Antimagic Labeling of Graphs

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Chapter 1

Introduction

It may seem strange to term a graph as having an "antimagic" labeling, but the term comes from its connection to magic labelings and magic squares. Magic squares can trace their origin back to ancient China somewhere around the 7th century BCE [4]. A magic square is an arrangement of numbers into a square such that the sum of each row, column and diagonal are equal. The term "antimagic" then comes from being the opposite of magic, or arranging numbers in a way such that no two sums are equal.

The interest in graph labelings can trace its roots back to a paper [9] by Alex Rosa in the late 1960's. Hartsfield and Ringel introduced the concept of antimagic labeling, which is an assignment of distinct values to different objects in a graph in such a way that when taking certain sums of the labels the sums will all be different. They conjectured in [7] that every graph except for K_2 has an antimagic edge labeling. Then Bodendiek and Walther proved in [3] that from some natural number n any connected graph other than K_2 will have a weak antimagic edge labelling if you allow the labels to be natural numbers with an upper bound of n. A weak antimagic labeling is simliar to an antimagic labeling except one does not require distinct labels. (For more precise definitions of antimagic and weak antimagic see Definitions 2.2 and 2.3). The work from these two papers are the motivation for my research. From this we look further into weak antimagic labelings of graphs, using the idea of inside-out polytopes to approach the problem in a different way [1]. We will show that bipartite graphs will have a weak antimagic labeling while limiting the labels of the edges to a value less than or equal to the number of edges in the graph. We will do this by using polynomials to count labels and show that they exist.

Chapter 2

Preliminaries

A **(simple) graph** $G = (V, E)$, is defined by a pair of finite sets V and E, which we denote as the **vertex set** and the **edge set** respectively. An element of the edge set is a two-element subset of the vertex set. In other words any edge e connecting vertex u to vertex v can be uniquely writen as $e = \{u, v\}$. Note that this is an unordered pair, so $\{u, v\} = \{v, u\}$. Two vertices are called **adjacent** if there is an edge between them. Typically, a graph is depicted as a set of dots for the vertices, joined by lines or curves for the edges. A graph G' is a **subgraph** of G if both the edge and vertex sets of G' are subsets of the edge and vertex sets of G respectively.

A **labeling** is an assignment of labels to edges, vertices, or both edges and vertices of a graph. An **edge labeling** is function $f : E \to \mathbb{Z}'$, where $\mathbb{Z}' \subset \mathbb{Z}$, in other words it is a labeling of all edges by integers. A graph with such a labeling is an **edge labeled graph**. For the remainer of this paper whenever refering to a graph we will be refering to an edge labeled graph. Similarly whenever refering to a labeling we will be refering to an edge labeling.

A **path** in a graph G is a sequence $v_0e_1v_1e_2v_2 \cdots v_{n-1}e_nv_n$ of edges and vertices, where $e_k = \{v_{k-1}, v_k\}$ and each e_k appears in the sequence only once. A **cycle** is a path that starts and ends on the same vertex in such a way that that vertex is the only edge or vertex that is repeated. A graph G is said to be **connected** if there exists a path between any two vertices in G. Since we are only concerned with edge labeled graphs in this paper we can restrict ourselves to connected graphs.

Definition 2.1. A graph G_{U_1,U_2} is **bipartite** if its vertex set can be divided into two disjoint sets U_1 and U_2, such that for all u_{1_i}, $u_{1_k} \in U_1$ and for all u_{2_i}, $u_{2_k} \in U_2$ there does not exist an edge $e \in E$, where $e = \{u_{1_i}, u_{1_k}\}$ or $e = \{u_{2_i}, u_{2_k}\}$.

Definition 2.2. A graph G is called **antimagic** if the n edges of G can be distinctly labeled 1 through n in such a way that when taking the sum of the edge labels incident to each vertex, the sums will all be different. (See Figure 2.1 for an example of an antimagic labeling for the graph K_4. Note that the red labels are the sums of the labels for the edges incident to the closest vertex).

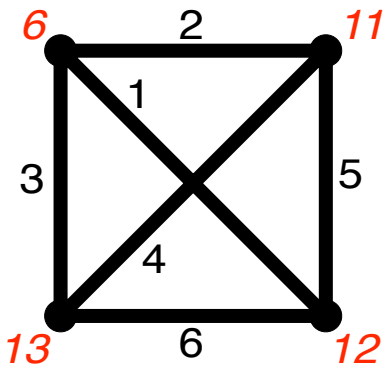


Figure 2.1: (Strong) antimagic labeling for K_4.

This type of labeling is sometimes refered to as a strong antimagic labeling due to the fact that there is also a weak antimagic labeling.

Definition 2.3. A graph is said to have a **weak antimagic labeling** if you can label the edges in an antimagic way, still allowing the edges to be integers less than or equal to the number of edges, without the edge labels necessarily being distinct. (See Figure 2.2 for an example of a weak antimagic labeling of K_4).

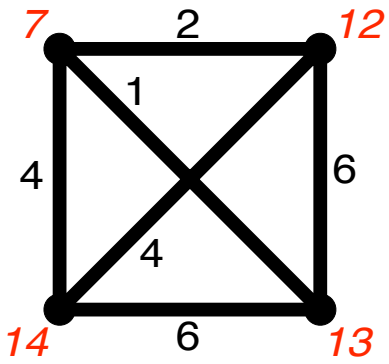


Figure 2.2: Weak antimagic labeling for K_4.

Any strong antimagic labeling is also a weak antimagic labeling.

Chapter 3

Polytopes and Inside-out Polytopes

3.1 Polytopes and Ehrhart Theory

A **convex polytope** is a convex hull of finitely many points in $\mathbb{R}^d$. More specifically, given a finite point set $\{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^d$, the polytope $\mathcal{P}$ is given by

$$\mathcal{P} = \left\{ \sum_{k=1}^{n} \lambda_k v_k \,\middle|\, \lambda_k \geq 0 \text{ and } \sum_{k=1}^{n} \lambda_k = 1 \right\}.$$

This is called the **vertex description** of the polytope and is written

$$\mathcal{P} = \text{conv}\{v_1, v_2, \ldots, v_n\}.$$

Equivalently a polytope can be defined as the bounded intersection of finitely many half spaces [11]. This is called the **hyperplane description** of $\mathcal{P}$; it is written

$$\mathcal{P} = \{x \in \mathbb{R}^d | Ax \leq b\}, \tag{3.1}$$

where the rows of $Ax \leq b$ are inequalities of the form $\sum_{j=1}^{d} a_{ij}x_j \leq b_i$ each of which decribes a half space in $\mathbb{R}^d$. The equation $\sum_{j=1}^{d} a_{ij}x_j = b_i$ describes a hyperplane in $\mathbb{R}^d$. The **dimension** of a polytope $\mathcal{P}$ is the dimension of the affine space spanned by its vertices. If a polytope has dimension d then we call it a d-polytope and write $\dim \mathcal{P} = d$. For a given polytope $\mathcal{P}$, we call a hyperplane H a **supporting hyperplane** if H bounds a half space that contains $\mathcal{P}$. A **face** of $\mathcal{P}$ is the intersection of $\mathcal{P}$ with some supporting hyperplane. An **edge** of a polytope is a face that is 1-dimensional and a **vertex** of a polytope is a face that is 0-dimensional. A polytope is called **integral** if its vertices all lie in $\mathbb{Z}^d$. The **interior** of a polytope $\mathcal{P}$ given by (3.1), denoted $\mathcal{P}^\circ$, is defined as

$$\mathcal{P}^\circ = \{x \in \mathbb{R}^d | Ax < b\}.$$

The **boundary** of a polytope $\mathcal{P}$ is the set $\partial\mathcal{P} := \mathcal{P} \backslash \mathcal{P}^\circ$. The t^{th} **dilate** of $\mathcal{P}$ is defined to be

$$t\mathcal{P} = \{tx | x \in \mathcal{P}\}$$

where $t \in \mathbb{Z}_{>0}$, [1]. Ehrhart Theory deals with finding the discrete volume of dilated polytopes. In other words we want to count the number of lattice points in a dilate of the polytope. For the t^{th} dilate of a given polytope $\mathcal{P} \subseteq \mathbb{R}^d$, we define the following two counting functions:

$$L_{\mathcal{P}}(t) = \#\{x \in t\mathcal{P} \cap \mathbb{Z}^d\}$$

and

$$L_{\mathcal{P}^\circ}(t) = \#\{x \in t\mathcal{P}^\circ \cap \mathbb{Z}^d\}.$$

These counting functions then give rise to the following theorems.

Theorem 3.1. *[5] Given a convex integral d-polytope $\mathcal{P}$, the counting function $L_{\mathcal{P}}(t)$ is a polynomial in t of degree d.*

Theorem 3.2. *[6, 8] For a convex integral d-polytope $\mathcal{P}$, $L_{\mathcal{P}}(-t) = (-1)^d L_{\mathcal{P}^\circ}(t)$.*

In the next section we will introduce the concept of inside-out polytopes, as well as show how antimagic labelings relate to both polytopes and inside-out polytopes.

3.2 Inside-out Polytopes

The theory of inside-out polytopes is concerned with counting the points of the integral lattice $\mathbb{Z}^d$ that lie within a polytope, but do not lie within a certain hyperplane arrangment. A **hyperplane arrangement** or **arrangement of hyperplanes** is a finite set of hyperplanes in an affine space.

Definition 3.1. An inside-out polytope is a polytope paired with an arrangement of hyperplanes that cut through the polytope acting as additional boundaries inside the polytope.

An inside-out polytope is denoted $(\mathcal{P}, H)$, where $\mathcal{P}$ is a convex polytope and H is an arrangement of hyperplanes. A **region** of the inside-out polytope $(\mathcal{P}, H)$ is a component of $\mathcal{P} \backslash \bigcup H$ or that component's closure. A **vertex** of $(\mathcal{P}, H)$ is a vertex of one of the regions of $(\mathcal{P}, H)$. An inside-out polytope is said to be **integral** if the vertices of that inside-out ploytope all lie in $\mathbb{Z}^d$. The lattice-point enumerator for an inside-out polytope $(\mathcal{P}, H)$ is

$$E^\circ_{(\mathcal{P}^\circ, H)}(t) = \#\{x \in [t\mathcal{P}^\circ \backslash \bigcup H] \cap \mathbb{Z}^d\}. \tag{3.2}$$

The lattice-point enumerator for the inside-out polytope starts with all possible points in $\mathcal{P}$. Then we subtract the lattice points in the hyperplane arangement H. Finally we use the inculsion-exclusion principle to add and substract the intersections of the hyperplanes so that each point is only counted once. The **inculsion-exclusion principle** states that if A and B are two finite sets then the number of of elements in $A \cup B$ is the number of elements in A plus the number of elements in B minus the number of elements in their intersection.

Theorem 3.3. *[2] If $(\mathcal{P}, H)$ is integral, then $E^\circ_{(\mathcal{P}^\circ, H)}(t)$ is a polynomial in t of degree dim$\mathcal{P}$.*

To relate inside-out polytopes to antimagic labelings, consider a graph with n edges. We label the n edges a_1 through a_n where each $a_i \in \mathbb{Z}_{>0}$. From the work done by in [3], we know that there exists a weak antimagic labeling for some upper bound on the labels. We let t be this upper bound so $0 < a_i < t$. Now we map this labeling to the n-tuple $(a_1, \ldots, a_n)$. Consider all possible labelings for this graph with these conditions as points in an n-dimensional space. The polytope that is constructed from these points is the t^{th} dilation of an n-dimensional unit cube. The lattice-point enumerator for the t^{th} dilation of the n-dimensional unit cube is $(t+1)^n$. Since we deal with strict inequalities, however, we want to look at the interior of this polytope for which the lattice-point enumerator is $(t-1)^n$. But that represents all possible labelings for that graph, we want the ones that are antimagic. So we construct an inside-out polytope $(\mathcal{P}^\circ, H)$ with $\mathcal{P}^\circ$ being the open n-dimensional unit cube and H consisting of the hyperplanes that contain the points of the form $(a_1, \ldots, a_n)$ such that $(a_1, \ldots, a_n)$ is not an antimagic labeling. Remember a labeling will not be antimagic if any of the sums of edges labels at two of more vertices are equal. We will call these hyperplanes **non-antimagic**. From this point forward we will use $\mathcal{P}_A$ to denote the antimagic inside-out polytope $(\mathcal{P}^\circ, H)$.

Since each point in $\mathcal{P}_A$ represents a labeling for our graph, the lattice-point enumerator of $\mathcal{P}_A$ will also be the counting function for the number of antimagic labelings. Thus the counting function $A(t)$ for the number of antimagic labelings is defined by

$$A(t) = E^\circ_{\mathcal{P}^\circ, H}(t). \tag{3.3}$$

For further clarification let's look at a specific graph. Let us consider the graph C_3 seen in Figure 3.1.

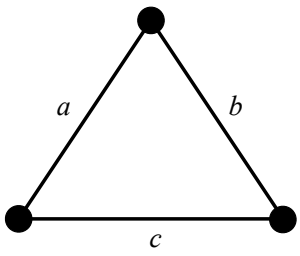


Figure 3.1: The graph C_3 with edge labels

If we consider all possible labels (a, b, c) with the contraints $0 < a, b, c < t$, then we know that the t^{th} dilate of the 3-dimensional unit cube will represent all possible labelings for this graph. Now using the non-antimagic hyperplanes we can complete the inside-out polytope for this graph. For this graph the only labelings that are not antimagic are if $a + b = b + c$, $a + b = a + c$, or $a + c = b + c$ (i.e., when the sums at the vertices are equal). When we simplify these we get the equations for the non-antimagic hyperplanes $a = b$, $a = c$, and $b = c$.

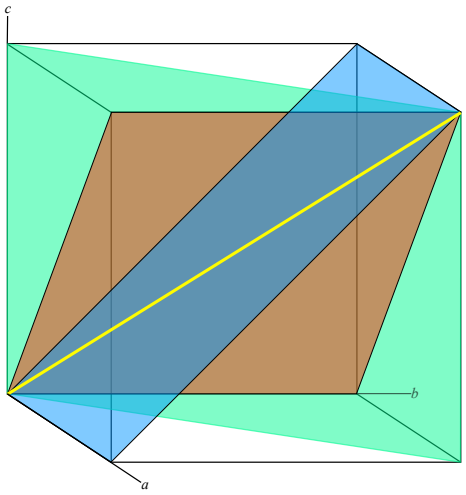


Figure 3.2: The inside-out polytope for the graph C_3

In Figure 3.2 the entire cube is the t^{th} dilate of the 3-dimensional unit cube. The green plane is the hyperplane $a = b$. The hyperplane $a = c$ is highlighted in red. The hyperplane $b = c$ is highlighted in blue. In the figure above, the line $a = b = c$ is highlighted in yellow.

If we want to find the lattice point enumerator for this inside-out polytope we can do it in parts. The number of lattice points in the interior of the cube is $(t-1)^3$. Then for each hyperplane the number of lattice points will be $(t-1)^2$. And finally the line segment of intersection will have $(t-1)$ lattice points. So starting with all possible lattice points in the interior of the cube we will remove the lattice points that are in the non-antimagic hyperplanes. However there is an overlap in the lattice points on the non-antimagic hyperplanes, along the line $a = b = c$. Thus, we add the lattice points along the line back in twice since it was the point of intersection of the non-antimagic hyperplanes and only needed to be removed once. Thus the lattice point enumerator for the interior of this inside-out polytope is

$$E^{\circ}_{\mathcal{P}^{\circ},H}(t) = (t-1)^3 - 3(t-1)^2 + 2(t-1).$$

Therefore the counting function for the antimagic labelings of C_3 will also be

$$A(t) = (t-1)^3 - 3(t-1)^2 + 2(t-1).$$

Unfortunately the antimagic inside-out polytope for other graphs is not easy to visualize, nor will the lattice point enumerator be as easy to calculate. Because of this we must find another way to understand the lattice-point enumerator for the inside-out polytopes of our graphs.

For the remainder of this paper when using the term antimagic it will mean weak antimagic. We will also shift from looking at all graphs to focus strictly on bipartite graphs.

Chapter 4

Main Results

4.1 Totally Unimodular Matrices

Consider a bipartite graph $G_{U,V}$. Label the vertices in U with the numbers 1 through m_1 and label the vertices in V with the numbers 1 through m_2. Now label the edges in the graph $a_{i,j}$, where i and j are the numbers of the vertices in U and V respectively, incident to the edge. Take the sum at each vertex of edge labels incident to that vertex. Let $V(i) = \{j \in V : j \text{ is adjacent to } i\}$ and $U(j) = \{i \in U : i \text{ is adjacent to } j\}$.
Then for each vertex $i \in U$ this sum is $\sum_{j \in V(i)} a_{i,j}$. Similarly, for each vertex $j \in V$ we have $\sum_{i \in U(j)} a_{i,j}$. In order for these labelings to be antimagic we want each of these sums to be distinct.
We construct a martix B, the **incidence matrix** for our graph, with entries $b_{v,a}$ that are defined by

$$b_{v,a} = \begin{cases} 1 & \text{if edge } a \text{ is incident to vertex } v, \\ 0 & \text{otherwise.} \end{cases}$$

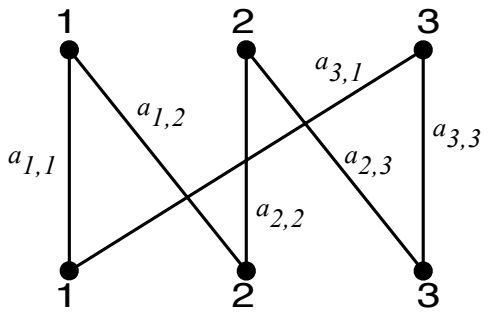
The matrix B is an $(m_1 + m_2) \times n$ matrix constructed in a way such that the first m_1 rows represent the vertices in U and the next m_2 rows the vertices in V. However since B can be rather large depending on the number of edges in the graph, it may be easier to look at an example with a specific graph in mind first. See Figure 4.1 for the incidence matrix for C_6.

Each row is representative of a different vertex, while the columns represent different edges. Since an edge can only be incident to two vertices, there will be two 1-entries in each column of this matrix while all the other entries will be 0. Now let us shift back to the general case.

Definition 4.1. A matrix is **totally unimodular** [10] if every one of its submatrices has a determinant of -1, 0, or 1.

Proposition 4.1. *The incidence matrix of a bipartite graph is totally unimodular.*

Proof. Take an arbitrary submatrix of an incidence matrix B of a bipartite graph. There are three possible cases for columns of submatrices of B. This submatrix can have a column



$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Figure 4.1: The graph C_6 with edge labels and its incidence matrix

of only entries equal to 0, only columns with two entries equal to 1, or at least one column that has one entry equal to 1. By induction on the size of the matrix, we will prove that this submatrix will have a determinant of ± 1 or 0.

For our base case let's consider 1×1 matrices with only 0 and 1 entries that follow the conditions for the submatrix. Since the determinant of (0) and (1) are 0 and 1 respectively, any 1×1 matrix under our conditions have a determinant of ± 1 or 0.

Now assume that any $k \times k$ matrix under these cases has determinant 0, or ± 1. This will be our inductive hypothesis.

Case 1: The submatrix has a column of only 0 entries.

If the matrix has a column of only 0 entries then its determinant is 0.

Case 2: The submatrix has only columns with two entries equal to 1.

If every column in the submatrix has two entries equal to 1, then every edge label for the subgraph which the submatrix represents will be incident to two vertices of the subgraph. Since the graph is bipartite, then for each column, one of the 1 entries will appear in a row that represents a vertex in the set U and the other in the set V. Because of this, if we multiply all the rows that represent the vertices in the set U by -1 and then add them to the rows of representative of vertices in the set V, all the rows will cancel. Hence this submatrix is singular and will have determinant 0.

Case 3: The submatrix has at least one column with only one entry equal to 1.

Consider a $(k+1) \times (k+1)$ matrix with a column with only one entry equal to 1, like the matrix below.

$$\begin{pmatrix} 1 & c \\ 0 & M \end{pmatrix}$$

This matrix was positioned in a way that the column that has only one entry equal to 1 is the first column, and that 1 was placed in the upper right corner. However, this column can appear anywhere in the matrix with the 1 being anywhere along that column. The only thing that may change as a result of this is the sign of the determinant. Since we are not concerned with the sign this will not affect the outcome and thus we can assume that the 1 is in the first row and column. The c in the matrix above represents the rest of the first row which can contain both entries of 0 and 1. The M represents the $k \times k$ submatrix that is not part of the first row or column. Now if we take the determinant of this matrix it is 1 times the determinant of the M submatrix. Since every entry except for the 1 of the first column is 0 we are not concerned with the remaining entries in c. Since M is a $k \times k$ matrix by our inductive hypothesis the determinant of M will be ± 1 or 0. Hence the determinant of the entire matrix is ± 1 or 0.

As we saw in all three cases the determinant of the submatrix was ± 1 or 0. Since these cases represent all possible submatrices of B, B is totally unimodular. □

Given a matrix A we contruct a new matrix A_1 in such a way that A_1 is made by taking the difference of two rows of A. In other words to construct A_1 we will replace row R_j in A with $R_j - R_k$. We keep track of this construction by creating a directed graph G_A such that each node in the graph represent a row in A. If we replace row R_j in A with $R_j - R_k$ in A_1 then we draw an edge from node R_j to node R_k. This is illistrated in the figure below.

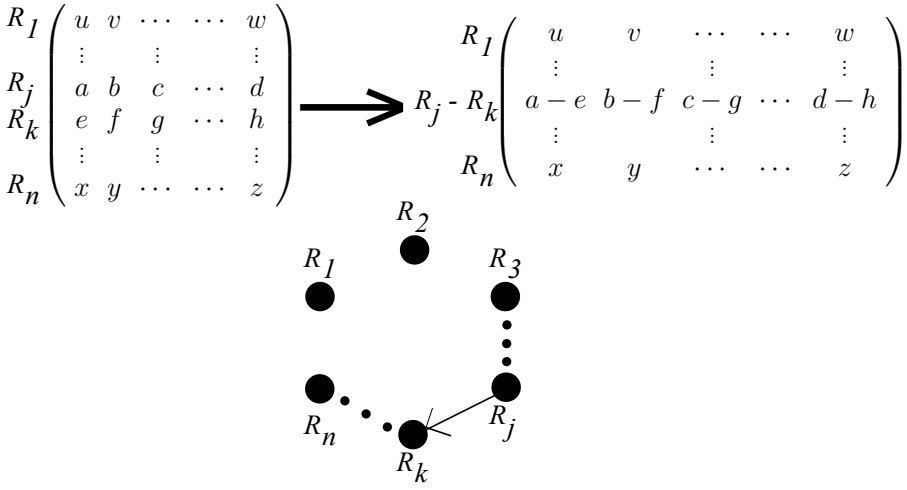


Figure 4.2: The matrices A and A_1 with graph G_A

Lemma 4.2. *If A is a totally unimodular matrix with entries 0 and 1 and G_A contains no cycles, then A_1 is also totally unimodular.*

Proof. We will prove this by induction on the number of row operations. For our base case consider a totally unimodular matrix A to which we perform none of the row operations described above. Then our graph contains no edges so there are no cycles and our matrix is unchanged, hence it remains totally unimodular.
Now assume that the lemma holds for n row operations.
Now consider A_1 being constructed by performing $n+1$ row operations on A. From our inductive hypothesis we know that for n of these operations our new matrix will be totally unimodular. If we perform one more operation in such a way that we are not creating a cycle in G_A then it will not effect the rank of our new matrix A_1. Then, taking the difference of two rows will not effect the determinant of our matrix or any of it submatrices. Hence A_1 will still be totally unimodular. □

4.2 The Polynomiality of $A(t)$

In order to better understand the inside-out polytope we are working with we need to take a closer look at its regions. Recall the inside-out polytope $\mathcal{P}_A$ consists of $\mathcal{P}^\circ$, the open n-dimensional unit cube, being cut by non-antimagic hyperplanes H. Since both the unit cube and the hyperplanes are open, each region of our inside-out polytope will be open. Each region is enclosed by some faces of $\mathcal{P}^\circ$ and some non-antimagic hyperplanes. The faces of $\mathcal{P}^\circ$ are defined by hyperplanes of the form $X_k = 1$ or $X_k = 0$. Since each vertex of a given region is just a point in an n-dimensional space, it can be described by the intersection of n hyperplanes. Since the vertices of the regions make up the vertices of $\mathcal{P}_A$, each vertex will be the intersection n hyperplanes from the set of hyperplanes of the form $X_k = 1$, $X_k = 0$, and the non-antimagic hyperplanes.

Theorem 4.3. *[10] Let A be an integral matrix of full row rank. Then the polytope $\mathcal{P} = \{x | x \geq 0 : Ax = b\}$ is integral for each integral vector b, if A is totally unimodular.*

Theorem 4.4. *The vertices of $\mathcal{P}_A$ are integral.*

Proof. Consider an arbitrary vertex of $\mathcal{P}_A$. This vertex is then the intersection on n hyperplanes from the set of hyperplanes of the form $X_k = 1$, $X_k = 0$, and the non-antimagic hyperplanes. Using the incidence matrix of our graph we can represent the non-antimagic hyperplanes by setting two rows equal to each other. Then if we take the difference of those two rows we can construct a system from the equations of these hyperplanes. In doing this we will leave out the hyperplanes of the form $X_k = 0$, for now. So a vertex is the solution to a system of the form

$$\begin{pmatrix} I & 0 \\ & B_\alpha \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

In this matrix the first j rows are formed by the equations $X_1 = X_2 = \cdots = X_j = 1$ the indices were relabelled for symplicity. Then B_α is a matrix constructed from taking differences of two rows from the incidence matrix of our graph. Since the incidence matrix is totally unimodular by Lemma 4.2 so is B_α. Now we look at the hyperplanes of the form $X_k = 0$. If $X_k = 0$, then we throw out column k. The resulting matrix is totally unimodular, and so the entire matrix will be totally unimodular, since adding an identity matrix does not change the determinant of the matrix or any submatrix. Then by Theorem 4.3 the solution must be integral. Hence the vertices of $\mathcal{P}_A$ are integral. □

Theorem 4.5. *The counting function for the antimagic labelings of a bipartite graph is polynomial of degree equal to the number of edges in the graph.*

Proof. Since the vertices of $\mathcal{P}_A$ are integral, Theorem 3.3 implies that $A(t) = E^\circ_{\mathcal{P}_A}(t)$, the counting function for antimagic labelings is a polynomial. The degree of this polynomial is $\dim\mathcal{P}$, which is the number of edges in the graph. □

Now that we have shown that the counting function for the number of antimagic labelings for our graph is a polynomial, we can use this information towards proving that the graph will have at least one antimagic labeling. Remember when we say antimagic labeling we mean weak antimagic labeling which does not require distinctness of the labels.

Theorem 4.6. *Any bipartite graph has an antimagic labeling.*

Proof. We know $A(t)$ is a polynomial for a bipartite graph with n edges. In order for this graph to have an antimagic labeling we can use only labels 1 to n. So we must only show that $A(n+1) \neq 0$. Suppose $A(n+1) = 0$, then that means that each number 1 to $n+1$ is a root of the polynomial $A(t)$: If some number k, such that $1 \leq k \leq n+1$ is not a root of $A(t)$ then there exits an antimagic labeling using labels 1 to $k-1$. Moreover, for each $h \geq k$, $A(h) \neq 0$ since an antimagic labeling can be attained using labels 1 to $k-1$ and adding more options for labels will not decrease the number of antimagic labelings, only increase. Since $A(t)$ is a polynomial of degree n it can only have n roots. Therefore 1 to $n+1$ can not all be roots of $A(t)$ and $A(n+1) \neq 0$. Hence any bipartite graph will have an antimagic labeling. □

Chapter 5

Open Problems

We have shown that any bipartite graph has a weak antimagic labeling. In general, the process we used to show this will hold true for any graph whose incidence matrix is totally unimodular. However, only bipartite graphs have totally unimodular incidence matrices [10]. There are also different types of antimagic according to what is being labelled, such as node antimagic. This method could possibly be used to prove a similar result for the different kinds of antimagic. For other forms of antimagic, see [2].

Furthermore, our method of proving bipartite graphs have a weak antimagic labeling could possibly be used to show that a graph also has a strong antimagic labeling. In order to do that we would have to add more restrictions that would result in adding more equations to the matrix for our graph. This process could also be extended to non-bipartite graphs. However in both circumstances our matrices become more complex and will not be unimodular. Because of this the counting function for antimagic labelings would only be guaranteed to be a quasi-polynomial and not a polynomial as in the case for a weak antimagic labeling of a bipartite graph. A **quasi-polynomial** is a "polynomial" with coefficients that are periodic functions with an integral period; unlike the coefficients of a polynomial which do not change. This then gives rise to another problem. If the counting function is a quasi-polynomial then for each period we get different coefficients and therefore a different polynomial. Then each of these polynomials will have its own set of roots. Because of this our argument using the roots of a polynomial will not work since a quasi-polynomial of degree d with a period of k can have kd roots. Hence a quasi-polynomial can have more roots than our graph has edges.

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