The Cover Time of Cartesian Product Graphs

Mohammed Abdullah, Colin Cooper, and Tomasz Radzik

Department of Computer Science, King's College London
{mohammed.abdullah,colin.cooper,tomasz.Radzik}@kcl.ac.uk

Abstract. Let $P = G\Box H$ be the cartesian product of graphs G, H. We relate the cover time $\mathbf{COV}[P]$ of P to the cover times of its factors. When one of the factors is in some sense larger than the other, its cover time dominates, and can become of the same order as the cover time of the product as a whole. Our main theorem effectively gives conditions for when this holds. The probabilistic technique which we introduce, based on the blanket time, is more general and may be of independent interest, as might some of our lemmas.

Keywords: Random walks, cover time, blanket time, effective resistance, cartesian product graphs.

1 Introduction

For a connected graph Let G, denote by $V(G)$ and $E(G)$ the vertex and edge set respectively. The *vertex cover time* $\mathbf{COV}[G]$ of G is defined as the expected time it takes a random walk to visit all vertices of the graph, maximised over all possible starting vertices. This quantity is a fundamental area in the study of random walks has been extensively studied giving rise to a large body of theory and application. Let $n = |V(G)|$ and $m = |E(G)|$. It is a classic result of Aleliunas, Karp, Lipton, Lovász and Rackoff [1] that $\mathbf{COV}[G] \leq 2m(n-1)$. It was shown by Feige [7], [8], that for any connected graph G, the cover time satisfies $(1-o(1))n \log n \leq \mathbf{COV}[G] \leq (1+o(1))\frac{4}{27}n^3$. Between these two extremal examples, the cover time, both exact and asymptotic, has been determined for a number of different classes of graphs.

In this work, we study the cover time of the *cartesian product* P of two graphs G, H defined as follows:

Definition 1. *The* cartesian product $P = G\Box H$ *of finite connected graphs* G, H*, is the graph such that*

- $V(P) = V(G) \times V(H)$
- $((a, x), (b, y)) \in E(P)$ *if and only if either*
  - $(a, b) \in E(G)$ *and* $x = y$*, or*
  - $a = b$ *and* $(x, y) \in E(H)$

For a natural number d, we denote by G^d the d'th cartesian power, that is, $G^d = G$ when $d = 1$ and $G^d = G^{d-1}\Box G$ when $d > 1$. We can think of $P = G\Box H$

in terms of the following construction: We make a copy of one of the graphs, say G, once for each vertex of the other, H. For the copy of G corresponding to vertex $x \in V(H)$, G_x, and a vertex $a \in V(G)$, we add an edge from $a \in G_x$ to $a \in G_y$ for all vertices $y \in V(H)$ such that $(x,y) \in E(H)$.

In the following, if z is a parameter, let z_G represent that parameter for a graph G. We have the following : n the number of vertices; m the number of edges; δ the minimum degree; θ average degree; Δ the maximum degree; D diameter.

In this paper we prove the following

Theorem 1. *Let $P = G\Box H$ where G, H are any connected, finite graphs. We have*

$$\max\left\{\left(\frac{\delta_G}{\Delta_H}+1\right)\mathbf{COV}[H], \left(\frac{\delta_H}{\Delta_G}+1\right)\mathbf{COV}[G]\right\} \leq \mathbf{COV}[P] \tag{1}$$

and

$$\mathbf{COV}[P] \leq K\min\left\{\left(1+\frac{\Delta_G}{\delta_H}\right)\mathbf{COV}[H]+\frac{Mm_Gm_Hn_Hl^2}{\mathbf{COV}[H]D_G}, \left(1+\frac{\Delta_H}{\delta_G}\right)\mathbf{COV}[G]+\frac{Mm_Gm_Hn_Gl^2}{\mathbf{COV}[G]D_H}\right\} \tag{2}$$

Where $M = |E(P)|$, $l = \log D_G \log(n_G D_G)$ and K is a universal constant.

This extends much work done on the particular case of the two-dimensional toroid on n^2 vertices, i.e., $\mathbb{Z}_n\Box\mathbb{Z}_n$ where $\mathbb{Z}_n$ is the n-vertex cycle, and on powers G^d done by [9]. To prove Theorem 1, we present a framework to analyse the cover time of a random walk on a graph which works by dividing the graph up into (possibly overlapping) regions, analysing the behaviour of the walk when locally observed on those regions, and then composing the analyses of all the regions over the whole graph. The technique facilitates the analysis of the local observation on a region by relating it to a walk on a graph derived from that region. Thus the analysis of the whole graph is reduced to analysis of outcomes on local regions and subsequent compositions of those outcomes. This framework can be applied more generally than cartesian products. Some of the lemmas we use may be of independent interest. In particular, Lemmas 7 and 8 provided bounds on effective resistances of graph products that extend well-known and commonly used bounds for the $n \times n$ lattice graph.

Our paper uses the very recently proved conjecture that the blanket time of a graph is within a universal constant factor of the cover time. The *blanket time* $\mathbf{B}[G]$ of a graph G, introduced in [12], is the expected time of the random walk on G not only to visit every vertex, but to visit all vertices more-less uniformly (the exact definition given in 2.2). Our analysis is an example of how to exploit the relation $\mathbf{B}[G] = O(\mathbf{COV}[G])$. The lower bound in Theorem 1 implies that $\mathbf{COV}[G\Box H] \geq \mathbf{COV}[H]$, and the upper bound can be viewed as providing conditions sufficient for $\mathbf{COV}[G\Box H] = O(\mathbf{COV}[H])$. For example, $\mathbf{COV}[\mathbb{Z}_p\Box\mathbb{Z}_q] = \Theta(\mathbf{COV}[\mathbb{Z}_q]) = \Theta(q^2)$ subject to the condition $p\log^2 p = O(q)$. Thus for this example, the lower and upper bounds in Theorem 1 are within a constant factor

2 Preliminaries

2.1 Some notation

We make use of the following notation: For a graph G let $V(G)$ and $E(G)$ denote the vertex and edge set of G respectively. For a random variable A representing a function of a walk, and a vertex $u \in V(G)$ let $\mathbf{E}_u[A]$ represent the expectation of A when the walk starts at u. Let $\tau(u)$ be a random variable representing the first time that u is visited by the walk and $\kappa(G)$ the first time every vertex in G has been visited by the walk. $\mathbf{H}[u,v] = \mathbf{E}_u[\tau(v)]$ is the hitting time from u to v, $\mathbf{COM}[u,v] = \mathbf{H}[u,v] + \mathbf{H}[v,u]$ is the commute time between u and v. $\mathbf{COV}[G] = \max_{u \in V(G)} \mathbf{E}_u[\kappa(G)]$ is the cover time of a graph G. Let $\mathbf{H}^+[G] = \max_{u,v} \mathbf{H}[u,v]$. The function $d(u)$ gives the degree of vertex u. For clarity, and because a vertex u may be considered in two different graphs, we may use $d_G(u)$ to explicitly denote the degree of u in graph G.

L_n denotes the n'th harmonic number, that is, $L_n = \sum_{i=1}^{n} 1/i$. Note $L_n = \log n + \gamma + O(1/n)$ Where $\gamma \approx 0.577$. In this paper all logarithms are base-e.

2.2 Blanket time

Definition 2 ([12]). *For a graph G, and $\delta \in [0,1)$ define the random variable $B_\delta(G) = \min\{t : (\forall v) N_v(t) > \delta \pi_v t\}$ where $N_v(t)$ is the number of times v has been visited by time t and π_v is the stationary probability of vertex v. The* blanket time $\mathbf{B}_\delta[G] = \max_{v \in V(G)} \mathbf{E}_v[B_\delta(G)]$.

The following was very recently proved.

Theorem 2 ([5]). *For any graph G, and any $\delta \in (0,1)$, we have*

$$\mathbf{B}_\delta[G] \leq c(\delta)\mathbf{COV}[G] \tag{3}$$

Where the constant $c(\delta)$ depend only on δ.

As stated in [12], this is equivalent to saying that the expected time until each vertex v is visited $\pi_v \mathbf{COV}[G]$ times - which we shall refer to as the *blanket-cover criterion* - is $O(\mathbf{COV}[G])$.

2.3 Random walks and electrical networks

We give some key facts and ideas relevant to this work, drawing on [10], which discusses electrical network theory in the wider context of Markov chains. [6] is the classical treatment. Consider a finite, connected graph $G = (V,E)$ with edge weights $\{c(e) : e \in E\}$. For a vertex u, let $c(u) = \sum_{v:(u,v)\in E} c(u,v)$ with each loop counted once, and let $c(G) = \sum_{u \in V} c(u)$. In the language of electrical network theory, the weight $c(u,v)$ is known as the *conductance* of the edge (u,v), and the *resistance* $r(u,v) = 1/c(u,v)$. The random walk on G defined by the transition matrix $P(u,v) = c(u,v)/c(u)$ defines a reversible Markov chain with the vertices of G as states and the transition matrix P. The stationary distribution is $\pi(u) = c(u)/c(G)$. Conversely, every reversible Markov chain can be shown to be a network. Thus the two are equivalent. A *flow* f is an asymmetric function on oriented edges, i.e., for $(u, v) \in E(G)$, $f(u, v) = -f(v, u)$ and *net flow* $f(u)$ at a vertex u is $\sum_{v:(u,v)\in E(G)} f(u, v)$. We note $\sum_{u\in V(G)} f(u) = 0$. For vertices a, z, a *flow from a to z f* is a flow with the additional properties that (i) $f(u) = 0$ for all $u \in V(G)\backslash\{a, z\}$ and (ii)the *strength* of the flow $f(a) \geq 0$. The *energy* $\mathcal{E}(f)$ of a flow from a to z, f is defined as $\mathcal{E}(f) = \sum_{e\in E} f^2(e)r(e)$ where the sum is over unoriented edges (i.e., each edge is considered once). We have the following

Lemma 1 (Thomson's principle). *For any finite connected graph, the* effective resistance $R(a, z)$ *between a and z is such that*

$$R(a, z) = \min\{\mathcal{E}(f) : f \text{ is a unit flow from } a \text{ to } z\}. \tag{4}$$

There is unique minimiser in the above known as the current flow.

This allows us to say that the energy of any unit flow we contruct is an upperbound on effective resistance. The following facts are useful. **Series law** Edges $(a, b), (b, c)$ with can be replaced by a single edge (a, c) with $r(a, c) = r(a, b) + r(b, c)$ if there are no other edge incident on b. **Parallel law** Parallel edges $(a, b)_1, (a, b)_2$ can be replaced by a single edge (a, b) with $c(a, b) = c((a, b)_1) + c((a, b)_2)$. **Shorting law** Adding an edge of zero resistance between two vertices is equivalent to merging them into one vertex, and cannot increase effective resistance anywhere in the network. **Cutting law** Removing an edge with positive conductance cannot decrease effective resistance anywhere in the network. **Monotonicity law** The effective resistance between two given vertices is monotonic in the resistances of the edges in the whole network.

The $k \times k$ lattice graph P_k^2, where P_k is the k-path, plays an important role in our work. We shall analyse random walks on subgraphs isomorphic to this structure. It is well known in the literature (see, e.g. [10]) that for any pair of vertices $u, v \in V(P_k^2)$, we have $R(u, v) \leq C \log k$ where C is some universal constant. We shall quote part of [9] Lemma 3.1 in our notation and refer the reader to the proof there.

Lemma 2 ([9], Lemma 3.1). *(a) Let u and v be any two vertices of P_k^2. Then $R(u, v) < 8L_k$, where L_k is the k'th harmonic number.*

The following important lemmas are widely used in the field

Lemma 3 ([3]). *For vertices $u, v \in V(G)$*

$$\mathbf{COM}[u, v] = c(G)R(u, v) \tag{5}$$

Lemma 4 ([11]). *For a finite connected graph G, (a)* $\mathbf{COV}[G] \leq \mathbf{H}^{+}[G]L_n$.

3 Related work

A d-demensional torus on $N = n^d$ vertices is the d'th power of an n-cycle, $\mathbb{Z}_n^d$. The behaviour of random walks on this structure is well studied. It is well-known (see, e.g., [10]) that $\mathbf{COV}[\mathbb{Z}_n^d] = \Theta(N(\log N)^2)$ when $d = 2$ and $\mathbf{COV}[\mathbb{Z}_n^d] = \Theta(N \log N)$ when $d \geq 3$. [4] gives $\mathbf{COV}[\mathbb{Z}_n^2] \sim \frac{1}{\pi} N(\log N)^2$. [9] extends the study of graph powers giving the following theorem

Theorem 3 ([9], Theorem 1.2). *Let $G = (V, E)$ be any connected graph on n vertices with $\theta_G = 2|E|/n$. Let $d \geq 2$ be an integer and let $N = n^d$. For $d = 2$, $\mathbf{COV}[G^d] = O(\theta_G N(\log N)^2)$ and for $d \geq 3$, $\mathbf{COV}[G^d] = O(\theta_G N \log N)$. These bounds are tight.*

[2] gives a number of theorems related to random walks and effective resistance between pairs of vertices in graph products. To give the reader a flavour we quote Theorem 1 of that paper, which is useful as a lemma implicitly in this paper and in the proof of [9] Theorem 1.2 to justify the intuition that the effective resistance is maximised between opposite corners of the square lattice.

Lemma 5 ([2], Theorem 1). *Let P_n be an n-vertex path with endpoints x and y. Let G be a graph and let a and b be any two distinct vertices of G. Consider the graph $G \times P_n$. The effective resistance R((a,x),(b,v)) is maximised over vertices v of P_n at $v = y$.*

For P_n^2 this is used twice: $R((0, 0), (r, s)) \leq R((0, 0), (n - 1, s)) \leq R((0, 0), (n - 1, n - 1))$.

4 Locally observed random walk

Let $G = (V, E)$ be a connected, unweighted (equiv., uniformly weighted) graph. Let $S \subset V$ and let $G[S]$ be the subgraph of G induced by S. Let $B = \{v \in S : \exists x \notin S, (v, x) \in E\}$. Call B the *boundary* of S, and the vertices of $V \backslash S$ *exterior vertices*. If $v \in S$ then $d_G(v)$ (the degree of v in G) is partitioned into $d(v, in) = |N(v, in)| = |N(v) \cap S|$ and $d(v, out) = |N(v, out)| = |N(v) \cap (V - S)|$, (*inside* and *outside* degree). Here $N(v)$ denotes the neighbour set of v.

Let $u, v \in B$. Say that u, v are *exterior-connected* if there is a (u, v)-path $u, x_1, ...x_k, v$ where $x_i \in V \backslash S, k \geq 1$. Thus all vertices of the path except u, v are exterior, and the path contains at least one exterior vertex. Let $A(B) = \{(u, v) :$ u, v are exterior-connected $\}$. Note $A(B)$ may include self-loops.

Call edges of $G[S]$ *interior*, edges of $A(B)$ *exterior*. We say that a walk $\omega = (u, x_1, ...x_k, v)$ on G is an exterior walk if $u, v \in S$ and $x_i \notin S$, $1 \leq i \leq k$.

We derive a weighted multi-graph H from G and S as follows: $V(H) = S$, $E(H) = E(G[S]) \cup A(B)$. Note if $u, v \in B$ and $(u, v) \in E$ then $(u, v) \in E(G[S])$, and if, furthermore, u, v are exterior connected, then $(u, v) \in A(B)$ *and these edges are distinct*, hence, H may not only have self-loops but also parallel edges, ie., $E(H)$ is a multiset.

Associate with an orientation $(\boldsymbol{u},\boldsymbol{v})$ of an edge $(u,v) \in A(B)$ the set of all exterior walks $\omega = (u, x_1, ...x_k, v)$, $k \geq 1$ that start at u and end at v, and associate with each such a walk the value $p(\omega) = 1/(d_G(u)d_G(x_1)...d_G(x_k))$ (note, the $d(x_i)$ is not ambiguous, since $x_i \notin E(H)$, but we leave the 'G' subscript in for clarity). This is precisely the probability that the walk ω is taken by a simple random walk on G starting at u. Let

$$p_H(\boldsymbol{u},\boldsymbol{v}) = \sum_{k\geq 1} \sum_{\omega=(u,x_1...x_k,v)} p(\omega), \tag{6}$$

where the sum is over all exterior walks ω.

We set the edge conductances (weights) of H as follows: If e is an interior edge, $c(e) = 1$. If it is an exterior edge $e = (u,v)$ define $c(e)$ as

$$c(e) = d_G(u)p_H(\boldsymbol{u},\boldsymbol{v}) = \sum_{k\geq 1} \sum_{\omega=(u,x_1...x_k,v)} \frac{1}{d_G(x_1)...d_G(x_k)} = d_G(v)p_H(\boldsymbol{v},\boldsymbol{u}) \tag{7}$$

Thus the edge weight is consistent. A weighted random walk on H is thus a finite reversible Markov chain with all the associated properties that this entails.

Definition 3. *The weighted graph H derived from (G,S) is termed* the local observation of G at S*, or* G locally observed at S*. We shall denote it as $H = Loc(G,S)$.*

The intuition in the above is that we wish to observe a random walk $\mathcal{W}(G)$ on a subset S of the vertices. When $\mathcal{W}(G)$ makes an external transition at the border B, we cease observing and resume observing if/when it returns to the border. It will thus appear to have transitioned a virtual edge between the vertex it left off and the one it returned on. It will therefore appear to be a weighted random walk on H. This equivalence is formalised thus

Definition 4. *Let G be a graph and $S \subset V(G)$. For a (unweighted) random walk $\mathcal{W}(G)$ on G starting at $x_0 \in S$, derive the Markov chain $\mathcal{M}(G,S)$ on the states of S as follows: (i) $\mathcal{M}(G,S)$ starts on x_0 (ii) If $\mathcal{W}(G)$ makes a transition through an internal edge (u,v) then so does $\mathcal{M}(G,S)$ (iii)If $\mathcal{W}(G)$ takes an exterior walk $\omega = (u, x_1...x_k, v)$ then $\mathcal{M}(G,S)$ remains at u until the walk is complete and subsequently transitions to v. We call $\mathcal{M}(G,S)$* the local observation of $\mathcal{W}(G)$ at S*, or* $\mathcal{W}(G)$ locally observed at S.

Lemma 6. *For a walk $\mathcal{W}(G)$ and a set $S \subset V(G)$, the local observation of $\mathcal{W}(G)$ at S, $\mathcal{M}(G,S)$ is equivalent to the weighted random walk $\mathcal{W}(H)$ where $H = Loc(G,S)$.*

Proof. See Appendix.

5 A general bound

We give **COV**$[P]$ bounds in terms of H, and by symmetry, Theorem 1 can be inferred. The lower bound is easy: It is clear that the H dimension needs to be covered - that is, each copy of G needs to be visited at least once. The probability of moving through the H dimension is at least $\frac{\Delta_H}{\Delta_H+\delta_G}$, and the lower bound follows.

For the upper bound, we first require the following lemmas. Denote by $R_{max}(G)$ the maximum effective resistance between any pair of vertices in a graph G.

Lemma 7. *For a graph G and tree T, $R_{max}(G\Box T) < 4R_{max}(G\Box P_r)$ where $|V(T)| \le r \le 2|V(T)|$ and P_r is the path on r vertices.*

Proof. Note first the following: (i) By the parallel law, an edge (a, b) of unit resistance can be replaced with two parallel edges between a, b, each of resistance 2. (ii) By the shorting law, a vertex a can be replaced with two vetices a_1, a_2 with a zero-resistance edge between them and the ends of edges incident on a disributed arbitrarily between a_1 and a_2. These transformations preserve electrical properties of a network.

Let $F = G\Box T$. Starting from some vertex v in T, perform a depth-first search (DFS) of T stopping when all vertices in T have been visited. Each edge of T is traversed at most twice; once in each orientation (though a particular vertex x will be visited up to $d(x)$ times). Let (e_i) be the sequence of oriented edges generated by the search. The idea is to use (e_i) to construct a transformation from $F = G\Box T$ to $G\Box P_r$. From (e_i), we derive another sequence (a_i), which is generated by following (e_i) and if we have edges e_i, e_{i+1} with $e_i = (a, b)$, $e_{i+1} = (b, c)$ such that it is neither the first time nor the last time b is visited in the DFS, then we replace e_i, e_{i+1} with (a, c). We term such an operation an *aggregation.* Consider F; by (i) above we can replace each unit-resistance edge by a pair of parallel edges each of resistance 2. For a pair of parallel edges in the T dimension, arbitrarily label one of them with an orientation, and label the other with the opposite orientation. Note, orientations are only an aid to the proof, and are not a flow restriction. We therefore see that (e_i) can be interpreted as a sequence of these parallel oriented edges. Now we modify F using (a_i): If $(a, b), (b, c)$ was aggregated to (a, c), then replace each pair of **oriented** edges $((x.a), (x, b))$ and $((x, b)(x, c))$ in F with an oriented edge $((x, a), (x, c))$ and set the resistance of it the sum of the resistances of the replaced edges. This operation is the same as restricting flow through $((x.a), (x, b))$ and $((x, b)(x, c))$ to only going from one to the other at vertex (x, b), without the possiblity of going through other edges, The infimum of this subset of flows is at least the infimum of the previous set and so by Thomson's principle, the effective resistance cannot be decreased by this operation.

For each copy of G in F excluding those that do not correspond to a leaf of T, by(ii), we can do the following: Create a "twin" copy by associating with each vertex $x \in V(F)$ (except those excluded) a twin vertex x', putting a zero-resistance edge between x and x'. We then (a) redistribute the parallel edges in the G dimension so as to preserve structural isomorphism between each copy and G and (b) redistribibute edges in the T dimension so as to respect the sequence (a_i). This means that when we trace (a_i) via any vertex $x \in V(G)$, then if we have $(a,b),(b,c)$ in (a_i), we must have the corresponding path of oriented edges $((x,a),(x,b)),((x,b),(x,c))$. We then remove the zero-resistance edges between each pair of twin vertices, and by Rayleigh's cutting law, this cannot decrease the effective resistance. Using the sequence (a_i) to trace a path of copies of G along the T dimension, we see that the resulting structure is isomorphic to $G\square P_r$. Since the aggregation process only aggregates edges that pass through a previously seen vertex, r is at least k. Also, because each edge is traversed at most once in each direction, r is at most $2k$. Each edge has resistance at most 4, and so the theorem follows.

Lemma 8. *For graphs G, H with with $D_G = k$ and $k \le n_H \le Rk$, $R_{max}(G\square H) < 32(3+R)L_k \le \zeta R \log D_G$ where L_k is the k'th harmonic number and ζ is some universal constant.*

Proof. Let $(a,x),(b,y)$ be any two vertices in $G\square H$. Let D be some diametric path of G. Let $\langle a, D\rangle$ represent the shortest path from a to D in G (which may trivially be a if it is on D). Similarly with $\langle b, D\rangle$. Let $T_D = D \cup \langle a, D\rangle \cup \langle b, D\rangle$. Note $k \le |V(T_D)| \le 3k$. Now let T_H be any spanning tree of H. Applying Lemma 7 twice we have

$$R_{max}(T_D\square T_H) < 4R_{max}(T_D\square P_r) < 16R_{max}(P_r\square P_s) \tag{8}$$

where $k \le r \le 6k$ and $k \le s \le 2Rk$. Considering a series of connected P_k^2 subgraphs and using the triangle inequality for effective resistance, we have $R_{max}(P_r\square P_s) \le 32(3+R)L_k$ and the theorem follows.

Lemma 8 gives us an upperbound of $\zeta \log D_G$ for the effective resistance in a block (definition below), which in turn allows us to bound the maximum hitting time within a block, and therefore the cover time via Matthews' technique.

The following proves the upperbound in Theorem 1.

Theorem 4. *Let $P = G\square H$ where G, H are any connected, finite graphs. We have*

$$\mathbf{COV}[P] \le K\left(\left(1+\frac{\Delta_G}{\delta_H}\right)\mathbf{COV}[H] + \frac{M m_G m_H n_H l^2}{\mathbf{COV}[H] D_G}\right) \tag{9}$$

where $l = \log D_G \log(n_G D_G)$ and K is some universal constant.

Proof. We group the vertices of H into sets such that for any set S and the subgraph of H induced by S, $H[S]$: (i)$|S| \ge D_G$, (ii)$H[S]$ is connected, (iii) The diameter of $H[S]$ is at most $4D_G$. We demonstrate this grouping is possible through the following algorithm on H: Choose some arbitrary vertex v as the root, and using a breadth-first search (BFS), descend from v at most distance D_G. The resulting tree $T(v)$ will have diameter at most $2D_G$. For each leaf l of $T(v)$, continue the BFS using l as a root. If $T(l)$ has fewer than D_G vertices, append it to $T(v)$. If not, recurse on the leaves of $T(l)$. Each tree then forms a group that satisfies the three conditions above. The root is part of a new group, unless it has been appended to another tree.

In the product P we refer to copies of G as *columns*. In P we have a natural association of each column with the set $S \subseteq V(H)$ defined above. We denote by $Block[S]$ the set of columns in P associated with S union with all edges incident on vertices in those columns. Therefore $Block[S] = (G\Box H[S]) \cup \{(u,v) \in E(P) : u \in V(G\Box H[S])\}$ (note, this is not a graph since it contains edges with free ends). For any two vertices $(.a), (.,b) \in Block[S]$ we can find a connected subgraph $T(a,b)$ of the tree T that generated S such that a and b are connected in $T(a,b)$ and $D_G \leq |V(T(a,b))| \leq 4D_G$. Then using Lemma 8, we can bound the effective resistance $R((.a),(.,b)) \leq 4\zeta \log D_G$.

Hence

$$R_{max}(Block[S]) \leq 4\zeta \log D_G \tag{10}$$

Similarly, $G\Box H[S] \subseteq Loc(P, V(Block[S]))$, and so

$$R_{max}(Loc(P, V(Block[S]))) \leq 4\zeta \log D_G \tag{11}$$

It is envisaged that the following is used with the idea in mind that G is small relative to H, and so the cover time of the product is essentially dominated by the cover time of H.

We use the following two-phase approach

Phase 1 Perform a random walk on $\mathcal{W}(P)$ until the blanket-cover criterion is satisfied for the H dimension.

Phase 2 Starting from the end of phase 1, perform a random walk on P until all vertices of P not visited in phase 1 are visited.

Phase 1 can be thought of in the following way: We couple $\mathcal{W}(P)$ with a walk $\mathcal{W}(H)$ such that (i)if $\mathcal{W}(P)$ starts at $(.,x)$, then $\mathcal{W}(H)$ starts at x, and (ii) $\mathcal{W}(H)$ moves to a new vertex y from a vertex x when and only when $\mathcal{W}(P)$ moves from $(.,x)$ to $(.,y)$ for the first time. This coupled process runs until $\mathcal{W}(H)$ satisfies the blanket-cover criteria for H, ie, when each vertex $u \in V(H)$ has been visited at least $\pi(u)\mathbf{COV}[H]$ times.

Having grouped P into blocks, we analyse the outcome of phase 1 by relating $\mathcal{W}(P)$ to the local observation on each block. A particular block B will have some vertices unvisited by $\mathcal{W}(P)$ if and only if $\mathcal{W}(P)$ locally observed on B fails to visit all vertices. We refer to such a block as *failed*. Consider the weighted random walk $\mathcal{W}(B')$ on $B' = Loc(P, V(B))$. This has the same law as $\mathcal{W}(P)$ locally observed on B. Hence, we bound the probability of $\mathcal{W}(P)$ failing to cover B by bounding the probability that $\mathcal{W}(B')$ fails to cover B'. Done for all blocks, we can bound the expected time it takes phase 2 to cover the failed blocks. We think of phase 1 as doing most of the "work", and phase 2 as a "mopping up" phase. Mopping up a block in phase 2 is costly, but if there are few of them, the overall cost is within a small factor of phase 1.

We bound $\mathbf{Pr}(\mathcal{W}(B')$ fails) by exploiting the fact that $\mathcal{W}(B')$ will have made some minimal number of transitions t. This is guaranteed because phase

1 terminates only when $\mathcal{W}(H)$ has statisfied the blanket-cover criterion on H, that is, each vertex $u \in V(H)$ has been visited at least $\pi(u)\mathbf{COV}[H]$ times, so each column G_u in P will have been visited at least that many times. If κ counts the number of steps of a walk $\mathcal{W}(B')$ until B' is covered, then $\mathbf{Pr}(\mathcal{W}(B')$ fails to cover$B') = \mathbf{Pr}(\kappa > t) \leq \mathbf{E}[\kappa]/t$ by Markov's inequality.

Definition 5. *For graphs $I = J \square K$, and $S \subset I$, denote by $S.J$ the* projection *of S* on *to J, that is, $S.J = \{u \in J : (u,.) \in S\}$.*

For a weighted graph G recall $c(G)$ is the total conductance (weight) of all edges of G.

Let B be a block and let $B' = Loc(P, V(B))$. By Section 4 $c(B') = c(B)$, given by the following

$$c(B) \leq m_G |\{u \in V(B).H\}| + n_G \sum_{u \in V(B).H} d(u) \tag{12}$$

Using 11 and Lemma 3 we therefore have for any $u, v \in V(B')$, $\mathbf{COM}[u,v] \leq Kc(B') \log D_G$ for some universal constant K. (In what follows K will change, but we shall keep the same symbol, with an understanding that what we finish with is a univeral constant). Hence, by Lemma 4

$$\mathbf{COV}[B'] \leq Kc(B') \log D_G \log(|V(B')|) = Kc(B) l_B \tag{13}$$

where $l_B = \log D_G \log(|V(B)|)$

For a block B, the number of transitions on the H dimension - and therefore the number of transitions on B - as demanded by the blanket-cover criterion is at least

$$\sum_{u \in V(B).H} \mathbf{COV}[H]\pi(u) = \frac{\mathbf{COV}[H]}{2m_H} \sum_{u \in V(B).H} d(u) \tag{14}$$

Now

$$\mathbf{Pr}(\mathcal{W}(P) \text{ fails on } B) = \mathbf{Pr}(\mathcal{W}(B') \text{ fails on } B') \tag{15}$$

$$\leq Kc(B) l_B \left(\frac{\mathbf{COV}[H]}{2m_H} \sum_{u \in V(B).H} d(u) \right)^{-1}. \tag{16}$$

The second equality by Markov's inequality.

Phase 2 consists of two components: movement between failed blocks, and covering a failed block it has arrived at. The total block-to-block movement is upperbounded by the time is takes to cover the H dimension of P (in other words, for each column to have been visited at least once). We denote this by $\mathbf{COV}_P[P.H]$. Denoting the covertime of a block B by the walk $\mathcal{W}(P)$ by $\mathbf{COV}_P[B]$,

$$\mathbf{E}[Ph2] \leq \mathbf{COV}_P[P.H] + \sum_{B\in P} \mathbf{Pr}(\mathcal{W}(P) \text{ fails on } B)\mathbf{COV}_P[B] \tag{17}$$

For $\mathcal{W}(H)$, the r.v. $\beta_H = \min\{t : (\forall v)N_v(t) > \pi(v)\mathbf{COV}[H]\}$ counts the time it takes to satisfy the blanket-cover criterion on H.

The expected number of movements on P per movement on the H dimension is at most $\frac{\Delta_G+\delta_H}{\delta_H}$. Therefore $\mathbf{E}[Ph1] \leq \frac{\Delta_G+\delta_H}{\delta_H}\mathbf{E}[\beta_H]$. Similarly, $\mathbf{COV}_P[P.H] \leq \frac{\Delta_G+\delta_H}{\delta_H}\mathbf{COV}[H]$.

Using 10 and Lemmas 3 and 4 again, we have $\mathbf{COV}_P[B] \leq K'c(P)l_B$ where $c(P) = |E(P)| = M$. Theorem 2 gives us $\mathbf{E}\beta_H \leq K\mathbf{COV}[H]$, for some universal constant K and so

$$\mathbf{COV}[P] \leq \mathbf{E}[Ph1] + \mathbf{E}[Ph2] \tag{18}$$

$$\leq K\frac{\Delta_G+\delta_H}{\delta_H}\mathbf{COV}[H] + \sum_{B\in P} \mathbf{Pr}(\mathcal{W}(P) \text{ fails on } B)\mathbf{COV}_P[B] \tag{19}$$

We have, using 16

$$\sum_{B\in P} \mathbf{Pr}(\mathcal{W}(P) \text{ fails on } B)\mathbf{COV}_P[B] \leq K\frac{Mm_H}{\mathbf{COV}[H]} \sum_{B\in P} \frac{c(B)l_B^2}{\sum_{u\in V(B).H} d(u)} \tag{20}$$

and

$$\sum_{B\in P} \frac{c(B)l_B^2}{\sum_{u\in V(B).H} d(u)} \leq \sum_{B\in P} \left(n_G + \frac{m_G|\{u \in V(B).H\}|}{\sum_{u\in V(B).H} d(u)} \right) l_B^2 \tag{21}$$

Since $\sum_{u\in V(B).H} d(u) \geq |\{u \in V(B).H\}|$, we have

$$\sum_{B\in P} \mathbf{Pr}(\mathcal{W}(P) \text{ fails on } B)\mathbf{COV}_P[B] \leq K\frac{Mm_Gm_H}{\mathbf{COV}[H]} \sum_{B\in P} l_B^2 \tag{22}$$

$$\leq K\frac{Mm_Gm_Hn_Hl^2}{\mathbf{COV}[H]D_G} \tag{23}$$

where $l = \log D_G \log(n_G D_G)$

References

1. R. Aleliunas, R.M. Karp, R.J. Lipton, L. Lovász, C. Rackoff, Random Walks, Universal Traversal Sequences, and the Complexity of Maze Problems, *Proceedings of the 20th Annual IEEE Symposium on Foundations of Computer Science* (1979) 218-223.
2. B. Bollobas, G. Brightwell, Random walks and electrical resistances in product graphs, *Discrete Applied Mathematics* 73 (1997), 69-79.

3. A.K. Chandra, P. Raghavan, W.L. Ruzzo, R. Smolensky, P. Tiwari, The electrical resistance of a graph captures its commute and cover times, *Computational Complexity*, 6 (1997) 312-340.
4. A. Dembo, Y. Peres, J. Rosen, O. Zeitouni, Cover times for Brownian motion and random walks in two dimensions, *Ann. Math.*, 160 (2004) 433-464.
5. J. Ding, J.R. Lee, Y. Peres, Cover times, blanket times and majorizing measures, manuscript, 2010
6. Peter G. Doyle and J. Laurie Snell, *Random walks and electrical networks*, 2006.
7. U. Feige, A tight upper bound for the cover time of random walks on graphs, *Random Structures and Algorithms*, 6 (1995) 51-54.
8. U. Feige, A tight lower bound for the cover time of random walks on graphs, *Random Structures and Algorithms*, 6 (1995) 433-438.
9. J. Jonasson, An upper Bound on the Cover Time for Powers of Graphs, *Discrete Mathematics* 222 (2000), 181-190.
10. David A. Levin, Yuval Peres, Elizabeth L. Wilmer, *Markov Chains and Mixing Times*, 2009.
11. P. Matthews, Covering problems for Brownian motion on spheres, *Ann. Prob.*, 16:189-199, 1988. C. St.J A. Nash-Williams, Random walk and electric currents in networks *Proc. Camb. Phil. Soc.*, 55:181-194, 1959.
12. P. Winkler, D. Zuckerman, Multiple Cover Time *Random Structures and Algorithms*, 9, 403-411, 1996.

A Appendix

A.1 Proof of Lemma 6

The states are clearly the same so it remains to show that the transition probability $P_{\mathcal{M}}(u,v)$ from u to v in $\mathcal{M}(G,S)$ is the same as $P_{\mathcal{W}(H)}(u,v)$ in $\mathcal{W}(H)$. Recall that B is the border of the induced subgraph $G[S]$. If $u \notin B$ then an edge $(u,v) \in E(H)$ is internal and so has unit conductance in H, as it does in G. Furthermore, for an internal edge e, $e \in E(H)$ if and only if $e \in E(G)$, thus $d_H(u) = d_G(u)$ when $u \notin B$. Therefore $P_{\mathcal{W}(H)}(u,v) = 1/d_H(u) = 1/d_G(u) = P_{\mathcal{M}}(u,v)$.

Now suppose $u \in B$. Let $E(u)$ denote the set of all edges incident with u in H and recall $A(B)$ above is the set of exterior edges. The total conductance (weight) of the exterior edges at u is

$$\begin{aligned}\sum_{e\in E(u)\cap A(B)} c_H(e) &= \sum_{x\in N(u,out)} \sum_{v\in B} \mathbf{Pr}(\text{walk from } x \text{ returns to } B \text{ at } v)\\ &= \sum_{x\in N(u,out)} 1\\ &= d(u,out).\end{aligned}$$

(Note the 'H' subscript in $c_H(e)$ above is redundant since exterior edges are only defined for H, but we leave it for clarity).

Thus for $u \in B$

$$\begin{aligned} c_H(u) = \sum_{e \in E(u)} c_H(e) &= \sum_{e \in E(u) \cap G[S]} 1 + \sum_{e \in E(u) \cap A(B)} c_H(e) \\ &= d(u, in) + d(u, out) \\ &= d_G(u) \end{aligned}$$

Now

$$P_{\mathcal{M}}(u,v) = \mathbf{1}_{\{(u,v) \in G[S]\}} \frac{1}{d_G(u)} + \sum_{k \geq 1} \sum_{\omega = (u, x_1 \dots x_k, v)} \frac{1}{d_G(u) d_G(x_1) \dots d_G(x_k)} \tag{24}$$

where the sum is over all exterior walks ω. Thus

$$P_{\mathcal{M}}(u,v) = \mathbf{1}_{\{(u,v) \in G[S]\}} \frac{1}{d_G(u)} + p_H(\boldsymbol{v}, \boldsymbol{u}) \tag{25}$$

$$\begin{aligned} P_{\mathcal{W}(H)}(u,v) &= \frac{1}{c_H(u)} \left[\mathbf{1}_{\{(u,v) \in G[S]\}} + \mathbf{1}_{\{(u,v) \in A(S)\}} c_H(u,v) \right] && (26) \\ &= \frac{1}{d_G(u)} \left[\mathbf{1}_{\{(u,v) \in G[S]\}} + \mathbf{1}_{\{(u,v) \in A(S)\}} d_G(u) p_H(\boldsymbol{v}, \boldsymbol{u}) \right] && (27) \\ &= \mathbf{1}_{\{(u,v) \in G[S]\}} \frac{1}{d_G(u)} + \mathbf{1}_{\{(u,v) \in A(S)\}} p_H(\boldsymbol{v}, \boldsymbol{u}) && (28) \\ &= P_{\mathcal{M}}(u,v) && (29) \end{aligned}$$