A “Survey” on Mixed-Integer Programming Techniques in Bilevel Optimization

Thomas Kleinert, Martine Labbé, Ivana Ljubic, **Martin Schmidt**

@schmaidt

July 4, 2022 — EURO 2022, Espoo, Finland

The Team




November 23: “I would like to invite you to give a semi-plenary keynote at our conference within the area of ”Discrete Optimization and Algorithms”. We think that your expertise in ”Bilevel Optimization” will make a valuable contribution to the conference.”

November 24: “Hi Arie, thank you very much for your email and your offer to give a keynote at the EURO 2022 in Espoo. I feel very honored - you can log me in!”

March 28: “Dear laureates, I am sorry I forgot one ”obligation” for one of you: ...”

March 28: “I am giving a keynote at EURO 2022 already but I am happy to give a talk on the paper if no other one wants to give this talk.”

End of March ...

March to end of June ...

End of June ...

What is Bilevel Optimization Anyway?

A Brief History of Mixed-Integer Techniques for Bilevel Optimization

What is Bilevel Optimization Anyway?

"Usual" optimization models

- a single decision maker
- one set of variables and constraints
- one objective function

Bilevel Optimization in a Nutshell

"Usual" optimization models

- a single decision maker
- one set of variables and constraints
- one objective function

Bilevel optimization

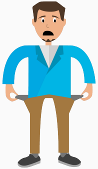
- two decision makers
- both interact in a hierarchical way

Hierarchical Decision Making

Leader: Alice x

decides first

anticipates follower (Bob)



Follower: Bob y

decides second (of course)

A Bit More Formal

Upper-level problem

$$\begin{aligned} \text{“}\min_x\text{”} \quad & F(x, y) \\ \text{s.t.} \quad & G(x, y) \geq 0 \end{aligned}$$

A Bit More Formal

Upper-level problem

$$\begin{aligned} \text{“}\min_x\text{”} \quad & F(x, y) \\ \text{s.t.} \quad & G(x, y) \geq 0, \quad y \in \mathcal{S}(x) \end{aligned}$$

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Lower-level problem

$$\begin{aligned} \min_{y} \quad & f(x, y) \\ \text{s.t.} \quad & g(x, y) \geq 0 \end{aligned}$$

A Bit More Formal

Upper-level problem

$$\begin{aligned} \text{“}\min_x\text{”} \quad & F(x,y) \\ \text{s.t.} \quad & G(x,y) \geq 0, \quad y \in \mathcal{S}(x) \end{aligned}$$

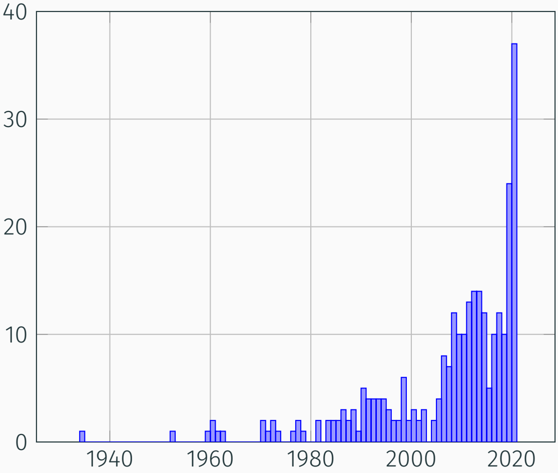
Lower-level problem

$$\begin{aligned} \min_y \quad & f(x,y) \\ \text{s.t.} \quad & g(x,y) \geq 0 \end{aligned}$$

- Different solution concepts: optimistic vs. pessimistic (Dempe 2002)
- Strongly NP-hard problem in general (Hansen, Jaumard, Savard 1992)
- Checking local optimality is NP-hard (Vicente et al. 1994)
- Mixed-integer linear bilevel problems are Σ_p^2-hard (Lodi et al. 2014)
- Optimistic variant

A Brief History of Mixed-Integer Techniques for Bilevel Optimization

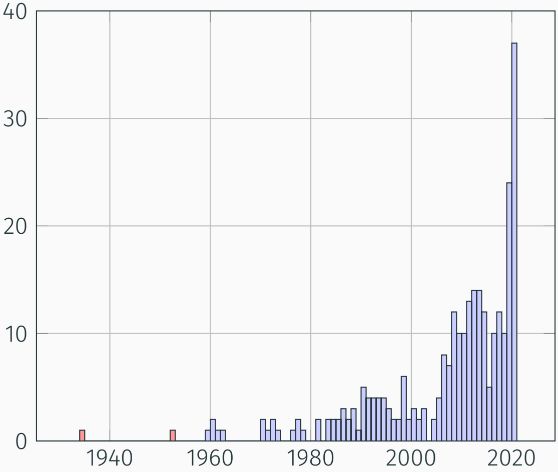
Research Activity in Bilevel Optimization



Based on the references in

A Survey on Mixed-Integer Programming Techniques in Bilevel Optimization

Where it all started

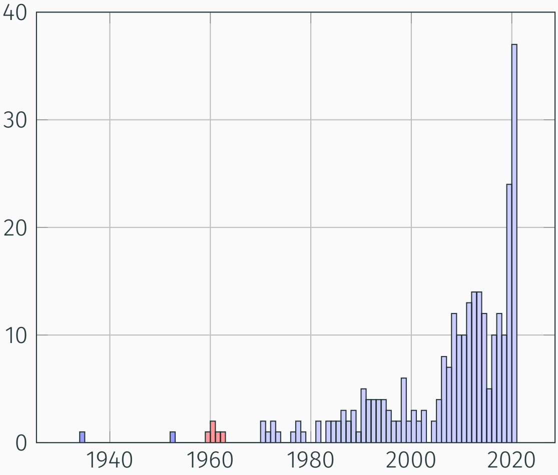


Heinrich Freiherr von Stackelberg

Hierarchy in decision making in markets

- 1934: Marktform und Gleichgewicht (Habilitation thesis)
- 1952: Theory of the market economy

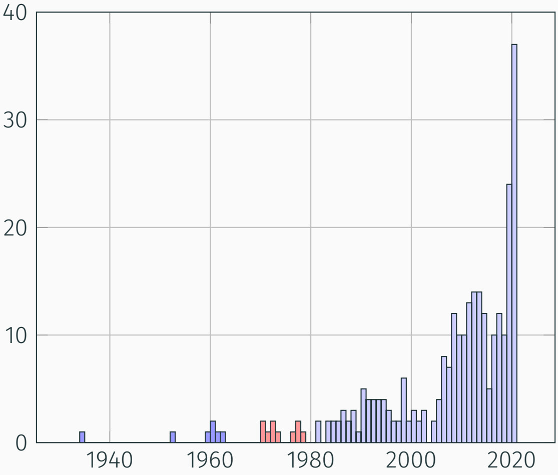
The 1960s: A Bilevel-Free Time



Bilevel-free time, but ...

- Land and Doig (1960): branch-and-bound
- Kelley (1960): cutting plane method
- Benders (1962): Benders decomposition
- Geoffrion (1972): generalized Benders decomposition
- Clark (1961) & Williams (1970): dual feasible set is unbounded for bounded primal feasible sets
- Beale and Tomlin (1970): special ordered sets (SOS) of type 1

The 1970s: Where it really started



The 1970s: Where it really started

- Bracken and McGill (1973)
- Military application
- Cost-minimal mix of weapons

Mathematical Programs with Optimization Problems in the Constraints

Jerome Bracken and James T. McGill

Institute for Defense Analyses, Arlington, Virginia

(Received October 5, 1971)

This paper considers a class of optimization problems characterized by constraints that themselves contain optimization problems. The problems in the constraints can be linear programs, nonlinear programs, or two-sided optimization problems, including certain types of games. The paper presents theory dealing primarily with properties of the relevant functions that result in convex programming problems, and discusses interpretations of this theory. It gives an application with linear programs in the constraints, and discusses computational methods for solving the problems.

Development Research Center

Discussion Papers

No. 20

MULTI-LEVEL PROGRAMMING

by

Wilfred Candler and Roger Norton

January 1977

- Candler and Norton (1977)
- First general discussion of multi-/two-level problems

Fortuny-Amat and McCarl (1981): Maybe the most influential bilevel paper

J. Opl Res. Soc. Vol. 32, pp. 783 to 792, 1981

0160-5682/81/090783-10$02.00/0

A Representation and Economic Interpretation of a Two-Level Programming Problem

JOSÉ FORTUNY-AMAT and BRUCE McCARL

Graduate School of Administration, University of California, Riverside, California, U.S.A. and Purdue University, West Lafayette, Indiana, U.S.A.

This paper first presents a formulation for a class of hierarchial problems that show a two-stage decision making process; this formulation is termed multilevel programming and could be defined, in general, as a mathematical programming problem (master) containing other multilevel programs in the constraints (subproblems). A two-level problem is analyzed in detail, and we develop a solution procedure that replaces the subproblem by its Kuhn–Tucker conditions and then further transforms it into a mixed integer quadratic programming problem by exploiting the disjunctive nature of the complementary slackness conditions.

An example problem is solved and the economic implications of the formulation and its solution are reviewed.

Fortuny-Amat and McCarl (1981): But at least (one of) the best cited papers in bilevel optimization

A representation and economic interpretation of a two-level programming problem

J **Fortuny-Amat**, B **McCarl** - Journal of the operational Research Society, **1981** - Springer

This paper first presents a formulation for a class of hierarchial problems that show a two-stage decision making process; this formulation is termed multilevel programming and could be defined, in general, as a mathematical programming problem (master) containing other multilevel programs in the constraints (subproblems). A two-level problem is analyzed in detail, and we develop a solution procedure that replaces the subproblem by its Kuhn-Tucker conditions and then further transforms it into a mixed integer quadratic programming …

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The Linear-Linear Case

$$\min_{x \in \mathbb{R}^n, y \in \mathbb{R}^m} \quad c^\top x + d^\top y \quad \text{s.t.} \quad Ax + By \geq a, \; y \in \mathcal{S}(x)$$

The Linear-Linear Case

$$\min_{x \in \mathbb{R}^n, y \in \mathbb{R}^m} \quad c^\top x + d^\top y \quad \text{s.t.} \quad Ax + By \geq a, \; y \in \mathcal{S}(x)$$

$\mathcal{S}(x)$: set of optimal solutions of the x-parameterized linear problem

$$\min_{y} \quad f^\top y \quad \text{s.t.} \quad Dy \geq b - Cx$$

Fortuny-Amat and McCarl (1981)

The lower-level problem is an LP:

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Fortuny-Amat and McCarl (1981)

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The KKT conditions

$$\begin{aligned} Cx + Dy &\geq b \\ \lambda \geq 0,\ D^\top \lambda &= f \\ \lambda^\top (Cx + Dy - b) &= 0 \end{aligned}$$

are both necessary and sufficient

Fortuny-Amat and McCarl (1981)

The lower-level problem is an LP:

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The KKT conditions

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are both necessary and sufficient

Single-level reformulation

$$\begin{aligned} \min_{x,y,\lambda} \quad & c^\top x + d^\top y \\ \text{s.t.} \quad & Ax + By \geq a, \quad Cx + Dy \geq b \\ & \lambda \geq 0, \quad D^\top \lambda = f \\ & \lambda^\top (Cx + Dy - b) = 0 \end{aligned}$$

Fortuny-Amat and McCarl (1981)

$$
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& \lambda \geq 0, \quad D^\top \lambda = f \\
& \lambda^\top (Cx + Dy - b) = 0
\end{aligned}$$

- Be careful if the dual multipliers are not unique (Dempe, Dutta 2012)
- Otherwise, all is nice ...
- ... except for the nasty KKT complementarity conditions

$$\lambda^\top (Cx + Dy - b) = 0$$

How to deal with KKT complementarity conditions

$$\lambda^\top (Cx + Dy - b) = 0$$

How to deal with KKT complementarity conditions

$$\lambda^\top (Cx + Dy - b) = 0$$

That's a disjunction

$$\lambda_i = 0 \quad \vee \quad (Cx + Dy - b)_i = 0, \quad i \in \{1, \ldots, \ell\}$$

Introduce a binary variable and some big-Ms ...

$$\begin{aligned} Cx + Dy - b &\leq M_{\mathrm{P}}(1 - u) \\ \lambda &\leq M_{\mathrm{D}} u \\ u &\in \{0, 1\}^\ell \end{aligned}$$

Mixed-Integer Linear Reformulation

$$
\begin{aligned}
\min_{x,y,\lambda} \quad & c^\top x + d^\top y \\
\text{s.t.} \quad & Ax + By \geq a, \quad Cx + Dy \geq b \\
& \lambda \geq 0, \quad D^\top \lambda = f \\
& Cx + Dy - b \leq M_{\mathrm{P}}(1 - u) \\
& \lambda \leq M_{\mathrm{D}} u \\
& u \in \{0, 1\}^\ell
\end{aligned}
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Mixed-Integer Linear Reformulation

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& Cx + Dy - b \leq M_{\mathrm{P}}(1 - u) \\
& \lambda \leq M_{\mathrm{D}} u \\
& u \in \{0, 1\}^\ell
\end{aligned}
$$

But how to choose the nasty big-Ms?

Pitfall #1: Heuristics do not work!

Solving Linear Bilevel Problems Using Big-Ms: Not All That Glitters Is Gold

Salvador Pineda and Juan Miguel Morales

***Abstract*—The most common procedure to solve a linear bilevel problem in the PES community is, by far, to transform it into an equivalent single-level problem by replacing the lower level with its KKT optimality conditions. Then, the complementarity conditions are reformulated using additional binary variables and large enough constants (big-Ms) to cast the single-level problem as a mixed-integer linear program that can be solved using optimization software. In most cases, such large constants are tuned by trial and error. We show, through a counterexample, that this widely used trial-and-error approach may lead to highly suboptimal solutions. Then, further research is required to properly select big-M values to solve linear bilevel problems.**

***Index Terms*—Bilevel programming, optimality conditions, mathematical program with equilibrium constraints (MPEC).**

in [5]. Dealing with the solution to this variant goes beyond the purposes of this letter and thus, we assume $d_i = 0$. This assumption is common in several applications of linear bilevel programming in the PES technical literature. For example, in long-term planning models formulated as bilevel problems [6], [7], [8], [9], the upper-level problem determines investment decisions to maximize investor's profit, while the lower-level problem yields the dispatch quantities to minimize operating cost. In most cases, upper-level constraints model maximum available capacities to be installed and/or budget limitations, but do not include lower-level dispatch variables.

Since the lower-level optimization problem is linear, it can be replaced with its KKT optimality conditions as follows:

Pitfall #2: It's really hard!

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Technical Note—There's No Free Lunch: On the Hardness of Choosing a Correct Big-M in Bilevel Optimization

Thomas Kleinert, Martine Labbé, Fr¨ank Plein, Martin Schmidt

Published Online: 30 Jun 2020 | https://doi.org/10.1287/opre.2019.1944

Abstract

One of the most frequently used approaches to solve linear bilevel optimization problems consists in replacing the lower-level problem with its Karush–Kuhn–Tucker (KKT) conditions and by reformulating the KKT complementarity conditions using techniques from mixed-integer linear optimization. The latter step requires to determine some big-*M* constant in order to bound the lower level's dual feasible set such that no bilevel-optimal solution is cut off. In practice, heuristics are often used to find a big-*M* although it is known that these approaches may fail. In this paper, we consider the hardness of two proxies for the above mentioned concept of a bilevel-correct big-*M*. First, we prove that verifying that a given big-*M* does not cut off any feasible vertex of the lower level's dual polyhedron cannot be done in polynomial time unless P = NP. Second, we show that verifying that a given big-*M* does not cut off any optimal point of the lower level's dual problem (for any point in the projection of the high-point relaxation onto the leader's decision space) is as hard as solving the original bilevel problem.

Remedy: It's not even required anymore!

Why there is no need to use a big-M in linear bilevel optimization: A computational study of two ready-to-use approaches

Thomas Kleinert[1,2] and Martin Schmidt[3]

Abstract. Linear bilevel optimization problems have gained increasing attention both in theory as well as in practical applications of Operations Research (OR) during the last years and decades. The latter is mainly due to the ability of this class of problems to model hierarchical decision processes. However, this ability makes bilevel problems also very hard to solve. Since no general-purpose solvers are available, a "best-practice" has developed in the applied OR community, in which not all people want to develop tailored algorithms but "just use" bilevel optimization as a modeling tool for practice. This best-practice is the big-M reformulation of the Karush–Kuhn–Tucker (KKT) conditions of the lower-level problem—an approach that has been shown to be highly problematic by Pineda and Morales (2019). Choosing invalid values for M yields solutions that may be arbitrarily bad. Checking the validity of the big-Ms is however shown to be as hard as solving the original bilevel problem in Kleinert et al. (2019). Nevertheless, due to its appealing simplicity, especially w.r.t. the required implementation effort, this ready-to-use approach still is the most popular method. Until now, there has been a lack of approaches that are competitive both in terms of implementation effort and computational cost.

In this note we demonstrate that there is indeed another competitive ready-to-use approach: If the SOS-1 technique is applied to the KKT complementarity conditions, adding the simple additional root-node inequality developed by Kleinert et al. (2020) leads to a competitive performance—without having all the possible theoretical disadvantages of the big-M approach.

The 1990s: Branch-and-Bound

- Bard and Moore (1990): Branch-and-bound for bilevel problems with continuous problems at both levels
    - Similar ideas and extensions: Bard (1988), Edmunds and Bard (1991)
- Hansen et al. (1992): New branching rules + strong NP hardness
- Moore and Bard (1990): First branch-and-bound for discrete bilevel problems
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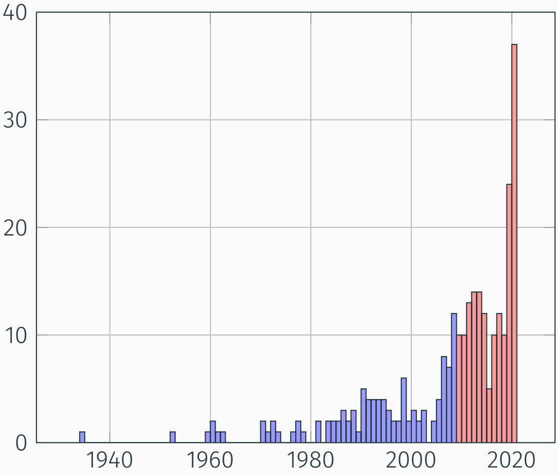
Cuts entered the stage later on:

- Wu et al. (1998): Tuy's cuts
- Audet, Haddad, et al. (2007): disjunctive cuts
- Audet, Savard, et al. (2007): Gomory-like cuts
- Kleinert, Labbé, et al. (2020): primal-dual coupling cuts

From 2009 on ...



Why 2009?

Moore and Bard (1990)

- First branch-and-bound for discrete bilevel problems
- Bad news: two of the three standard branch-and-bound fathoming rules for mixed-integer optimization are not valid in the bilevel context

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The Redemption

- DeNegre and Ralphs (2009): "A branch-and-cut algorithm for integer bilevel linear programs"
- MILP-based branch-and-cut approach

This pushed the research again

Branch-and-bound

- Fischetti, Ljubić, et al. (2018): branch-and-bound method for mixed-integer upper- and lower-level problems + coupling constraints at the upper level
- Xu and Wang (2014): multi-way branching
- Wang and Xu (2017): watermelon algorithm

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Branch-and-Cut

- Tahernejad et al. (2020): generalized no-good cuts
- Caramia and Mari (2015): another variant of no-good-cuts
- Fischetti, Ljubić, et al. (2018): intersection cuts to separate integer bilevel infeasible points
- Fischetti, Ljubić, et al. (2017): Follow-up with improved computational techniques + available code
- Tahernejad et al. (2020): another available code

Not touched here but in the survey

- Bilinear lower levels
  - pricing problems
  - toll setting problems
- Stackelberg bimatrix games
- Interdiction games
- Pessimistic setting
- Mixed-integer nonlinear bilevel problems

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A Survey on Mixed-Integer Programming Techniques in Bilevel Optimization

Check for updates

Thomas Kleinert[a,b], Martine Labbé[c,d], Ivana Ljubić[e], Martin Schmidt[f,*]

[a] Friedrich-Alexander-Universität Erlangen-Nürnberg, Discrete Optimization, Cauerstr. 11, Erlangen 91058, Germany
[b] Energie Campus Nürnberg, Fürther Str. 250, Nürnberg 90429, Germany
[c] Université Libre de Bruxelles, Department of Computer Science, Boulevard du Triomphe, CP212, Brussels 1050, Belgium
[d] Inria Lille - Nord Europe, Parc scientifique de la Haute Borne, 40, av. Halley - Bât A - Park Plaza, Villeneuve d'Ascq 59650, France
[e] ESSEC Business School of Paris, 95021 Cergy-Pontoise, France
[f] Trier University, Department of Mathematics, Universitätsring 15, Trier 54296, Germany

Discussion of the state-of-the art

More than 250 references

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