentioned to evaluate this integral, but Binet did not obtain a closed fo The result reads as follows. Let $\alpha = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$ Then

$$\begin{aligned}\Gamma(\alpha, x; b) &= b^{\alpha/2}\{K_\alpha(2\sqrt{b}) + \pi(-1)^{\alpha-1/2} I_\alpha(2\sqrt{b})\}\operatorname{erfc}(\sqrt{x} + \sqrt{b/x}) \\ &\quad + b^{\alpha/2} K_\alpha(2\sqrt{b})\operatorname{erfc}(\sqrt{x} - \sqrt{b/x}) + 2e^{-x-b/x} \sum_{j=0}^{\alpha-3/2} x^{j+1/2} b^{\alpha/2-1/2j-1/4} \\ &\quad \times \{(-1)^{\alpha+j+1/2} K_{j+1/2}(2\sqrt{b}) I_\alpha(2\sqrt{b}) + I_{j+1/2}(2\sqrt{b}) K_\alpha(2\sqrt{b})\}. \qquad (\end{aligned}$$

Proof. Firstly, we mention the result (see [1, (7.4.33)], [4, (23)])

$$\Gamma(\tfrac{1}{2},x;b)=\tfrac{1}{2}\sqrt{\pi}\,e^{2\sqrt{b}}\operatorname{erfc}(\sqrt{x}+\sqrt{b/x})+\tfrac{1}{2}\sqrt{\pi}\,e^{-2\sqrt{b}}\operatorname{erfc}(\sqrt{x}-\sqrt{b/x}).$$

The integral becomes easy if one realizes that the factor $1/\sqrt{t}$ in the integrand can be written as

$$\frac{1}{\sqrt{t}}=\frac{1}{2}\left(\frac{1}{\sqrt{t}}-\frac{1}{t}\sqrt{b/t}\right)+\frac{1}{2}\left(\frac{1}{\sqrt{t}}+\frac{1}{t}\sqrt{b/t}\right).$$

Similarly, [4, (28)],

$$\Gamma(-\tfrac{1}{2},x;b)=-\tfrac{1}{2}\sqrt{\pi/b}\,e^{2\sqrt{b}}\operatorname{erfc}(\sqrt{x}+\sqrt{b/x})+\tfrac{1}{2}\sqrt{\pi/b}\,e^{-2\sqrt{b}}\operatorname{erfc}(\sqrt{x}-\sqrt{b/x}),$$

by writing in the integrand

$$\frac{1}{t\sqrt{t}}=-\frac{1}{2\sqrt{b}}\left(\frac{1}{\sqrt{t}}-\frac{1}{t}\sqrt{b/t}\right)+\frac{1}{2\sqrt{b}}\left(\frac{1}{\sqrt{t}}+\frac{1}{t}\sqrt{b/t}\right).$$

By examining the recursion (see [4, (15)])

$$\Gamma(\alpha+1,x;b)=\alpha\Gamma(\alpha,x;b)+b\Gamma(\alpha-1,x;b)+x^{\alpha}e^{-x-b/x},\quad \alpha\geqslant\tfrac{1}{2},$$

it is clear that for $\alpha=-\frac{1}{2},\frac{1}{2},\frac{3}{2},\ldots$, the function $\Gamma(\alpha,x;b)$ has the general form

$$\Gamma(\alpha,x;b)=P^{+}(\alpha;b)\sqrt{\pi}\,e^{2\sqrt{b}}\operatorname{erfc}(\sqrt{x}+\sqrt{b/x})$$
$$+P^{-}(\alpha;b)\sqrt{\pi}\,e^{-2\sqrt{b}}\operatorname{erfc}(\sqrt{x}-\sqrt{b/x})+G(\alpha,x;b)e^{-x-b/x}. \tag{6.2}$$

P^{+}, P^{-}, G are quantities satisfying the following recursions:

$$P^{+}(\alpha+1;b)=\alpha P^{+}(\alpha;b)+bP^{+}(\alpha-1;b),\qquad P^{+}(-\tfrac{1}{2};b)=-\frac{1}{2\sqrt{b}},\qquad P^{+}(\tfrac{1}{2};b)=\tfrac{1}{2},$$

$$P^{-}(\alpha+1;b)=\alpha P^{-}(\alpha;b)+bP^{-}(\alpha-1;b),\qquad P^{-}(-\tfrac{1}{2};b)=\frac{1}{2\sqrt{b}},\qquad P^{-}(\tfrac{1}{2};b)=\tfrac{1}{2},$$

$$G(\alpha+1,x;b)=\alpha G(\alpha,x;b)+bG(\alpha-1,x;b)+x^{\alpha},\qquad G(-\tfrac{1}{2},x;b)=0,\qquad G(\tfrac{1}{2},x;b)=0.$$

Write

$$P^{\pm}(\alpha;b)=(-1)^{k}b^{b/2}Q_{k}^{\pm}(b),\quad k=\alpha-\tfrac{1}{2},\ k=-1,0,1,\ldots,$$

then $Q_k^{\pm}(b)$ satisfy the recurrence relation

$$Q_{k+1}^{\pm}(b)-Q_{k-1}^{\pm}(b)=\frac{2k+1}{2\sqrt{b}}Q_{k}^{\pm}(b).$$

A set of independent solutions of this recurrence relation is (see [1, (10.2.18)])

$$u_k^1(z) = (-1)^{k+1}\frac{\sqrt{\pi/2}}{\sqrt{z}}K_{k+1/2}(z), \quad u_k^2(z) = \frac{\sqrt{\pi/2}}{\sqrt{z}}I_{k+1/2}(z), \qquad z = 2\sqrt{b}.$$

Using the initial values to start the recursion, we find

$$P^+(\alpha; b) = \frac{1}{\sqrt{\pi}} b^{k/2+1/4} e^{-2\sqrt{b}}(K_{k+1/2}(2\sqrt{b}) + \pi(-1)^k I_{k+1/2}(2\sqrt{b})),$$

$$P^-(\alpha; b) = \frac{1}{\sqrt{\pi}} b^{k/2+1/4} e^{2\sqrt{b}}\, K_{k+1/2}(2\sqrt{b}). \tag{6.3}$$

The inhomogeneous recurrence relation will be solved by the method of undetermined coefficients. Write

$$G(\alpha, x; b) = (-1)^k b^{k/2} \tilde{G}_k(x; b), \quad k = \alpha - \tfrac{1}{2}, \ k = -1, 0, 1, \ldots$$

$\tilde{G}_k$ satisfies

$$\tilde{G}_{k+2} + \frac{2k+3}{2\sqrt{b}}\tilde{G}_{k+1} - \tilde{G}_k = (-1)^k b^{-k/2-1} x^{k+3/2}, \qquad \tilde{G}_{-1} = 0, \quad \tilde{G}_0 = 0, \qquad k \geqslant -1.$$

Write $\tilde{G}_k(x; b) = A_k u_k^1(z) + B_k u_k^2(z)$. Then this method gives the equations

$$(A_{k+1} - A_k)u_{k+1}^1 + (B_{k+1} - B_k)u_{k+1}^2 = 0,$$

$$(A_{k+1} - A_k)(u_{k+2}^1 - u_{k+1}^1) + (B_{k+1} - B_k)(u_{k+2}^2 - u_{k+1}^2) = (-1)^k b^{-k/2-1} x^{k+3/2}.$$

From these two equations we can solve for $a_k = A_{k+1} - A_k$, $b_k = B_{k+1} - B_k$, which yields

$$a_k = A_{k+1} - A_k = -2^{3/2} b^{-k/2-1/4} x^{k+3/2}\sqrt{2/\pi}\, I_{k+3/2}(2\sqrt{b}),$$

$$b_k = B_{k+1} - B_k = (-1)^k 2^{3/2} b^{-k/2-1/4} x^{k+3/2}\sqrt{2/\pi}\, K_{k+3/2}(2\sqrt{b}).$$

Furthermore, $\tilde{G}_{-1} = 0$ implies $A_{-1} = cI_{-1/2}(z)$, $B_{-1} = -cK_{-1/2}(z)$, for some constant c. If one evaluates $\tilde{G}_0$:

$$\tilde{G}_0 = (a_{-1} + A_{-1})u_0^1 + (b_{-1} + B_{-1})u_0^2$$

$$= -\frac{\sqrt{\pi/2}}{\sqrt{z}} c(K_{1/2}(z)I_{-1/2}(z) + K_{-1/2}(z)I_{1/2}(z)) = -\frac{\sqrt{\pi/2}}{\sqrt{z}}\frac{c}{z}$$

(see [1, (9.6.15)]) one finds $c = 0$, by the requirement $\tilde{G}_0 = 0$. It follows that $A_{-1} = 0$, $B_{-1} = 0$. With this information A_k, B_k can be found by a summation of $a_{j'}$, $b_{j'}$ over $j' = -1, \ldots, k-1$. The last term in the summation cancels. By substituting $j = j' + 1$, $k = \alpha - \frac{1}{2}$, we find the expression for $G(\alpha, x; b)$. Together with (6.2), (6.3) the announced result (6.1) follows. □

If one lets b tend to 0, the function $\Gamma(\alpha, x; b)$ reduces to $\Gamma(\alpha, x)$, and (6.1) reduces to formula (A8) in [6]. The right-hand side of (6.1) is regular at $b = 0$, because the singularities cancel each other. We have the following results

$$\Gamma(-\tfrac{1}{2}, x) = -2\sqrt{\pi}\,\operatorname{erfc}\sqrt{x} + 2\frac{\mathrm{e}^{-x}}{\sqrt{x}}, \qquad \Gamma(\tfrac{1}{2}, x) = \sqrt{\pi}\,\operatorname{erfc}(\surd),$$

$$\Gamma(\alpha, x) = \Gamma(\alpha)\left\{\operatorname{erfc}(\sqrt{x}) + \mathrm{e}^{-x}\sum_{j=0}^{\alpha-3/2}\frac{x^{j+1/2}}{(j+\frac{1}{2})\Gamma(j+\frac{1}{2})}\right\}, \quad \alpha = \frac{3}{2}, \frac{5}{2}, \ldots,$$

where we have used $K_{-1/2}(z) = K_{1/2}(z)$ and the limit behavior of $K_\alpha(z)$, $I_\alpha(z)$ as $z \downarrow 0$:

$$K_\alpha(z) \sim \tfrac{1}{2}\Gamma(\alpha)\left(\frac{2}{z}\right)^\alpha, \quad \Re\alpha > 0, \quad I_\alpha(z) \sim \frac{1}{\Gamma(\alpha+1)}\left(\frac{z}{2}\right)^\alpha, \qquad \alpha \neq -1, -2, \ldots$$

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