Reliability-Based Design Optimization of Structural Systems under Stochastic Excitation: An Overview

D. J. Jerez[1], H. A. Jensen[2,3], and M. Beer[1,3,4]

[1] *Institute for Risk and Reliability, Leibniz Universität Hannover, 30167 Hannover, Germany.*

[2] *Department of Civil Engineering, Federico Santa Maria Technical University, Valparaiso, Chile.*

[3] *International Joint Center for Engineering Reliability and Stochastic Mechanics, Tongji University, Shanghai 200092, China.*

[4] *Institute for Risk and Uncertainty and School of Engineering, University of Liverpool, Liverpool L69 72F, United Kingdom of Great Britain and Northern Ireland.*

Abstract

This article presents a brief survey on some of the latest developments in the area of reliability-based design optimization of structural systems under stochastic excitation. The contributions are grouped into three main categories, namely, sequential optimization approaches, stochastic search based techniques, and schemes based on augmented reliability spaces. The different approaches are described and summarized. In addition, remarks are provided about their range of application, advantages, disadvantages, relevance, and potential research directions. The literature review indicates that computational aspects play a key role in the solution of this class of optimization problems. Besides, this overview suggests that methods for optimal design in stochastic structural dynamics are no longer restricted to academic-type of problems but they can be used as tools in a class of engineering design problems as well.

Keywords: Metamodels, Optimization techniques, Reliability analysis, Sensitivity analysis, Simulation techniques, Stochastic dynamical systems, Stochastic optimization.

1. Introduction

One of the most important contributions of structural engineering to modern society is the design of safe and efficient systems to accomplish a wide variety of goals, including industrial requirements, needs of private users, and the provision of critical functions to the public. Structures are usually devised to be optimum in terms of a given criterion, while satisfying a number of design requirements under certain loading conditions [1]. Of particular importance are dynamic loads associated with environmental actions, such as wind effects, earthquakes, sea waves, etc. [2, 3, 4]. One of the main

Email address: `hector.jensen@usm.cl` (D. J. Jerez[1], H. A. Jensen[2,3], and M. Beer[1,3,4])

characteristics of this class of excitations is their uncertainty, since it is not possible to accurately predict future loading conditions. In this context, stochastic excitation models are a viable and common means to represent these uncertainties in an explicit manner during the design process [5, 6, 7]. Moreover, proper design procedures must take into account all uncertainties associated with the system under consideration as they may lead to significant deviations from the expected behavior of final designs [8, 9, 10]. In this regard, probabilistic approaches such as reliability-based formulations provide a realistic and rational framework for structural optimization which explicitly accounts for the uncertainties during the design process [11, 12].

In the framework of this contribution, attention is directed to reliability-based optimization (RBO) problems involving structural dynamical systems under stochastic excitation. In this case, the number of basic random variables involved in the characterization of the problem may be high (tens, hundreds or even thousands), and first-passage probabilities over a certain reference period are considered for reliability assessment [13, 14, 15]. Then, the evaluation of reliability integrals constitutes a high-dimensional problem that is extremely challenging from the numerical viewpoint. Even though some early approaches have considered approximate reliability formulations in order to circumvent integral evaluations [16, 17], they are mostly limited to simple systems and their accuracy is usually disputed. The use of simulation methods [18, 19, 20], however, is the most widely accepted approach for the evaluation of high-dimensional reliability integrals due to their generality and ability to obtain accurate and robust estimates. The application of these techniques generally requires hundreds or thousands of dynamic analyses, which can lead to significant computational efforts. In summary, it is the objective of this contribution to provide a systematic review on recent developments addressing reliability-based optimization problems of structures under stochastic excitation where the system reliability is characterized in terms of first-passage probabilities. The contributions under study have been categorized into three groups, namely, (i) sequential optimization approaches, (ii) stochastic search based techniques, and (iii) formulations in augmented reliability spaces. In each category, the main contributions are described and summarized. In addition, some remarks are provided on the range of application, relevance, general advantages, possible disadvantages, and potential future research efforts associated with the different techniques.

The structure of the work is as follows. Section 2 provides the formulation of the problem and highlights its main challenges. A general description of the proposed categories is given in Section 3. Section 4 describes sequential optimization approaches. Contributions based on stochastic search based techniques are presented in Section 5. Optimization frameworks relying on augmented reliability formulations are examined in Section 6. The paper closes with some conclusions and potential research directions.

2. Problem Description

2.1. Mechanical Modeling

The general class of structural systems considered in the different contributions of this overview is characterized by a multi-degree of freedom model satisfying the equation of motion

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) + \mathbf{k}_{NL}\left(\mathbf{u}(t), \dot{\mathbf{u}}(t), \mathbf{y}(t)\right) = \mathbf{f}(t) \tag{1}$$

where $\mathbf{u}(t)$ denotes the displacement vector, $\mathbf{k}_{NL}(\mathbf{u}(t), \dot{\mathbf{u}}(t), \mathbf{y}(t))$ the vector of nonlinear restoring forces, $\mathbf{y}(t)$ the vector of variables that describe the state of the nonlinear components, and $\mathbf{f}(t)$ the external force vector. The matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ describe the mass, damping, and stiffness of the system, respectively. The evolution of the set of variables $\mathbf{y}(t)$ is described by an appropriate nonlinear model which depends on the nature of the nonlinearity. The equation of motion for the displacement vector $\mathbf{u}(t)$ and the equation for the evolution of the set of variables $\mathbf{y}(t)$ constitute a system of coupled nonlinear equations. Note that for linear systems the vector of nonlinear restoring forces verifies $\mathbf{k}_{NL}(\mathbf{u}(t), \dot{\mathbf{u}}(t), \mathbf{y}(t)) = \mathbf{0}$. For realistic applications the solution of Eq. (1) relies on complex black-box computational procedures such as the finite element method.

2.2. Formulation

The reliability-based design optimization problem is stated as

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & c(\mathbf{x}) \\
\text{subject to} \quad & r_j(\mathbf{x}) \leq 0, \qquad j = 1, \ldots, n_r \\
& g_j(\mathbf{x}) \leq 0, \qquad j = 1, \ldots, n_g \\
& \mathbf{x} \in X \subset \mathbb{R}^{n_x}
\end{aligned}
\tag{2}
$$

where $\mathbf{x} = \langle x_1, \ldots, x_{n_x} \rangle^T \in X \subset \mathbb{R}^{n_x}$ is the vector of n_x design or control variables (continuous and/or discrete), $c(\mathbf{x})$ is a general cost function, $r_j(\mathbf{x}) \leq 0, j = 1, \ldots, n_r$ correspond to n_r constraints on the system reliability, and $g_j(\mathbf{x}) \leq 0, j = 1, \ldots, n_g$ represent n_g standard constraints. The set X represents the constraints on the design variables. For each continuous design variable x_i, the constraints are given in terms of its lower and upper bounds such that $x_i^L \leq x_i \leq x_i^R$, whereas for each discrete variable x_i the constraints are characterized by a finite set of possible values. The objective function $c(\mathbf{x})$ can quantify initial, construction, maintenance, repair or downtime costs, structural performance, users' comfort, cost of failure, life-cycle cost, etc. Moreover, the constraints $g_j(\mathbf{x}) \leq 0, j = 1, \ldots, n_g$ are associated with design conditions such as material availability, geometric requirements, budget restrictions, etc., that do not require structural reliability assessment. Hence, it is assumed that the standard constraints are relatively inexpensive to evaluate. Finally, the reliability constraints $r_j(\mathbf{x}) \leq 0, j = 1, \ldots, n_r$ represent structural requirements expressed in a probabilistic manner and can be defined in terms of different criteria such as serviceability and partial or total collapse failure. The reliability constraint functions are written in terms of failure probabilities as

$$
r_j(\mathbf{x}) = P_{F_j}(\mathbf{x}) - P_{F_j}^*, \quad j = 1, \ldots, n_r \tag{3}
$$

where $P_{F_j}(\mathbf{x})$ is the probability of failure event F_j, evaluated at design $\mathbf{x}$, and $P_{F_j}^*$ is the corresponding maximum allowable value. Note that, according to the formulation of the RBO problem, failure probabilities can be associated with the objective function, the constraint functions, or both. As previously pointed out, reliability assessment (i.e., evaluation of failure probabilities) for structural systems under stochastic excitation is an involved task from the numerical viewpoint [13, 14, 15, 21]. A more thorough description of the challenges arising in this context is presented in Section 2.5. Based on the previous formulation, it is seen that the optimization problem stated in equation (2) is quite general in the sense that different formulations can be devised for the RBO of structural systems under stochastic excitation [11, 12]. In fact, different applications have been considered in this context, including the seismic design of fluid filled tanks [22], the mitigation of seismic pounding risk between buildings [23], the design of wind-excited cable-stayed masts [24] and high-rise buildings [25, 26, 27], the design of nonlinear devices for seismic protection [28, 29], the topology optimization of building systems [30], and the design of large-scale linear systems [31, 32]. Finally, it is noted that the previous formulation can be extended to multi-objective optimization problems, where two or more objective functions are simultaneously minimized while satisfying a number of design requirements [33, 34].

2.3. Reliability Measures

System reliability measures are usually expressed in terms of equivalent failure probability measures. In the context of structural systems under stochastic excitation, the probability that certain performance conditions are not satisfied within a given reference period provides a useful measure for the likelihood of failure events. This quantity is referred to as *first-passage probability* and quantifies the plausibility of occurrence of unacceptable behavior of the structural system [13]. In this framework, consider a vector $\boldsymbol{\theta} \in \Theta \subset \mathbb{R}^{n_\theta}$ of random variables involved in the characterization of the system. This vector comprises the variables associated with the representation of the stochastic excitation and uncertain system parameters. The random variable vector follows a multivariate probability density function (PDF) $q(\boldsymbol{\theta}|\mathbf{x})$, that is, $\boldsymbol{\theta} \sim q(\boldsymbol{\theta}|\mathbf{x})$. It is noted that this PDF can depend on the design variables $\mathbf{x}$. This is the case when some distribution parameters, e.g. mean values, are associated with the design variables. In case no design variable influences the distribution of the basic random variables, they are simply distributed as $\boldsymbol{\theta} \sim q(\boldsymbol{\theta})$. A failure event F that indicates if certain design requirements or desired performance conditions are not met within a given reference period T, can be written as

$$F(\mathbf{x}) = \{\boldsymbol{\theta} \in \Theta \subset \mathbb{R}^{n_\theta} : d(\mathbf{x}, \boldsymbol{\theta}) > 1\} \tag{4}$$

where $d(\mathbf{x}, \boldsymbol{\theta})$ is the so-called normalized demand function evaluated at design $\mathbf{x}$ and at a given realization of $\boldsymbol{\theta}$. This function is usually defined as

$$d(\mathbf{x}, \boldsymbol{\theta}) = \max_{t \in [0,T]} \max_{\ell=1,\ldots,n_h} \frac{|h_\ell(t; \mathbf{x}, \boldsymbol{\theta})|}{h_\ell^*} \tag{5}$$

where $h_\ell(t; \mathbf{x}, \boldsymbol{\theta})$, $\ell = 1, \ldots, n_h$ are the response functions of interest with corresponding maximum allowable values $h_\ell^* > 0$. These responses are computed from the solution of Eq. (1) and they are time-dependent, due to the dynamic nature of the excitation, and also depend on the design variables, $\mathbf{x}$, and the random variables, $\boldsymbol{\theta}$. Thus, the normalized demand function quantifies the maximum demand-to-capacity ratio observed during the reference period T across all the responses of interest. It is noted that, however, alternative definitions of the normalized demand function can also be considered. In the previous setting, the failure probability function $P_F(\mathbf{x})$ measures the plausibility of unacceptable structural behavior at the design $\mathbf{x}$ according to given performance criteria or design requirements. The first-excursion probability can be written in terms of a multidimensional integral as

$$P_F(\mathbf{x}) = \int_{d(\mathbf{x},\boldsymbol{\theta})>1} q(\boldsymbol{\theta}|\mathbf{x})d\boldsymbol{\theta} = \int_{\boldsymbol{\theta}\in\Theta} I_F(\mathbf{x},\boldsymbol{\theta})q(\boldsymbol{\theta}|\mathbf{x})d\boldsymbol{\theta} \tag{6}$$

where $I_F(\mathbf{x},\boldsymbol{\theta})$ is the indicator function, with $I_F(\mathbf{x},\boldsymbol{\theta}) = 1$ if $d(\mathbf{x},\boldsymbol{\theta}) > 1$ and $I_F(\mathbf{x},\boldsymbol{\theta}) = 0$ otherwise. As previously pointed out, $\boldsymbol{\theta}$ is high-dimensional for the type of systems under consideration. Therefore, the previous integral represents a high-dimensional problem whose evaluation at each design constitutes a demanding task from the numerical point of view [13, 14, 15]. Although some simplifications can be made in order to obtain approximate expressions that reduce the computational cost of evaluating the previous multidimensional integral, they are mostly limited to simple linear systems subject to stationary white noise excitation [16, 17]. Thus, the evaluation of (6) relies on the use of advanced stochastic simulation techniques for realistic and practical cases. Finally, it is noted that reliability measures can also be defined in terms of time-varying reliability. Such formulations, in the context of RBO problems, are not considered in the present overview.

2.4. Stochastic Simulation Methods

Stochastic simulation techniques are widely accepted as an effective means for the reliability assessment of general structural systems subject to stochastic excitation [14, 15]. This class of approaches relies on the generation of samples of the basic random variables $\boldsymbol{\theta}$, and the evaluation of the corresponding normalized demand function values $d(\mathbf{x},\boldsymbol{\theta})$ in order to populate the important regions of the failure domain F. The most well known stochastic simulation technique is Monte Carlo simulation (MCS) [18]. Generally, a large number of samples is required by MCS in order to reach a certain level of accuracy. Thus, the corresponding computational burden can be prohibitive for involved structural systems in which a single analysis requires significant computational effort. This difficulty, which is the main drawback of MCS for reliability assessment, has motivated the development of alternative simulation tools.

Several advanced simulation methods have been developed to address the reliability assessment of involved systems. The distinctive feature of these approaches is the implementation of specialized sampling strategies that allow to obtain sufficiently accurate estimates of the failure probability with a reduced number of samples. Examples of these techniques in the context of complex high-dimensional reliability problems include Subset simulation [35, 36], Subset simulation based on hidden variables [37, 38], Importance sampling [39], Line sampling [40], Horseracing simulation [41], Domain decomposition method [42], Directional importance sampling [43], and the Probability density evolution method [44, 45]. It is noted that even though advanced simulation methods provide improved efficiency for reliability assessment, a significant number of system re-analyses (usually in the order of hundreds or thousands) are still required to obtain failure probability estimates.

2.5. Challenges

As indicated in previous sections, the RBO of structural systems under stochastic excitation is a challenging task. The difficulties arising in this type of problems are associated with the computational cost, noisy behavior and sensitivity evaluation of the functions involved in the problem. In fact, as already pointed out, the high dimension of the uncertain parameter space for the type of systems under consideration leads to the use of advanced simulation techniques in order to estimate failure probabilities. Thus, a large number of dynamic analyses should be carried out, in principle, at any given design during the optimization process. As a consequence, the corresponding computational efforts can be significantly high specially for involved structural systems where a single dynamic analysis can take significant computational time. In this regard, parallelization techniques or existing computational power can be exploited to increase the computational efficiency of the overall design process. In addition, the estimation of failure probability functions relies on stochastic simulation procedures and, therefore, these estimates exhibit some variability. In other words, any simulation-based failure probability estimate inherently possesses some variability that must be taken into account. Finally, it is noted that several optimization procedures make use of the gradients (i.e., the sensitivities) of the objective and constraint functions in order to explore the design space in an effective manner [1, 46]. However, sensitivity evaluation of failure probability functions is a challenging task [47, 48]. The previous difficulties must be properly addressed by any RBO approach. In fact, the inadequate treatment of these features could lead to the identification of sub-optimal solutions or to choose final designs that are actually unfeasible.

3. General Classification of Approaches

The contributions studied in this work have been classified based on the search strategy and the type of information required during the optimization process. In particular, three general categories are considered: sequential optimization schemes, stochastic search based techniques, and formulations based on augmented reliability spaces. Sequential optimization schemes (see Section 4) consider iterative schemes in which surrogates for the failure probability functions are introduced at each stage. Then, based on these surrogates, an ordinary optimization problem is solved using any standard search technique to obtain a new candidate solution. In general, these methods require the full assessment of only few designs during the entire design process. The corresponding failure probability surrogates usually require the evaluation of both the failure probability functions and their derivatives. On the other hand, techniques based on stochastic search schemes (see Section 5) rely on randomized search in the design space. The randomization principle is known to be, in general, an effective means to escape a local optimum as well as to make the design process less sensitive to the noisy nature of failure probability functions. These approaches commonly require only information on the failure probability function values, avoiding sensitivity evaluation procedures. Finally, formulations based on augmented reliability spaces (see Section 6) simultaneously consider the design variables and the basic random variables. An instrumental variability is artificially introduced to the design variables. Failure probability functions are then replaced by marginal probability density functions in the augmented space, avoiding nested reliability assessment.

4. Sequential Optimization Schemes

As previously pointed out, one of the challenges of solving RBO problems involving structural systems under stochastic excitation is the high computational cost. One strategy that has gained considerable attention for circumventing this issue is the formulation of sequential optimization approaches. During each optimization cycle, the failure probability functions are replaced by surrogates that are relatively inexpensive to evaluate and make use of information gathered around the current solution [12]. Then, the new approximate problem is solved by means of a suitable search technique in order to identify a new candidate solution. The process is repeated until some convergence criterion is verified.

4.1. Exponential-Type of Approximations

In this class of approaches, the original optimization problem is replaced by a sequence of approximate sub-optimization problems. Each sub-optimization problem involves explicit closed-form approximations for the reliability constraints in terms of the design variables and, therefore, it can be efficiently solved by any suitable standard search algorithm, such as sequential quadratic programming (SQP), nonlinear programming by quadratic Lagragian (NLPQL), etc. In addition, move limits on the design variables are imposed in order to control the quality of the approximations. The problems of interest correspond to the minimization of a deterministic cost function subject to reliability constraints. As originally proposed in [49] for deterministic linear systems, the failure probability functions are locally approximated around the current candidate solution $\mathbf{x}^k$ during each optimization cycle as

$$P_F(\mathbf{x}) \approx \tilde{P}_F(\mathbf{x};\mathbf{x}^k) = \exp\left(a_0 + \sum_{i=1}^{n_x} a_i(x_i - x_i^k)\right) \tag{7}$$

where $a_i, i = 0, 1, \ldots, n_x$ are polynomial coefficients obtained in terms of $n_x + 1$ direct evaluations of $P_F(\mathbf{x})$ around the current candidate design. An efficient importance sampling technique [39] is integrated to assess the failure probability with reduced computational effort. The approach is extended to uncertain linear systems under stochastic excitation in [50]. For increased efficiency, approximate system responses instead of full structural analyses are considered to evaluate structural reliability measures. In particular, modal participation factors and the corresponding natural frequencies are approximated using a convex linearization scheme [51]. This requires the computation of the derivatives of the system's eigenvectors and eigenvalues with respect to the design variables and uncertain structural parameters, which is carried out using an efficient method [52]. In this manner, a single structural and sensitivity analysis is required during each cycle of the proposed approach to formulate the approximate sub-optimization problem.

The previous contributions involved the repeated evaluation of the failure probability function in the vicinity of the current candidate solution to compute the polynomial coefficients. An alternative approach is proposed in [53] for linear systems with random structural parameters subject to general Gaussian excitation. The sought coefficients are obtained by solving a set of nonlinear equations in order to match the average and first-order moments of the failure probability function in a vicinity Ω^k of the current candidate solution $\mathbf{x}^k$, that is,

$$P_F^{\text{average}} = \frac{1}{|\Omega^k|}\int_{\Omega^k} P_F(\mathbf{x})d\mathbf{x}, \quad m_{P_F}^i = \frac{1}{|\Omega^k|}\int_{\Omega^k} x_i P_F(\mathbf{x})d\mathbf{x}, \quad i = 1, \ldots, n_x \tag{8}$$

where $|\Omega^k|$ is the hyper-volume of Ω^k, and $P_F(\mathbf{x})$ is written as in Eq. (7). Moreover, the *augmented reliability concept* (see Section 6 for more details) is considered. The idea is to artificially treat the design variables $\mathbf{x}$ as uncertain. Then, a single simulation run in the joint space $\langle \mathbf{x}, \boldsymbol{\theta} \rangle$ can be used to estimate P_F^{average} and $m^i_{P_F}$. Moreover, similar to [50], approximate responses are used and, therefore, a single dynamical and sensitivity analysis of the system is required during each optimization cycle. A different approach is proposed in [54]. In this case, the polynomial coefficients are defined using information obtained from a reliability sensitivity analysis. The idea is to match the partial derivatives of the failure probability function with those of the exponential approximation. The required quantities are computed using the augmented reliability concept and direct Monte Carlo simulation.

4.2. Convex and Conservative Approximations

The contributions presented in the previous subsection are based on linear estimations for the logarithm of the failure probability and move limits to control their accuracy. A different class of methods considers the implementation of *convex global approximations* of all functions involved in the optimization problem. During each optimization cycle, an approximate optimization problem is generated by replacing the objective and constraint functions with expansions around the current candidate design in terms of direct and reciprocal variables. No move limits are imposed on the design variables. Due to the simple explicit algebraic structure of each sub-optimization problem, it can be efficiently solved with standard search techniques to find a new candidate solution. The process is repeated until a certain stopping criterion is verified. The sequential optimization framework based on convex global approximations is initially proposed for the RBO of dynamical systems under stochastic excitation in [55]. In this setting, each function $f(\mathbf{x})$ involved in the optimization problem is approximated around the current candidate solution $\mathbf{x}^k$ during each optimization cycle as

$$f(\mathbf{x}) \approx \tilde{f}(\mathbf{x};\mathbf{x}^k) = f(\mathbf{x}^k) + \sum_{(i^+)} \frac{\partial f(\mathbf{x}^k)}{\partial x_i}(x_i - x_i^k) + \sum_{(i^-)} \frac{\partial f(\mathbf{x}^k)}{\partial x_i}\frac{x_i^k}{x_i}(x_i - x_i^k) \quad (9)$$

where $\sum_{(i^+)}$ and $\sum_{(i^-)}$ indicate summation over the variables belonging to the groups (i^+) and (i^-), respectively. Group (i^+) contains the variables for which $\partial f/\partial x_i(\mathbf{x}^k)$ is positive, and group (i^-) includes the remaining variables. This expansion corresponds to a linearization in terms of the direct variables (x_i) for group (i^+) and of the reciprocal variables $(1/x_i)$ for group (i^-). An attractive property of this mixed linearization, which is also referred to as *convex linearization*, is that it yields the most conservative approximation among all possible combinations of direct/reciprocal variables [51]. Moreover, the expansion is a convex and separable function, which is beneficial from the optimization viewpoint. However, this linearization is not guaranteed to be conservative in an absolute sense. In other words, the approximations of the different functions involved in the RBO problem are not necessarily more conservative than the original ones. In this regard, conservatism of the approximations can be forced by including second-order terms in Eq. (9) [56]. The use of convex and conservative approximations has been demonstrated in RBO problems involving complex structural systems equipped with nonlinear devices. In addition, the approach has been also applied to handle mixed discrete-continuous design variables [56, 57]. In this case, a dual formulation is introduced at each optimization cycle and then solved by means of a standard first-order algorithm to obtain a new candidate solution.

The optimization framework based on convex and conservative approximations requires the evaluation of the objective and constraint functions at the current candidate solution as well as their partial derivatives. In particular, failure probability functions and their sensitivities must be computed. In [55, 57] linear approximations for the logarithm of the failure probability functions are considered, where the corresponding coefficients are obtained by matching the average and first-order moments of the failure probability function in a vicinity of the current design (see [53]). On the other hand, in [56, 58, 59], reliability sensitivities are estimated using a two-level approximation framework embedded in subset simulation [60, 61]. The main features of this framework are discussed in the next subsection. Alternatively, the approach proposed in [62] integrates conservative approximations with the probability density evolution method (PDEM) [44, 45] and the change of probability measure (COM) technique [63], allowing to obtain the required sensitivity information as a by-product of the reliability assessment step.

4.3. Line Search Methods

The integration of approximation schemes for the failure probability functions into well established line search methods is a promising research venue for the RBO of stochastic dynamical systems under stochastic excitation. The main idea of this class of approaches, initially proposed in [60, 61], is that failure probability functions are only required along the search direction during each optimization cycle. Thus, failure probability surrogates need to be formulated only in one dimension instead of an n_x-dimensional space. This feature can certainly help to obtain high-quality approximations without excessive computational efforts. In this context, each cycle of the optimization process considers the following steps [64]. First, a search direction is identified based on the values of the objective function, standard constraints and reliability constraints, as well as their sensitivities. This step generally involves the solution of a system or systems of linear equations associated with first-order optimality conditions [46, 65]. Then, one-dimensional surrogates for the reliability constraints along the search direction are initially established. After that, a new candidate solution is identified by means of a standard line search procedure based on these approximations. During this process, new information on the actual failure probability functions and their sensitivities along the search direction is gathered. This is used to adaptively improve the metamodels and provide more accurate approximations for the failure probability functions along the search direction. The one-dimensional surrogate of any failure probability function formulated about the current candidate solution $\mathbf{x}^k$ and along the search direction $\mathbf{v}^k$, as originally introduced in [60], is given by

$$P_F(\mathbf{x}^k + \tau\mathbf{v}^k) \approx \tilde{P}_F(\tau) = \exp\left(a_0 + a_1\tau + a_2\tau^2\right), \quad \tau \geq 0 \tag{10}$$

where $\tau \geq 0$ is a step size along $\mathbf{v}^k$. The coefficients $a_\ell, \ell = 0, 1, 2$ are computed by means of a least squares problem that takes into account information on the failure probability functions and their directional derivatives [66]. These coefficients are continuously updated during the line search process as new candidate designs are evaluated.

The implementation of this framework requires the computation of the gradients of the failure probability functions. As already pointed out, this is a challenging task especially in nonlinear stochastic dynamics. An efficient approach for reliability sensitivity estimation embedded in the framework of subset simulation [35, 36] has been considered to this end [60, 61]. The most salient feature of this technique is that it requires a single subset simulation run plus some additional structural analyses in order to estimate reliability sensitivity. First, the failure probability is expressed as a function of a threshold d^* for the normalized demand, that is,

$$P\left[d(\mathbf{x},\boldsymbol{\theta}) \geq d^*\right] = P_F(d^*) \approx \exp\left[\psi_0 + \psi_1(d^* - 1)\right] \tag{11}$$

where $d^* \in [1-\epsilon, 1+\epsilon]$, ε represents a small tolerance, and ψ_0 and ψ_1 are real constants. These coefficients are obtained by means of a least squares fit based on the relationship between P_F and the normalized demand threshold d^*, which is obtained from subset simulation. In addition, a linear surrogate of the normalized demand function in the vicinity of the current candidate solution $\mathbf{x}^k$ is defined as

$$d(\mathbf{x},\boldsymbol{\theta}) \approx \tilde{d}(\mathbf{x},\boldsymbol{\theta}) = d(\mathbf{x}^k,\boldsymbol{\theta}) + \sum_{i=1}^{n_x} \delta_i (x_i - x_i^k) \tag{12}$$

where the coefficients $\delta_i, i = 1, \ldots n_x$, are computed by means of a least squares fit. The corresponding training points are generated by perturbing the design variables and reusing samples drawn from subset simulation that are near the failure boundary. In this manner, improved accuracy for the limit state surface is expected in the vicinity of $\mathbf{x}^k$. Based on the previous approximations, the gradient of the failure probability function can be estimated as [60, 61]

$$\left.\frac{\partial P_F(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\mathbf{x}^k} \approx \psi_1 \delta_i P_F(\mathbf{x}^k), \quad i = 1, \ldots, n_x \tag{13}$$

Generally, relatively few additional model evaluations are required to obtain sufficiently accurate estimates of the reliability sensitivities [60, 61]. The strategy is further developed in [67] by including uncertain structural parameters as well as an explicit quantification of the effects of the uncertainty in system properties on the final design. An alternative approach based on an interior point algorithm together with the integration of the PDEM [44, 45], metamodels at the structural response level and a finite difference scheme has been proposed in [68]. Interior point schemes were also applied to multi-objective optimization problems in [69]. In particular, the efficient determination of specific compromise solutions (Pareto solutions) is carried out by a compromise programming approach [34]. The application of reduced-order models based on substructure coupling for dynamic analysis [70, 71] is demonstrated in [72] as a means of additional efficiency improvement in the context of interior point methods.

4.4. A Threshold Based Local Approximation

The idea of the approach proposed in [73] is to avoid the evaluation of failure probabilities during each optimization cycle by reusing the reliability analysis results at the previous candidate design $\mathbf{x}^k$. To this end, a linear representation similar to the one given in Eq. (12) is considered to approximate the normalized demand function. Then, the failure probability function is written as

$$P_F(\mathbf{x}) = P[d(\mathbf{x},\boldsymbol{\theta}) \geq 1] \approx P\left[d(\mathbf{x}^k,\boldsymbol{\theta}) \geq 1 - \sum_{i=1}^{n_x} \delta_i (x_i - x_i^k)\right] \tag{14}$$

which is estimated using the results obtained from a subset simulation run at $\mathbf{x}^k$. In other words, each approximate sub-optimization problem replaces $P_F(\mathbf{x})$ with the failure probability at the previous candidate design corresponding to the threshold $(1 - \sum_{i=1}^{n_x} \delta_i(x_i - x_i^k))$. Move limits on the design variables are imposed to control the quality of the approximations. Any appropriate search technique can be implemented to solve the sequence of sub-optimization problems. In particular, genetic algorithms [74] are considered in [73]. The method is demonstrated in linear and nonlinear examples involving few discrete design variables.

4.5. Heuristic Framework Based on Operator Norm Optimization

This approach deals with reliability optimization subject to standard constraints, where the main idea is to heuristically replace the original objective function, i.e. the failure probability function, with a function defined in terms of a matrix norm [75]. The contribution is tailored to linear systems subject to stochastic excitation where all model parameters are deterministic. In this framework, a vector containing n_t discrete values of the ℓ^{th} normalized response of interest, $\tilde{\mathbf{h}}_\ell(\mathbf{x}, \boldsymbol{\theta}) = \langle h_\ell(t_1; \mathbf{x}, \boldsymbol{\theta}), \ldots, h_\ell(t_{n_t}; \mathbf{x}, \boldsymbol{\theta})\rangle^T / h_\ell^*$ (see Section 2.3), is computed and written as

$$\tilde{\mathbf{h}}_\ell(\mathbf{x}, \boldsymbol{\theta}) = \tilde{\mathbf{A}}_\ell(\mathbf{x})\boldsymbol{\theta}, \quad \ell = 1, \ldots, n_h \tag{15}$$

where the matrices $\mathbf{A}_\ell(\mathbf{x}) \in \mathbb{R}^{n_t \times n_\theta}, \ell = 1, \ldots, n_h$, are constructed in terms of response thresholds, Karhunen-Loève representations, and convolution integrals. Hence, all matrices involved in the dynamical characterization of the system must be available. Then, the original RBO problem is heuristically replaced with the auxiliary optimization problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & \max_{\ell=1,\ldots,n_h} \|\tilde{\mathbf{A}}_\ell(\mathbf{x})\|_{p_1,p_2} \\ \text{subject to} \quad & g_j(\mathbf{x}) \leq 0, \qquad j = 1, \ldots, n_g \\ & \mathbf{x} \in X \subset \mathbb{R}^{n_x} \end{aligned} \tag{16}$$

where $\|\cdot\|_{p_1,p_2}$ denotes the induced (p_1, p_2)-norm of a matrix, which is defined as

$$\|\tilde{\mathbf{A}}_\ell(\mathbf{x})\|_{p_1,p_2} = \sup_{\boldsymbol{\theta} \neq \mathbf{0}} \frac{\|\tilde{\mathbf{A}}_\ell(\mathbf{x})\boldsymbol{\theta}\|_{p_1}}{\|\boldsymbol{\theta}\|_{p_2}} = \sup_{\boldsymbol{\theta} \neq \mathbf{0}} \frac{\|\tilde{\mathbf{h}}_\ell(\mathbf{x}, \boldsymbol{\theta})\|_{p_1}}{\|\boldsymbol{\theta}\|_{p_2}} \tag{17}$$

where $\|\cdot\|_p$ denotes the p-norm of a vector. The (p_1, p_2)-norm can be interpreted as the maximum amplification of the response's measure (according to the p_1-norm) with respect to the measure of the basic random variables vector (according to the p_2-norm). The solution of the original RBO problem is replaced by the solution of (16) with $p_1 = \infty$ and $p_2 = 2$. The proposed solution scheme involves a single deterministic optimization problem followed by a single reliability analysis. The previous approach has been also demonstrated in cases involving discrete design variables [76]. Concerning the solution of the auxiliary optimization problem, any appropriate strategy can be adopted, including sequential optimization schemes, interior point algorithms, and evolutionary strategies [1, 46, 74].

4.6. Summary and Comparison: Sequential Optimization Schemes

The contributions presented in this section rely on three main concepts: (i) the introduction of surrogates for the failure probability functions, (ii) the formulation of approximate optimization problems in terms of those surrogates, and (iii) the implementation of suitable search techniques to solve such approximate optimization problems. In this manner, subsequent reliability assessment and standard optimization steps are carried out to obtain candidate solutions with reduced computational efforts. The distinctive feature of each class of sequential optimization techniques corresponds to the type of failure probability surrogate under consideration. In order to provide a general outlook of sequential optimization approaches, Table 1 compares the optimization strategies described in each subsection based on the types of problems that have been addressed and relevant implementation aspects.

The contributions reported in Sections 4.1 to 4.4 address cost optimization subject to reliability constraints involving a low to moderate number of continuous or discrete design variables. They rely on the introduction of metamodels for the failure probability functions based on information gathered near each candidate solution. The methods reported in Section 4.1 propose linear surrogates for the logarithm of the failure probability functions, whereas those in Section 4.2 are based on convex/conservative expansions of all functions involved in the RBO problem. The approaches presented in Section 4.3 formulate one-dimensional surrogates of the failure probability functions coupled with line search strategies. A threshold-based technique that reuses reliability analysis results at the previous candidate design is described in Section 4.4. The previous approaches can be applied to a wide class of structural systems since they have been coupled with general simulation methods such as subset simulation and the PDEM. However, some of them require the computation of failure probability gradients, which has been addressed with ad-hoc approaches embedded in the reliability assessment technique under consideration. Regarding the class of search algorithms considered to solve the approximate sub-optimization problems, the general rule is that it should exploit the specific characteristics of the approximate problem in order to solve it most efficiently. All previous contributions can be regarded as successive decoupling strategies due to the local nature of the approximations. A different scheme was introduced by the operator norm optimization framework (see Section 4.5), whose scope is the reliability optimization of deterministic linear systems subject to Gaussian excitation. The operator norm of the structural system is used as a global proxy for the failure probability function, which allows the total decoupling of the RBO problem. As a final remark, it is noted that most of the approaches described in this section have been demonstrated in realistic applications, including complex finite element models and nonlinear structural systems.

5. Stochastic search based techniques

Optimization algorithms based on stochastic search techniques introduce randomization in the exploration of the design space as a means to avoid local minima [77]. Generally, these methodologies are quite flexible and general. In addition, they do not usually require information about the sensitivity (i.e., derivatives) of the functions involved in the optimization problem and most of them can directly deal with discrete design variables. Nevertheless, a common drawback of these approaches is that they require a large number of function evaluations, i.e. failure probability estimations. Recently, several stochastic optimization algorithms based on advanced simulation methods have been

Criterion	Approaches (Section 4.1)	Approaches (Section 4.2)	Approaches (Section 4.3)	Approach (Section 4.4)	Approaches (Section 4.5)
Role of failure probabilities	Constraints	Constraints	Constraints	Constraints	Objective
Nature of design variables	Continuous	Continuous or discrete	Continuous	Discrete	Continuous or discrete
Dimension of design space	Intermediate ($n_x = 2 - 7$)	Intermediate ($n_x = 2 - 10$)	Intermediate ($n_x = 2 - 10$)	Low ($n_x = 3 - 4$)	High ($n_x = 3 - 20$)
Structural behavior	Linear	Nonlinear	Nonlinear	Nonlinear	Linear
Failure probability surrogates	Exponential function	Convex or conservative expansion	Quadratic one-dimensional approximation	Local-type approximation	Operator norm
Failure probability sensitivities	Not required	Required	Required	Not required	Not required
Optimization method	SQP and NLPQL	Dual methods and Genetic algorithms	Line search techniques	Genetic algorithms	Genetic algorithms, Interior point
Simulation method	Importance sampling and MCS	Subset simulation and PDEM	Subset simulation and PDEM	Subset simulation	Directional importance sampling
Decoupling strategy	Successive	Successive	Successive	Successive	Total

Table 1: Comparison of the different types of approaches based on sequential optimization schemes.

developed specifically in the context of structural systems under stochastic excitation.

5.1. Asymptotically Independent Markov Sampling Based Approach

The asymptotically independent Markov sampling method for global optimization (AIMS-OPT) is a stochastic optimization method proposed in [78], which is based on a sampling technique originally developed for Bayesian inference problems [79]. Three main concepts are involved in the formulation of the optimization approach: annealing, importance sampling, and Markov chain Monte Carlo (MCMC). The optimization approach is targeted to the unconstrained global optimization of general expected performance measures and, in particular, to unconstrained global reliability optimization problems.

Based on the concept of *annealing* (or tempering), the problem of finding the minimum value of $P_F(\mathbf{x})$ is equivalent to find the maximum value of $\exp(-P_F(\mathbf{x})/T)$ for any given *annealing temperature* $T > 0$. Next, treating the design variables as uncertain and uniformly distributed over the feasible domain, a non-normalized *tempered distribution* is defined as [80]

$$p_T(\mathbf{x}) \propto \exp\left(-\frac{P_F(\mathbf{x})}{T}\right) I_X(\mathbf{x}), \quad T > 0 \tag{18}$$

where $I_X(\mathbf{x})$ represents the indicator function on the set $X = \{\mathbf{x} \in \mathbb{R}^{n_x} : x_i^L \leq x_i \leq x_i^U\}$, which defines the side constraints of the design variables. It is observed that $\lim_{T\to\infty} p_T(\mathbf{x}) = U_X(\mathbf{x})$, where $U_X(\mathbf{x})$ is a uniform distribution over X. On the other hand, as T decreases and tends to zero, the distribution $p_T(\mathbf{x})$ becomes spikier, and it puts more and more of its probability mass into the set that maximizes $\exp(-P_F(\mathbf{x})/T)$. Then, $\lim_{T\to 0} p_T(\mathbf{x}) = U_{X^*_{P_F}}(\mathbf{x})$, where $X^*_{P_F}$ is the optimal solution set. In other words, if T is close to zero, then a sample drawn from $p_T(\mathbf{x})$ will be in a vicinity of $X^*_{P_F}$ with very high probability. The idea is to generate a sequence of tempered distributions $\{p_{T_j}(\mathbf{x}), j = 0, 1, \ldots\}$ according to (18) with monotonically decreasing temperatures $\infty = T_0 > T_1 > \ldots > T_j > \ldots$, where the initial distribution is uniform over X, that is, $p_{T_0}(\mathbf{x}) = U_X(\mathbf{x})$. The samples at each level are generated based on samples from the previous one using importance sampling concepts and standard MCMC procedures [81, 82]. For improved efficiency, the sequence of temperatures is defined to ensure a smooth transition between subsequent distributions based on an effective sample size criterion [83]. The initial Markov chain state at each stage is randomly drawn near the best design obtained during the previous level. In addition, the corresponding proposal distribution for MCMC depends only on samples from the previous level. Thus, at each stage, AIMS-OPT explores local neighborhoods of the samples generated at the previous annealing level. Moreover, all samples can be generated independently and, therefore, the corresponding computations can be fully scheduled in parallel to improve the computational efficiency. The approach has been demonstrated in the RBO of a nonlinear structure subject to stochastic seismic excitation involving few design variables.

5.2. A Transitional Markov Chain Monte Carlo Based Approach

The approach introduced in [84] is tailored to deterministic linear structural systems under Gaussian excitation. Cost minimization subject to a single reliability constraint and standard constraints is addressed, where continuous design variables are considered. The contribution integrates (i) the Domain Decomposition Method (DDM) [42] for efficient reliability assessment, (ii) the transitional Markov chain Monte Carlo (TMCMC) method [85, 86] for exploration of the design space, and (iii) subset simulation [35, 36] to obtain an initial set of feasible designs. Similar to AIMS-OPT (see Section 5.1), the fundamental idea is to replace the optimization problem with the equivalent problem of obtaining a sample from the distribution

$$p_T(\mathbf{x}) \propto I_{X_{\text{feasible}}}(\mathbf{x}) \exp\left(-\frac{\ln\left(c(\mathbf{x})/c_0\right)}{T}\right), \qquad T \to 0 \tag{19}$$

where $c(\mathbf{x})$ is a cost function, c_0 is a scaling factor, T is the annealing temperature [80], and $I_{X_{\text{feasible}}}(\mathbf{x})$ is the indicator function corresponding to the feasible set X_{feasible} given by

$$X_{\text{feasible}} = \{\mathbf{x} \in X \subset \mathbb{R}^{n_x} : P_F(\mathbf{x}) \leq P_F^* \wedge g_j(\mathbf{x}) \leq 0, j = 1, \ldots, n_g\} \tag{20}$$

In order to draw samples from $p_T(\mathbf{x}), T \to 0$, the TMCMC method [85, 86, 87] is implemented. The corresponding sequence of non-normalized intermediate distributions is given by

$$p_{T_j}(\mathbf{x}) \propto I_{X_{\text{feasible}}}(\mathbf{x}) \exp\left(-\frac{\ln\left(c(\mathbf{x})/c_0\right)}{T_j}\right), \qquad j = 0, 1, \ldots, m \tag{21}$$

where $\infty = T_0 > T_1 > \ldots > T_m \to 0$. Based on the previous definition, the distribution $p_{T_0}(\mathbf{x})$ is uniform over the feasible space. In other words, a set of uniformly distributed designs over the feasible design space must be generated at the initial stage of the TMCMC method. This is not straightforward for general systems, since the samples must satisfy the reliability constraint, $P_F(\mathbf{x}) \leq$

P_F^*, and the standard constraints. To address this task, subset simulation [35, 36] is implemented considering an auxiliary failure domain defined in the design space as $F^{aux} = \{\mathbf{x} \in X_g : P_F(\mathbf{x}) \leq P_F^*\}$, where the set X_g contains the designs that satisfy the standard constraints $g_j(\mathbf{x}) \leq 0, j = 1, \ldots, n_g$, and the side constraints. The samples in X_g are obtained directly in an efficient manner since the deterministic constraints are easy to evaluate. In addition, the DDM [42] is implemented to evaluate efficiently the failure probability. Since the accuracy level of the reliability assessment step is only required to decide about the feasibility of each design, an adaptive scheme that exploits specific characteristics of the DDM to allocate the computational effort is proposed in [84]. The capabilities of the approach are demonstrated in a relatively simple system. Overall, the approach provides an effective strategy to deal with optimization problems involving deterministic linear systems subject to Gaussian excitation.

5.3. Two-Phase Bayesian Model Updating Framework

An approach based on a Bayesian model updating problem has been proposed in [88] for unconstrained optimization and extended in [89, 90] to constrained optimization. The method is based on the formulation of an equivalent Bayesian model updating problem. In addition, an adaptive surrogate model for the failure probability functions is implemented to improve the computational efficiency of the optimization procedure. Similar to the contributions reported in the previous subsections, a non-normalized distribution is defined as

$$p_T(\mathbf{x}) \propto U_{X_{\text{feasible}}}(\mathbf{x}) \, \exp\left(-\frac{c(\mathbf{x})}{T}\right), \quad T > 0 \tag{22}$$

where $U_{X_{\text{feasible}}}(\mathbf{x})$ is a uniform distribution over the feasible design space X_{feasible}. For unconstrained optimization problems, X_{feasible} comprises the side constraints on the design variables. The distribution $p_T(\mathbf{x})$ can be interpreted as a posterior distribution where $U_{X_{\text{feasible}}}(\mathbf{x})$ is the prior distribution and $\exp(-c(\mathbf{x})/T)$ is the likelihood function. Moreover, it is noted that $\lim_{T\to\infty} p_T(\mathbf{x}) = U_{X_{\text{feasible}}}(\mathbf{x})$. In addition, a sample drawn from $p_T(\mathbf{x}), T \to 0$, will be in a vicinity of the optimal solution set X_c^* with very high probability [78, 84]. In other words, finding the solution to the RBO problem is equivalent to solve a Bayesian model updating problem with posterior distribution $\lim_{T\to 0} p_T(\mathbf{x})$. To solve the model updating problem, a sequence of non-normalized intermediate distributions $\{p_{T_j}(\mathbf{x}),\, j = 0, 1, \ldots, m\}$ is defined as

$$\begin{aligned} p_{T_0}(\mathbf{x}) &= U_{X_{\text{feasible}}}(\mathbf{x}) && (T_0 = \infty) \\ p_{T_j}(\mathbf{x}) &\propto U_{X_{\text{feasible}}}(\mathbf{x}) \exp\left(-\frac{c(\mathbf{x})}{T_j}\right), && j = 1, 2, \ldots, m \end{aligned} \tag{23}$$

with annealing temperatures $\infty = T_0 > T_1 > \ldots > T_m \to 0$. The initial distribution is uniform over the design space, whereas the probability mass of the final distribution is concentrated in a vicinity of the optimum solution set. The idea of the method is to iteratively generate samples from the intermediate distributions, as they theoretically converge to the optimum solution set. To this end, the TMCMC method [85, 86, 87] is adopted and implemented. The initial stage of the TMCMC method requires a set of designs uniformly distributed over the feasible region. For unconstrained optimization problems, this is a trivial task since direct Monte Carlo simulation can be used. However, when general constraints are considered, standard sampling procedures such as the accept-reject method may be inefficient since the geometry of the feasible domain can be quite complex. In order to overcome these issues, an auxiliary unconstrained optimization problem is introduced as

$$\begin{aligned} \min_{\mathbf{x}} \quad & h(\mathbf{x}) = \max\left\{0,\ g_1(\mathbf{x}), \ldots, g_{n_g}(\mathbf{x}),\ r_1(\mathbf{x}), \ldots, r_{n_r}(\mathbf{x})\right\} \\ \text{subject to} \quad & x_i^L \leq x_i \leq x_i^U, \quad i = 1, \ldots, n_x \end{aligned} \tag{24}$$

The minimum value of this function, if the feasible set is not empty, is given by $h(\mathbf{x}) = 0$, and the corresponding optimum solution set verifies $X_h^* = X_{\text{feasible}}$. Thus, solving the auxiliary optimization problem (24) using the TMCMC method provides a set of designs that are uniformly distributed over the feasible design space. In addition, it can be shown that all feasible designs generated during the different stages of the TMCMC method are also uniformly distributed [88]. This leads to a two-phase approach where the same stochastic simulation technique is implemented to successively explore the feasible design space and the optimum solution set. The numerical implementation of the optimization scheme depends on few control parameters, which is advantageous from the practical viewpoint. The same framework has been extended in [90] to RBO problems involving discrete-continuous design variables by introducing a suitable proposal distribution to explore the design space in the context of the TMCMC method.

Due to its theoretical foundations, the approach has high chances of reaching a vicinity of the optimum solution set. Additionally, valuable sensitivity information on the objective and constraint functions can be obtained. However, the population-based nature of the optimization technique leads to a high number of function calls in order to effectively explore the search space, which can be computationally very demanding for involved structural systems. Kriging-based adaptive metamodels have been implemented in [88, 89] to approximate the failure probability functions, providing a noticeable efficiency improvement to the overall design process without sacrificing the quality of the optimization results. The capabilities of the two-phase framework have been demonstrated in applications involving nonlinear structural systems under general stochastic excitation.

5.4. Summary and Comparison: Stochastic Search Based Techniques

Contributions based on the implementation of stochastic search techniques have been described in this section. Their general idea is to transform the original RBO problem into the task of obtaining samples following a target distribution whose probability mass is concentrated around the optimal solution set. In this setting, simulated annealing concepts play a key role in the formulation and implementation of the different approaches. In order to summarize the methods presented in the different subsections, Table 2 highlights some characteristics associated with their application scope and relevant implementation details. According to this table, the common feature of these approaches is that they do not require reliability sensitivity assessment. This is advantageous in cases where such procedures are not available or are difficult to implement, although a high number of function calls is usually required to explore the design space in an effective manner. At the same time, the main difference between these methods corresponds to the role of the failure probabilities in the problem. In this sense, the optimization strategies can address unconstrained global reliability optimization (see Section 5.1), cost optimization of deterministic linear systems subject to a single reliability constraint (see Section 5.2), or general constrained RBO problems (see Section 5.3).

The approach presented in Section 5.1 proposes the use of the Asymptotically Independent Markov Sampling method as stochastic search technique, in which continuous design variables and nested reliability assessment are considered. The method described in Section 5.2 proposes the TMCMC method as search technique and subset simulation to generate an initial set of feasible designs. Nested reliability assessment based on the DDM is considered in the formulation. In this case, the number of samples required by the simulation technique is adaptively tuned to obtain sufficiently accurate estimates while reducing the computational burden as much as possible. This type of strategy has proved quite effective in reducing overall computational costs and, additionally, it shows that the consideration of particular features of advanced simulation methods can be quite beneficial for RBO procedures. Finally, the two-phase framework presented in Section 5.3 proposes the use of the TMCMC method as a search technique to explore both the feasible design space and the optimum solution set. Subset simulation has been considered as reliability analysis method, although alternative techniques can be integrated as well. A successive decoupling strategy based on adaptive kriging surrogates for the failure probability functions is implemented, which can provide substantial computational savings. Furthermore, this illustrates that the integration of suitable failure probability surrogates into stochastic search techniques can effectively improve their efficiency by avoiding full reliability assessment at every new design. As in the case of sequential optimization schemes, the previous approaches have been employed in a number of linear and nonlinear structural models.

Criterion	Approach (Section 5.1)	Approach (Section 5.2)	Approach (Section 5.3)
Role of failure probabilities	Objective	Constraints	Objective or constraints
Nature of design variables	Continuous	Continuous	Continuous, discrete or mixed
Dimension of design space	Low ($n_x = 2$)	Low ($n_x = 2$)	Intermediate ($n_x = 2 - 8$)
Structural behavior	Nonlinear	Linear	Nonlinear
Failure probability surrogates	None	None	Kriging metamodel
Failure probability sensitivities	Not required	Not required	Not required
Optimization method	AIMS-OPT	TMCMC	TMCMC
Simulation method	MCS	DDM	Subset simulation
Decoupling strategy	None	None	Successive

Table 2: Comparison of the different types of approaches based on stochastic search techniques.

6. Augmented Reliability Space Formulations

An alternative framework for solving reliability-based optimization problems involving structural systems under stochastic excitation is based on the *augmented reliability concept* [91, 92]. In this framework, the design variables $\mathbf{x}$ are artificially treated as random variables following an arbitrary distribution $p(\mathbf{x})$, that is, $\mathbf{x} \sim p(\mathbf{x})$. This distribution is usually taken as uniform over the design space, although alternative distributions can be selected as well. The *augmented reliability space* simultaneously considers the design variables $\mathbf{x}$ and the basic random variables $\boldsymbol{\theta}$, that is, $\langle \mathbf{x}, \boldsymbol{\theta} \rangle$. Then, the *augmented reliability problem* for any failure event F corresponds to the evaluation of

$$P(F) = \int_{\mathbf{x} \in \mathbb{R}^{n_x}} \int_{\boldsymbol{\theta} \in \mathbb{R}^{n_\theta}} I_F(\mathbf{x}, \boldsymbol{\theta}) q(\boldsymbol{\theta}|\mathbf{x}) p(\mathbf{x}) d\boldsymbol{\theta} d\mathbf{x} \tag{25}$$

which is a quantity that measures the plausibility of failure when the uncertainties in the basic random variables and the artificial uncertainties in the design variables are jointly considered. The definition of $P(F)$ is purely instrumental within the augmented reliability space framework. Then, according to Bayes' theorem [93], the failure probability function $P_F(\mathbf{x})$ for any given design in the augmented reliability space is given by

$$P_F(\mathbf{x}) = P(F|\mathbf{x}) = \frac{P(F) p(\mathbf{x}|F)}{p(\mathbf{x})} \tag{26}$$

where $p(\mathbf{x}|F)$ is the marginal distribution of $\mathbf{x}$ conditioned on failure event F. If $p(\mathbf{x})$ is taken, without any loss of generality, as a uniform distribution over the design space, the failure probability function verifies

$$P_F(\mathbf{x}) \propto p(\mathbf{x}|F) \tag{27}$$

since $P(F)$ and $p(\mathbf{x})$ are constant values. Equation (26) shows that the computation of the failure probability function $P_F(\mathbf{x})$ requires the marginal probability density function $p(\mathbf{x}|F)$. The different approaches reported in this section take advantage of this basic relationship in order to improve the overall efficiency of reliability-based optimization procedures.

6.1. Stochastic Subset Optimization

Stochastic Subset Optimization (SSO) is an iterative method to deal with global reliability optimization, which was initially introduced in [94, 95]. The algorithm effectively avoids nested reliability evaluations by iteratively shrinking the search domain in the augmented reliability space in order to reduce its average failure probability value. In this framework, a *class of admissible subsets* $A \subset X$ is considered, where X is the set comprising the side constraints on the design variables. This class contains subsets with some predetermined characteristics such as size or shape. The new search region identified during each iteration corresponds to the *optimal subset* $I^* \in A$ which verifies

$$I^* = \arg\min_{I \in A} P(F|\mathbf{x} \in I) \tag{28}$$

where $P(F|\mathbf{x} \in I)$ is the average failure probability on a subset $I \subset X$ given by

$$P(F|\mathbf{x} \in I) = P(F)\frac{P(\mathbf{x} \in I|F)}{P(\mathbf{x} \in I)} = P(F)\frac{\int_I p(\mathbf{x}|F)d\mathbf{x}}{\int_I p(\mathbf{x})d\mathbf{x}} \tag{29}$$

where $P(\mathbf{x} \in I|F)$ is the probability mass of subset I conditioned on failure event F, $P(\mathbf{x} \in I)$ is the probability mass of subset I, and $p(\mathbf{x})$ is the uniform distribution. All these quantities can be estimated, in principle, by means of simulation techniques such as MCMC. During each optimization cycle, N_T failure samples are available in the $\langle \mathbf{x}, \boldsymbol{\theta} \rangle$-space, among which N_I belong to each admissible subset I. Then, problem (28) is approximately given by

$$I^* = \arg\min_{I \in A} \frac{N_I}{V_I} \tag{30}$$

where V_I is the volume of $I \in A$. Appropriate methods for solving non-smooth optimization problems [96] must be implemented. The performance of the SSO algorithm highly depends on the definition of the admissible subsets. Hyper-rectangles and hyper-ellipses with adjustable ratio between dimensions have been considered [94, 95]. The correct choice of the class of admissible subsets remains one of the main challenges in SSO, specially for higher dimensions and disjoint regions [97]. The method has been demonstrated in applications involving linear and nonlinear systems, considering a relatively small number of design variables [98, 99].

6.2. Non-Parametric Stochastic Subset Optimization

The Non-Parametric Stochastic Subset Optimization (NP-SSO) method is an extension of SSO for solving unconstrained reliability optimization problems [100] and further developed in [101, 102]. The method avoids the parametric description of subsets and the search for the one that has the smallest average value. Moreover, NP-SSO is focused on the estimation of the marginal PDF $p(\mathbf{x}|F)$ by means of boundary-corrected kernel density estimation (KDE) methods [103, 104, 105, 106]. Their implementation requires independent and identically distributed (i.i.d.) failure samples which are generated by rejection sampling [19]. An iterative framework is proposed to improve the computational efficiency of the optimization process. The idea is to continuously shrink the search domain to regions with lower values of the objective function. When the updated search region is composed of multiple clusters, suitable techniques are implemented to characterize them [107, 108]. At the end of the iterative procedure, a reduced search space and a KDE approximation of the failure probability function are obtained. To improve the numerical efficiency of the overall procedure, the NP-SSO technique is coupled in [101, 102] with various soft-computing techniques [109, 110].

The previous method has been extended in [102] to the minimization of a cost function subject to reliability constraints. The idea is to obtain a surrogate model $P_F(\mathbf{x}) \approx \tilde{P}_F(\mathbf{x})$ that is sufficiently accurate near the boundaries of the feasible design space. The iterations of the NP-SSO method are carried out until the new search region lies within the feasible design space. Then, a refinement stage is implemented to improve the quality of the failure probability surrogate near the boundary of the feasible domain. The resulting approximate optimization problem can be solved using any suitable optimization technique. Although NP-SSO circumvents the main difficulties encountered in the original SSO formulation, the robustness of KDE approaches for density fitting decreases in high dimensions [111]. Thus, the range of applications is somewhat limited in terms of the number of design variables.

6.3. Approach Based on Partitioned Design Space

The approach proposed in [112] addresses the minimization of a cost function subject to a single reliability constraint. A surrogate for the failure probability function is formulated in terms of an augmented reliability space and a partitioning of the design space. The design space X is partitioned into several subspaces $D_i, i = 1, \ldots, n_p$ in an iterative manner. At the i^{th} iteration, a sufficient number of failure samples within the current subdomain D_i are generated in the augmented reliability space using MCMC. With these samples, the marginal distribution $p(\mathbf{x}|F, D_i)$ is approximated in the current subdomain D_i, i.e. $p(\mathbf{x}|F, D_i) \approx \tilde{p}(\mathbf{x}|F, D_i)$, using a rectangular binning approach and least squares estimates based on second-order polynomials [113, 114]. Then, a new subspace $D_{i+1} \subset D_i$ ($D_1 = X$) is defined as $D_{i+1} = \{\mathbf{x} \in D_i : \tilde{p}(\mathbf{x}|F, D_i) \leq p_i^*\}$, where p_i^* is adaptively chosen as in subset simulation [35, 36]. At the end of the iterative process, the failure probability function evaluated at a design $\mathbf{x}$, such that $\mathbf{x} \in D_i$ and $\mathbf{x} \notin D_{i+1}$, can be estimated as

$$\tilde{P}_F(\mathbf{x}) = \frac{P\left(D_i|F\right)\hat{P}(F)}{p(\mathbf{x})}\,\tilde{p}(\mathbf{x}|F, D_i) \tag{31}$$

where $P\left(D_i|F\right)$ is obtained as a by-product of the partitioning process and $\hat{P}(F)$ can be estimated from the samples obtained during the first iteration. The method provides a sequence of least squares estimates for each subspace rather than a unique fit over the complete design space, which can lead to increased accuracy. Moreover, the surrogate allows to completely decouple the reliability assessment cycle from the optimization process. Nonetheless, the approach seems to be restricted to low-dimensional design spaces due to the current limitations of density fitting procedures.

6.4. Maximum Entropy Based Methods

The approach proposed in [115] deals with cost minimization under a single reliability constraint. The main idea of the approach is to generate surrogates of the failure probability function based on the augmented reliability formulation and the maximum entropy (ME) method [116, 117]. The objective is to obtain the distribution that maximizes the entropy subject to constraints on the distribution's moments. In particular, the ME estimate of $p(\mathbf{x}|F)$ under first moment constraints is implemented and given by [115, 118]

$$\tilde{p}(\mathbf{x}|F) = \exp\left(-\alpha - \boldsymbol{\lambda}^T\mathbf{x}\right) \tag{32}$$

where α and $\boldsymbol{\lambda} = \langle\lambda_1, \ldots, \lambda_{n_x}\rangle^T$ are the optimal parameters. The formulation leads to linear surrogates for $\ln(P_F(\mathbf{x}))$, as proposed in earlier works [12, 49]. However, no move limits are imposed for the ME estimate in this case. The required failure samples for defining the surrogates are generated by means of subset simulation [35, 36], which also provides an estimate $P(F) \approx \hat{P}(F)$. Then, the failure probability function is estimated as

$$\tilde{P}_F(\mathbf{x}) = \frac{\hat{P}(F)\tilde{p}(\mathbf{x}|F)}{p(\mathbf{x})} \tag{33}$$

The variability of this estimate is due to the variability in $\hat{P}(F)$ and $\tilde{p}(\mathbf{x}|F)$. An approach based on confidence intervals (CIs) is implemented to consider this issue [115, 118, 119]. The goal of the approach is to solve a set of explicit *approximate RBO problems*. First, each approximate problem is defined by means of an approximate representation of $\tilde{P}_F(\mathbf{x})$, which is given by a realization of $\hat{P}(F)$ and $\boldsymbol{\lambda}$ drawn from their corresponding CIs. Second, these approximate RBO problems are solved in order to obtain a *set of approximate optimal designs*. Finally, a screening procedure is carried out to identify the final solution. According to [115], it is expected that the performance of the approach may decrease when the number of uncertain parameters is too large or when the behavior of $\ln(P_F(\mathbf{x}))$ is highly nonlinear. The method is demonstrated in several examples involving linear and nonlinear systems, considering few design variables.

6.5. Scheme Based on Equivalent Safety-Factor Constraints

The approach proposed in [120] addresses the minimization of a cost function subject to reliability and standard constraints. The original reliability constraints are replaced by safety-factor constraints in order to formulate a standard optimization problem. The equivalent safety-factor constraints are defined as

$$\eta_j^* \bar{d}_j(\mathbf{x}) \leq 1, \qquad j = 1, \ldots, n_r \tag{34}$$

where $\eta_j^* \geq 1$ is the designated safety factor and $\bar{d}_j(\mathbf{x}) > 0$ is a nominal normalized demand function which can be defined as $\bar{d}_j(\mathbf{x}) = d_j(\mathbf{x}, E[\boldsymbol{\theta}])$ or $\bar{d}_j(\mathbf{x}) = E_{\boldsymbol{\theta}}\left[d_j(\mathbf{x}, \boldsymbol{\theta})\right]$. Under certain conditions [120], the functional relationship between η_j^* and $P_{F_j}^*$ is given by

$$P\left[d_j(\mathbf{x}, \boldsymbol{\theta}) - \eta_j^* \bar{d}_j(\mathbf{x}) > 0\right] = P_{F_j}^* \iff P\left[\frac{d_j(\mathbf{x}, \boldsymbol{\theta})}{\bar{d}_j(\mathbf{x})} > \eta_j^*\right] = P_{F_j}^* \tag{35}$$

The designated safety factor η_j^* is estimated using simulation techniques. For improved efficiency, the values $\eta_j^*, j = 1, \ldots, n_r$ are simultaneously computed from a single simulation run in the augmented reliability space. To this end, direct Monte Carlo simulation and parallel subset simulation [121] are implemented in [120]. Finally, the original RBO problem is transformed into a nonlinear optimization problem which can be solved by standard optimization schemes. The applicability of the approach is demonstrated on the RBO of a linear system under stochastic excitation, involving relatively few design variables.

6.6. Summary and Comparison: Formulations Based on Augmented Reliability Spaces

Contributions based on augmented reliability spaces allow to treat reliability assessment in the joint space of random and design variables. This is possible by introducing an instrumental variability to the design variables. Thus, in principle, a single simulation run could provide the necessary information to solve the RBO problem. The common idea is to take advantage of the relationship between the failure probability function and the marginal conditional PDF of the design variables. To illustrate the characteristics of the different approaches reported in this section, Table 3 presents their main features in terms of their application range and implementation aspects. It is seen that all of the contributions avoid reliability sensitivity assessment. In addition, their performance has been demonstrated in low-dimensional design spaces involving continuous design variables.

The different approaches reported in this section exploit in a different way the structure of the augmented reliability problem to avoid nested reliability assessment. The approach presented in Section 6.1 focuses on improving the average failure probability value by iteratively selecting smaller subsets in the search domain. Building on this idea, the contributions of Section 6.2 introduce kernel density estimation and machine learning techniques to obtain simultaneously a failure probability surrogate and a reduced search space, respectively. The previous strategies allow successive decoupling, since the sampling and subset identification steps are sequential. The rest of contributions correspond to total decoupling strategies. Section 6.3 presents a method based on the partition of the design domain to obtain a sequence of surrogates for the failure probability function in terms of second-order polynomials. The contribution in Section 6.4 proposes a maximum entropy estimate for the marginal conditional distribution, whereas equivalent safety-factor constraints obtained by a single simulation run are considered in Section 6.5. It is noted that the selection of adequate sampling schemes is one of the most relevant implementation aspects in the augmented reliability framework. In general, the chosen method must explore the augmented space to obtain the required information in a robust and efficient manner. Finally, approaches based on augmented reliability spaces have been demonstrated in several applications involving linear and nonlinear structural models.

Criterion	Approaches (Section 6.1)	Approaches (Section 6.2)	Approach (Section 6.3)	Approach (Section 6.4)	Approach (Section 6.5)
Role of failure probabilities	Objective	Objective or constraints	Constraints	Constraints	Constraints
Nature of design variables	Continuous	Continuous	Continuous	Continuous	Continuous
Dimension of design space	Intermediate ($n_x = 2 - 6$)	Low ($n_x = 2 - 4$)	Low ($n_x = 2$)	Low ($n_x = 2$)	Low ($n_x = 3$)
Structural behavior	Nonlinear	Nonlinear	Linear	Linear	Linear
Failure probability surrogates	Marginal PDF average	Kernel density estimates	Second-order polynomials	Exponential	Safety-factor constraints
Failure probability sensitivities	Not required	Not required	Not required	Not required	Not required
Optimization method	SSO	NP-SSO	SQP	Standard scheme	Genetic algorithms
Simulation method	MCS and MCMC	Rejection sampling	MCMC	Subset simulation	MCS or parallel subset simulation
Decoupling strategy	Successive	Successive	Total	Total	Total

Table 3: Comparison of the different types of approaches based on augmented reliability space formulations.

7. Conclusions and Outlook

This work has summarized and discussed some of the latest developments in the context of reliability-based design optimization of structural systems under stochastic excitation. The contributions have been grouped into three categories: sequential optimization approaches, stochastic search based techniques, and formulations based on augmented reliability spaces.

Sequential optimization approaches involve consecutive reliability assessment, construction of failure probability surrogates and exploration of the search space in order to reduce the overall computational effort. Surrogates for the failure probability functions are developed to formulate approximate optimization problems which are solved by means of any suitable optimization technique. Thus, these approaches can handle, in principle, high-dimensional design spaces. However, the quality of the failure probability surrogates usually tends to decrease as the number of design parameters increases. One way to circumvent this problem is the development of line search techniques, in which one-dimensional surrogates are required. Some of the reported methods involve the evaluation of both, failure probabilities and their sensitivities, which can be very challenging. In this regard, an important task is to develop general approaches to efficiently evaluate reliability sensitivities based on, for example, advanced simulation techniques. Although global optimization schemes can be used in the context of sequential optimization approaches, the different contributions are mainly based on local search due to the inclusion of move limits and the type of optimization techniques under consideration. As a result, they may not be appropriate for problems involving several local optima or disconnected feasible design regions. Thus, the use of global optimizers offers a practical and important extension of sequential approaches.

The second category considers stochastic search based techniques. These techniques introduce randomization in the exploration of the design space as a means to avoid local minima. The contributions examined in this work are based on the combination of annealing concepts and Markov chain Monte Carlo methods. They present high theoretical chances of reaching a vicinity of the optimum solution set. In addition, the computation of the derivatives of the failure probability functions is not required by these algorithms. Generally, a relatively large number of function calls (failure probability estimates) is required to obtain an adequate solution and, therefore, the corresponding computational efforts can be significant. Several strategies have been proposed to overcome this issue, including the implementation of surrogates for the failure probability functions and adaptive allocation of the number of samples for reliability assessment. The ability to obtain a set of close-to-optimal designs rather than a single candidate solution can provide more flexibility to the overall design process. This is especially important in complex design problems involving multiple optima as well as to cope with the inherent uncertainty arising in reliability assessment. Some of the important challenges in the context of these techniques are the effective integration of sensitivity information during the design process, the consideration of multi-objective optimization problems, the treatment of mixed discrete-continuous design variables, the extension to higher-dimensional design spaces, and the reduction of computational efforts associated with the number of function calls or the construction of sufficiently accurate surrogates.

The third category is associated with formulations based on augmented reliability spaces. In this framework, failure probability surrogates are constructed using failure samples associated with an augmented reliability problem. The reliability and optimization processes are usually combined.

Then, in principle, a single simulation run in the augmented space could provide the required information to solve the optimization problem. A number of low-dimensional problems have been addressed, including global reliability optimization and cost minimization subject to reliability constraints. The implementation of these approaches has mostly relied on the characterization of the marginal probability density function in terms of different quantities, including subsets in the design space and density fitting techniques such as kernel density estimation, maximum entropy, and least squares estimates. Some extensions of these formulations include the consideration of higher-dimensional design spaces, the integration of sensitivity information, the treatment of discrete design variables, and the development and implementation of accurate and effective metamodels.

From the previous discussion, it is clear that the different approaches provide different advantages and difficulties to carry out the optimization process. Even though they have been generally tested in a variety of realistic applications, including complex structural systems, the choice of a particular method is problem-dependent. Some characteristics that must be taken into account to choose a particular optimization technique include the number of design variables, available computational power, possibility of having multiple optima, discrete or continuous nature of the design variables, linearity or non-linearity of the structural system, and the role of the failure probability functions in the characterization of the optimization problem (as objective and/or constraint functions). The user must be able to carefully select the most appropriate method for the problem under consideration. In this manner, adequate candidate solutions can be established and, more importantly, further insight about the system behavior can be obtained. As previously pointed out, reliability-based optimization procedures for structural systems under stochastic excitation are problem-dependent. However, it is believed that future research efforts can provide a general improvement to these methodologies. For example, the use of model reduction techniques combined with parametrization schemes can certainly benefit optimization procedures by increasing the efficiency of basic structural analyses without compromising their accuracy. Another topic that is under continuous development and has received great attention lately corresponds to the implementation of adaptive metamodels. This can improve optimization procedures by reducing the computational overhead at the structural response or failure probability function levels. Furthermore, the development of new simulation schemes for reliability and sensitivity assessment provides additional opportunities to develop novel RBO methods that can provide further options to engineering practice. Finally, parallelization features at the reliability and sensitivity assessment level as well as at the physical model level can be exploited to increase the efficiency of the different approaches. These can be implemented either for efficient and effective construction of metamodels, or for direct analyses in surrogate-free schemes.

In conclusion, the arguments presented in this brief overview suggest that computational aspects play a key role in designing realistic systems and structures. Moreover, the preceding sections indicate that more developments and research are needed in the area of reliability-based design optimization of structural systems under stochastic excitation. Future efforts should focus on making approaches in this area more efficient by providing and implementing effective and robust numerical procedures. This emphasizes the necessity for devising not only sound and improved theoretical algorithms but also the appropriate tools needed for applying such procedures. Overcoming these challenges can lead to significant advancements in this area and, ultimately, assist complex decision-making processes in real-life situations.

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References

[1] R. T. Haftka and Z. Gürdal. *Elements of Structural Optimization, 3rd Ed.* Kluwer, Dordrecht, The Netherlands, 1992.

[2] E. Simiu. *Design of buildings for wind: A guide for ASCE 7-10 standard users and designers of special structures.* John Wiley & Sons, Inc., 2011.

[3] A. Elghazouli. *Seismic design of buildings to Eurocode 8.* Spon Press, 2009.

[4] S. Chandrasekaran. *Dynamic analysis and design of offshore structures.* Springer India, 2015.

[5] G. M. Atkinson. Stochastic modeling of California ground motions. *Bulletin of the Seismological Society of America*, 90(2):255–274, 2000.

[6] J. Li, Q. Yan, and J. Chen. Stochastic modeling of engineering dynamic excitations for stochastic dynamics of structures. *Probabilistic Engineering Mechanics*, 27(1):19–28, 2012.

[7] Q. S. Ding, L. D. Zhu, and H. F. Xiang. Simulation of stationary Gaussian stochastic wind velocity field. *Wind and Structures*, 9(3):231–243, 2006.

[8] H. O. Madsen, N. C. Lind, and S. Krenk. *Methods of Structural Safety.* Dover Publications, 2006.

[9] S. Okazawa, K. Oide, K. Ikeda, and K. Terada. Imperfection sensitivity and probabilistic variation of tensile strength of steel members. *International Journal of Solids and Structures*, 39(6):1651–1671, 2002.

[10] C. A. Schenk and G. I. Schuëller. *Uncertainty assessment of large finite element systems.* Springer-Verlag, 2005.

[11] I. Enevoldsen and J.D. Sørensen. Reliability-based optimization in structural engineering. *Structural Safety*, 15(3):169–196, 1994.

[12] M. Gasser and G. I. Schuëller. Reliability-based optimization of structural systems. *Mathematical Methods of Operations Research*, 46(3):287–307, 1997.

[13] A. Der Kiureghian. The geometry of random vibrations and solutions by FORM and SORM. *Probabilistic Engineering Mechanics*, 15(1):81–90, 2000.

[14] G. I. Schuëller, H. J. Pradlwarter, and P. S. Koutsourelakis. A critical appraisal of reliability estimation procedures for high dimensions. *Probabilistic Engineering Mechanics*, 19(4):463–474, 2004.

[15] G. I. Schuëller and H. J. Pradlwarter. Benchmark study on reliability estimation in higher dimensions of structural systems An overview. *Structural Safety*, 29(3):167–182, 2007.

[16] N. C. Nigam. Structural optimization in random vibration environment. *AIAA Journal*, 10(4):551–553, 1972.

[17] L. D. Lutes and S. Sarkani. *Random vibrations: Analysis of structural and mechanical systems.* Elsevier Butterworth-Heinemann, 2004.

[18] G. S. Fishman. *Monte Carlo.* Springer New York, 1996.

[19] C. Robert and G. Casella. *Monte Carlo Statistical Methods.* Springer New York, 2010.
[20] R. Y. Rubinstein and D. P. Kroese. *Simulation and the Monte Carlo Method.* John Wiley & Sons, Inc., 2016.
[21] J. Song, W.-H. Kang, Y.-J. Lee, and J. Chun. Structural system reliability: Overview of theories and applications to optimization. *ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems, Part A: Civil Engineering*, 7(2):03121001, 2021.
[22] C. Theodosiou, P. Aichouh, S. Natsiavas, and C. Papadimitriou. Reliability-based optimal design of fluid filled tanks under seismic excitation. In *Volume 5: 19th Biennial Conference on Mechanical Vibration and Noise, Parts A, B, and C*, 2003.
[23] M. Barbato and E. Tubaldi. A probabilistic performance-based approach for mitigating the seismic pounding risk between adjacent buildings. *Earthquake Engineering & Structural Dynamics*, 42(8):1203–1219, 2012.
[24] I. Venanzi, A. L. Materazzi, and L. Ierimonti. Robust and reliable optimization of wind-excited cable-stayed masts. *Journal of Wind Engineering and Industrial Aerodynamics*, 147:368–379, 2015.
[25] X. Zeng, Y. Peng, and J. Chen. Serviceability-based damping optimization of randomly wind-excited high-rise buildings. *The Structural Design of Tall and Special Buildings*, 26(11):e1371, 2017.
[26] S. M. J. Spence and M. Gioffrè. Large scale reliability-based design optimization of wind excited tall buildings. *Probabilistic Engineering Mechanics*, 28:206–215, 2012.
[27] S. M. J. Spence and A. Kareem. Performance-based design and optimization of uncertain wind-excited dynamic building systems. *Engineering Structures*, 78:133–144, 2014.
[28] D. Altieri, E. Tubaldi, M. De Angelis, E. Patelli, and A. Dall'Asta. Reliability-based optimal design of nonlinear viscous dampers for the seismic protection of structural systems. *Bulletin of Earthquake Engineering*, 16(2):963–982, 2018.
[29] Y. Peng, Y. Ma, T. Huang, and D. De Domenico. Reliability-based design optimization of adaptive sliding base isolation system for improving seismic performance of structures. *Reliability Engineering & System Safety*, 205:107167, 2021.
[30] S. Bobby, A. Suksuwan, S. M. J. Spence, and A. Kareem. Reliability-based topology optimization of uncertain building systems subject to stochastic excitation. *Structural Safety*, 66:1–16, 2017.
[31] S. M. J. Spence, M. Gioffr, and A. Kareem. An efficient framework for the reliability-based design optimization of large-scale uncertain and stochastic linear systems. *Probabilistic Engineering Mechanics*, 44:174–182, 2016.
[32] S.M.J. Spence and M. Gioffr. Efficient algorithms for the reliability optimization of tall buildings. *Journal of Wind Engineering and Industrial Aerodynamics*, 99(6):691–699, 2011.
[33] V. Pareto. *Manual of political economy.* A. M. Kelley, New York, 1971.
[34] R. T. Marler and J. S. Arora. Survey of multi-objective optimization methods for engineering. *Structural and Multidisciplinary Optimization*, 26(6):369–395, 2004.
[35] S.-K. Au and J. L. Beck. Estimation of small failure probabilities in high dimensions by subset simulation. *Probabilistic Engineering Mechanics*, 16(4):263–277, 2001.
[36] S.-K. Au and Y. Wang *Engineering risk assessment with Subset Simulation.* John Wiley & Sons, Inc., 2014.
[37] S.-K. Au and E. Patelli. Rare event simulation in finite-infinite dimensional space. *Reliability Engineering & System Safety*, 148(Supplement C):67–77, 2016.
[38] I. Papaioannou, W. Betz, K. Zwirglmaier, and D. Straub. MCMC algorithms for subset simu-

lation. *Probabilistic Engineering Mechanics*, 41(Supplement C):89–103, 2015.
[39] S.-K. Au and J. L. Beck. First excursion probabilities for linear systems by very efficient importance sampling. *Probabilistic Engineering Mechanics*, 16(3):193–207, 2001.
[40] P. S. Koutsourelakis, H. J. Pradlwarter, and G. I. Schuëller. Reliability of structures in high dimensions, part I: Algorithms and applications. *Probabilistic Engineering Mechanics*, 19(4):409–417, 2004.
[41] K. M. Zuev and L. S. Katafygiotis. The Horseracing Simulation algorithm for evaluation of small failure probabilities. *Probabilistic Engineering Mechanics*, 26(2):157–164, 2011.
[42] L. Katafygiotis and S. H. Cheung. Domain Decomposition Method for calculating the failure probability of linear dynamic systems subjected to Gaussian stochastic loads. *Journal of Engineering Mechanics*, 132(5):475–486, 2006.
[43] M. A. Misraji, M. A. Valdebenito, H. A. Jensen, and C. F. Mayorga. Application of directional importance sampling for estimation of first excursion probabilities of linear structural systems subject to stochastic Gaussian loading. *Mechanical Systems and Signal Processing*, 139:106621, 2020.
[44] J. Li and J. Chen. Probability density evolution method for dynamic response analysis of structures with uncertain parameters. *Computational Mechanics*, 34(5):400–409, 2004.
[45] J. Li and J. Chen. *Stochastic Dynamics of Structures.* John Wiley & Sons, Ltd., 2009.
[46] J. Nocedal and S. J. Wright. *Numerical optimization.* Springer New York, 2006.
[47] R. E. Melchers and M. Ahammed. A fast approximate method for parameter sensitivity estimation in Monte Carlo structural reliability. *Computers & Structures*, 82(1):55–61, 2004.
[48] S. Song, Z. Lu, and H. Qiao. Subset simulation for structural reliability sensitivity analysis. *Reliability Engineering & System Safety*, 94(2):658–665, 2009.
[49] H. A. Jensen. Structural optimization of linear dynamical systems under stochastic excitation: a moving reliability database approach. *Computer Methods in Applied Mechanics and Engineering*, 194(12-16):1757–1778, 2005.
[50] H. A. Jensen. Design and sensitivity analysis of dynamical systems subjected to stochastic loading. *Computers & Structures*, 83(14):1062–1075, 2005.
[51] C. Fleury and V. Braibant. Structural optimization: a new dual method using mixed variables. *International Journal for Numerical Methods in Engineering*, 23(3):409–428, 1986.
[52] R. B. Nelson. Simplified calculation of eigenvector derivatives. *AIAA Journal*, 14(9):1201–1205, 1976.
[53] H. A. Jensen, M. A. Valdebenito, and G. I. Schuller. An efficient reliability-based optimization scheme for uncertain linear systems subject to general Gaussian excitation. *Computer Methods in Applied Mechanics and Engineering*, 198(1):72–87, 2008.
[54] H. Yu, F. Gillot, and M. Ichchou. Reliability based robust design optimization for tuned mass damper in passive vibration control of deterministic/uncertain structures. *Journal of Sound and Vibration*, 332(9):2222–2238, 2013.
[55] H. A. Jensen, M. S. Ferre, and D. S. Kusanovic. Reliability-based synthesis of non-linear stochastic dynamical systems: a global approximation approach. *International Journal of Reliability and Safety*, 4(2-3):139–165, 2010.
[56] H. A. Jensen and J. G. Sepulveda. Structural optimization of uncertain dynamical systems considering mixed-design variables. *Probabilistic Engineering Mechanics*, 26(2):269–280, 2011.
[57] H. A. Jensen and M. Beer. Discretecontinuous variable structural optimization of systems under stochastic loading. *Structural Safety*, 32(5):293–304, 2010.

[58] H. A. Jensen and J. G. Sepulveda. On the reliability-based design of structures including passive energy dissipation systems. *Structural Safety*, 34(1):390–400, 2012.
[59] H. Jensen, J. Sepulveda, and L. Becerra. Robust stochastic design of base-isolated structural systems. *International Journal for Uncertainty Quantification*, 2(2):95–110, 2012.
[60] H. A. Jensen, M. A. Valdebenito, G. I. Schuller, and D. S. Kusanovic. Reliability-based optimization of stochastic systems using line search. *Computer Methods in Applied Mechanics and Engineering*, 198(49):3915–3924, 2009.
[61] M. A. Valdebenito and G. I. Schuëller Efficient strategies for reliability-based optimization involving non-linear, dynamical structures. *Computers & Structures*, 89(19):1797–1811, 2011.
[62] J. Chen, J. Yang, and H. Jensen. Structural optimization considering dynamic reliability constraints via probability density evolution method and change of probability measure. *Structural and Multidisciplinary Optimization*, 62(5):2499–2516, 2012.
[63] Z. Wan, J. Chen, J. Li, and A. H.-S. Ang. An efficient new PDEM-COM based approach for time-variant reliability assessment of structures with monotonically deteriorating materials. *Structural Safety*, 82:101878, 2020.
[64] H. A. Jensen, L. G. Becerra, and M. A. Valdebenito. On the use of a class of interior point algorithms in stochastic structural optimization. *Computers & Structures*, 126:69–85, 2013.
[65] J. Herskovits and G. Santos. On the computer implementation of feasible direction interior point algorithms for nonlinear optimization. *Structural optimization*, 14(2):165–172, 1997.
[66] F. van Keulen and K. Vervenne. Gradient-enhanced response surface building. *Structural and Multidisciplinary Optimization*, 27(5):337–351, 2004.
[67] H. A. Jensen, D. S. Kusanovic, M. A. Valdebenito, and G. I. Schuller. Reliability-based design optimization of uncertain stochastic systems: gradient-based scheme. *Journal of Engineering Mechanics*, 138(1):60–70, 2012.
[68] J. Yang, H. Jensen, and J. Chen. Structural optimization under dynamic reliability constraints utilizing probability density evolution method and metamodels in augmented input space. *Manuscript submitted for publication*, 2021.
[69] H. A. Jensen, D. S. Kusanovic, and M. A. Valdebenito. Compromise design of stochastic dynamical systems: a reliability-based approach. *Probabilistic Engineering Mechanics*, 29:40–52, 2012.
[70] H. Jensen and C. Papadimitriou. *Sub-structure coupling for dynamic analysis.* Springer-Verlag GmbH, 2019.
[71] R. Craig. *Structural dynamics: an introduction to computer methods.* Wiley, New York, 1981.
[72] H. A. Jensen, A. Muñoz, C. Papadimitriou, and E. Millas. Model-reduction techniques for reliability-based design problems of complex structural systems. *Reliability Engineering & System Safety*, 149:204–217, 2016.
[73] M. A. Valdebenito and G. I. Schuller. Reliability-based optimization considering design variables of discrete size. *Engineering Structures*, 32(9):2919–2930, 2010.
[74] J. H. Holland. *Adaptation in Natural and Artificial Systems: An Introductory Analysis with Applications to Biology, Control and Artificial Intelligence.* MIT Press, 1992.
[75] M. G. R. Faes and M. A. Valdebenito. Fully decoupled reliability-based design optimization of structural systems subject to uncertain loads. *Computer Methods in Applied Mechanics and Engineering*, 371:113313, 2020.
[76] M. G. R. Faes and M. A. Valdebenito. Fully decoupled reliability-based optimization of linear structures subject to Gaussian dynamic loading considering discrete design variables. *Mechanical*

Systems and Signal Processing, 156:107616, 2021.
[77] J. C. Spall. *Introduction to Stochastic Search and Optimization.* John Wiley & Sons, Inc., 2003.
[78] K. M. Zuev and J. L. Beck. Global optimization using the asymptotically independent Markov sampling method. *Computers & Structures*, 126:107–119, 2013.
[79] J. L. Beck and K. M. Zuev Asymptotically independent Markov sampling: A new Markov chain Monte Carlo scheme for Bayesian inference. *International Journal for Uncertainty Quantification*, 3(5):445–474, 2013.
[80] S. Kirkpatrick, C. D. Gelatt, and M. P. Vecchi. Optimization by simulated annealing. *Science*, 220(4598):671–680, 1983.
[81] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller. Equation of state calculations by fast computing machines. *The Journal of Chemical Physics*, 21(6):1087–1092, 1953.
[82] W. K. Hastings. Monte Carlo sampling methods using Markov chains and their applications. *Biometrika*, 57(1):97–109, 1970.
[83] A. Kong, J. S. Liu, and W. H. Wong. Sequential imputations and Bayesian missing data problems. *Journal of the American Statistical Association*, 89(425):278–288, 1994.
[84] J. Wang and L. S. Katafygiotis. Reliability-based optimal design of linear structures subjected to stochastic excitations. *Structural Safety*, 47:29–38, 2014.
[85] J. Ching and Y.-C. Chen. Transitional Markov chain Monte Carlo method for Bayesian model updating, model class selection, and model averaging. *Journal of Engineering Mechanics*, 133(7):816–832, 2007.
[86] A. Lye, A. Cicirello, and E. Patelli. Sampling methods for solving Bayesian model updating problems: A tutorial. *Mechanical Systems and Signal Processing*, 159:107760, 2021.
[87] W. Betz, I. Papaioannou, and D. Straub. Transitional Markov chain Monte Carlo: observations and improvements. *Journal of Engineering Mechanics*, 142(5):04016016, 2016.
[88] H. A. Jensen, D. J. Jerez, and M. Valdebenito. An adaptive scheme for reliability-based global design optimization: A Markov chain Monte Carlo approach. *Mechanical Systems and Signal Processing*, 143:106836, 2020.
[89] H. A. Jensen, D. J. Jerez, and M. Beer. A general two-phase Markov chain Monte Carlo approach for constrained design optimization: application to stochastic structural optimization. *Computer Methods in Applied Mechanics and Engineering*, 373:113487, 2021.
[90] H. Jensen, D. Jerez, and M. Beer. Structural synthesis considering mixed discrete-continuous design variables: a Bayesian framework. *Mechanical Systems and Signal Processing*, 162:108042, 2021.
[91] S.-K. Au. Reliability-based design sensitivity by efficient simulation. *Computers & Structures*, 83(14):1048–1061, 2005.
[92] P. S. Koutsourelakis. Design of complex systems in the presence of large uncertainties: a statistical approach. *Computer Methods in Applied Mechanics and Engineering*, 197(49-50):4092–4103, 2008.
[93] E. T. Jaynes. *Probability theory: The logic of science.* Cambridge University Press, 2003.
[94] A. A. Taflanidis and J. L. Beck. Stochastic Subset Optimization for optimal reliability problems. *Probabilistic Engineering Mechanics*, 23(2):324–338, 2008.
[95] A. A. Taflanidis and J. L. Beck. An efficient framework for optimal robust stochastic system design using stochastic simulation. *Computer Methods in Applied Mechanics and Engineering*, 198(1):88–101, 2008.

[96] P. M. Pardalos and M. G. C. Resende. *Handbook of applied optimization.* Oxford University Press, 2002.
[97] A. A. Taflanidis and J. L. Beck. Stochastic Subset Optimization for reliability optimization and sensitivity analysis in system design. *Computers & Structures*, 87(5):318–331, 2009.
[98] A. A. Taflanidis and J. L. Beck. Reliability-based design using two-stage stochastic optimization with a treatment of model prediction errors. *Journal of Engineering Mechanics*, 136(12):1460–1473, 2010.
[99] P.-R. Wagner, V. K. Dertimanis, E. N. Chatzi, and J. L. Beck. Robust-to-uncertainties optimal design of seismic metamaterials. *Journal of Engineering Mechanics*, 144(3):04017181, 2018.
[100] G. Jia and A. A. Taflanidis. Non-parametric stochastic subset optimization for optimal-reliability design problems. *Computers & Structures*, 126:86–99, 2013.
[101] G. Jia and A. A. Taflanidis. Non-parametric stochastic subset optimization utilizing multivariate boundary kernels and adaptive stochastic sampling. *Advances in Engineering Software*, 89:3–16, 2015.
[102] G. Jia, A. A. Taflanidis, and J. L. Beck. Non-parametric stochastic subset optimization for design problems with reliability constraints. *Structural and Multidisciplinary Optimization*, 52(6):1185–1204, 2015.
[103] J. Beirlant, E. J. Dudewicz, L. Györfi, and E. C. Van Der Meulen. Nonparametric entropy estimation: an overview. *International Journal of Mathematical and Statistical Sciences*, 6(1):17–39, 1997.
[104] D. W. Scott and S. R. Sain. Multidimensional density estimation. In *Handbook of Statistics*, 229–261. Elsevier, 2005.
[105] R. J. Karunamuni and T. Alberts. On boundary correction in kernel density estimation. *Statistical Methodology*, 2(3):191–212, 2005.
[106] R. J. Karunamuni and S. Zhang. Some improvements on a boundary corrected kernel density estimator. *Statistics & Probability Letters*, 78(5):499–507, 2008.
[107] R. Tibshirani, G. Walther, and T. Hastie. Estimating the number of clusters in a data set via the gap statistic. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 63(2):411–423, 2001.
[108] J. A. Hartigan and M. A. Wong. Algorithm AS 136: A K-Means clustering algorithm. *Applied Statistics*, 28(1):100–108, 1979.
[109] J. P. Neilsen. Multivariate boundary kernels from local linear estimation. *Scandinavian Actuarial Journal*, 1999(1):93–95, 1999.
[110] B. Schölkopf and A. J. Smola. *Learning with kernels: support vector machines, regularization, optimization, and beyond.* MIT Press Ltd., 2018.
[111] D. W. Scott. *Multivariate Density Estimation: Theory, Practice and Visualization.* John Wiley & Sons, Inc., 1992.
[112] W.-S. Liu and S. H. Cheung. Reliability based design optimization with approximate failure probability function in partitioned design space. *Reliability Engineering & System Safety*, 167:602–611, 2017.
[113] J. N. Yang and S. Lin. Identification of parametric variations of structures based on least squares estimation and adaptive tracking technique. *Journal of Engineering Mechanics*, 131(3):290–298, 2005.
[114] S.-C. Kang, H.-M. Koh, and J. F. Choo. An efficient response surface method using moving least squares approximation for structural reliability analysis. *Probabilistic Engineering Mechanics*,

25(4):365–371, 2010.

[115] J. Ching and Y.-H. Hsieh. Approximate reliability-based optimization using a three-step approach based on subset simulation. *Journal of Engineering Mechanics*, 133(4):481–493, 2007.

[116] D. Ormoneit and H. White. An efficient algorithm to compute maximum entropy densities. *Econometric Reviews*, 18(2):127–140, 1999.

[117] A. Zellner and R. A. Highfield. Calculation of maximum entropy distributions and approximation of marginal posterior distributions. *Journal of Econometrics*, 37(2):195–209, 1988.

[118] J. Ching and Y.-H. Hsieh. Local estimation of failure probability function and its confidence interval with maximum entropy principle. *Probabilistic Engineering Mechanics*, 22(1):39–49, 2007.

[119] E. T. Jaynes. Information theory and statistical mechanics. *Physical Review*, 106(4):620–630, 1957.

[120] J. Ching and W.-C. Hsu. Approximate optimization of systems with high-dimensional uncertainties and multiple reliability constraints. *Computer Methods in Applied Mechanics and Engineering*, 198(1):52–71, 2008.

[121] W.-C. Hsu and J. Ching. Evaluating small failure probabilities of multiple limit states by parallel subset simulation. *Probabilistic Engineering Mechanics*, 25(3):291–304, 2010.